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CONDENSED-STATE PHYSICS

STRESS-STRAIN STATE OF ELASTIC PLATE WITH A CRACK

E. E. Deryugin UDC 538.9

The paper considers an infinite plate with a crack in the form of a narrow hole with a certain curvature at the tip of the crack. The stress-strain state parameters of this plate under uniaxial load are studied, such as the stress-concentration factor, crack-driving force, and elastic strain energy. Determined are the elastic energy consumption during the crack propagation, general laws of the mechanical state of the cracked plate, and curvature at the tip of the crack. It is shown that, in fact, the Griffith crack has no stress singularity at the end. The stress-strain state of the plate with an elliptical shaped crack is similar to that with a uniform plastic deformation zone.

Keywords: stress-strain state, crack, elliptical shaped crack, stress concentration, crack-driving force, curvature, energy.

INTRODUCTION

Fracture of solids is caused by the formation and development of macroscopic cracks. In fracture mechanics, this process is schematized by replacing a crack with a hole of a zero thickness. This hole is an ellipse with its small semi-axis approaching to zero. A study of the localized stress-strain state in the vicinity of elliptical shaped cracks is therefore very interesting.

The stress analysis for a plate with an elliptical hole was first proposed by Inglis [1]. The linear elastic solution proposed by Inglis for the stress field surrounding the ellipse, was an important step in the development of the linear elastic fracture mechanics. Similar to the solution proposed by Kirsch [2] for a circular hole, it was applied to an infinite isotropic plate under uniaxial tension. In contrast to Kirsch's solution, the solution proposed by Inglis, could be applied to an infinite number of different scenarios of ellipses with different ratios of semi-axes.

An important characteristic of an ellipse is its curvature at the semi-major axis, i.e., $\eta = 1/r_a$, where r_a is the curvature at the semi-major axis. The solution of the boundary value problem concerning the holes shows that only two geometrical parameters have a significant influence on the displacement of the crack periphery and the stress concentration during tension, namely: the crack length in the direction perpendicular to the tensile axis and the maximum curvature at the semi-major axis along this direction [3, 4]. Therefore, stresses at the tip of the crack $2a$ long and the curvature η can be determined, if considering the crack as an ellipse with the semi-major axis a and the semi-minor axis $b = (a/\eta)^{1/2}$.

The aim of this work is to study an infinite plate with a crack in the form of a narrow hole with a certain curvature at the tip. The stress-strain state and the mechanical state characteristics are determined for this plate under the uniaxial load, such as stress-concentration factor, crack-driving force, and elastic strain energy. The dependences between the stress-strain curves and the crack curvature are analyzed under the tensile load.

Institute of Strength Physics and Materials Science of the Siberian Branch of the Russian Academy of Sciences, Tomsk, Russia, e-mail: dee@ms.tsc.ru. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 11, pp. 3–8, November, 2020. Original article submitted February 12, 2020.

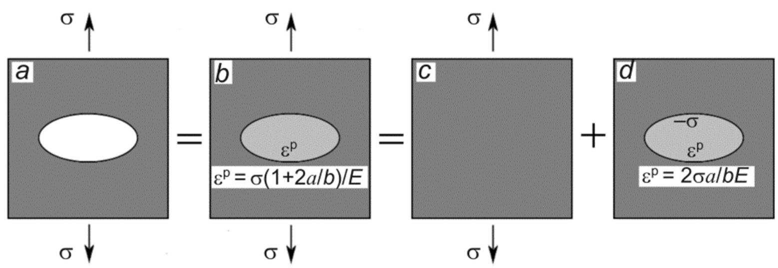

Fig. 1. Schematic of stress-strain state identity between the plate with (*a*) and without the elliptic hole, but with plastic deformation zone (*b*), *c* – uniform stress field σ, *d* – plate with plastic deformation zone ε^p.

INFINITE PLATE WITH ELLIPTICAL HOLE UNDER TENSION

The general solution of the problem of the infinite plate with an elliptical hole under tension can be found in the work of Musheliŝvili [5]. In the Cartesian coordinate system with the origin of the semi-major axis *a* of the ellipse under σ tension along the *y*-axis, the stress tensor components along the *x*-axis can be written as

$$\sigma_x = \frac{\sigma a}{a-b}\left[\frac{-a}{a-b} + \frac{x+a}{c}\left[\frac{a}{a-b} - \frac{b^2}{c^2}\right]\right], \quad \sigma_y = \frac{\sigma a}{a-b}\left[\frac{b^2}{a(a-b)} + \frac{x+a}{c}\left[1 - \frac{b}{a-b} + \frac{b^2}{c^2}\right]\right], \tag{1}$$

where $c = \sqrt{x^2 + 2xa + b^2}$, *b* is the semi-minor axis of the ellipse. The tensor component is $d\tau_{xy} = 0$.

Equations (1) describe the nonuniform stress field outside the hole. All the stress tensor components jump to zero at the ellipse periphery. The significant elastic strain energy concentrates in a small region nearby the ellipse periphery. The stress field (Eq. (1)) identically determines the displacement of the ellipse periphery:

$$u_x = -\frac{\sigma}{E}y, \qquad u_y = \frac{\sigma}{E}\left(1 + 2\frac{a}{b}\right)x\,, \tag{2}$$

where *E* is Young's modulus. One can see that the boundary conditions (Eq. (2)) at the ellipse periphery satisfy the uniform plastic deformation zone:

$$\varepsilon_x^{\mathrm{p}} = \frac{\partial u_x}{\partial y} = \frac{-\sigma}{E}, \qquad \varepsilon_y^{\mathrm{p}} = \frac{\partial u_y}{\partial x} = \frac{\sigma}{E}\left(1 + 2\frac{a}{b}\right). \tag{3}$$

In accordance with the consistent continuous defect theory [6], the uniform plastic deformation zone is not associated with stresses. This means that the stress σ inside the ellipse is zero. Consequently, the stress-strain state beyond the hole accurately reproduces the stress-strain state of the plate with the plastic deformation zone (Eq. (3)).

Figure 1 illustrates the identity of the stress-strain state of the plate with and without the elliptic hole, but with the plastic deformation zone. The case shown in Fig. 1*b* can be considered as superposition of the uniform stress field σ and the plate with plastic deformation zone $\varepsilon^{\mathrm{p}} = 2\sigma a/(bE)$, any external forces being absent. In the last case, the nonuniform and uniform (–σ) fields of internal stresses are observed outside and inside the plastic deformation zone, respectively.

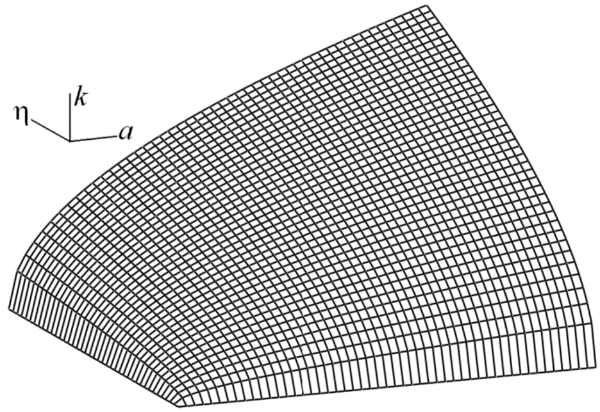

Fig. 2. The function $k = f(a, \eta)$.

It is much more convenient to consider the stress field outside the hole without the external uniform stress field σ. It is clear that the case shown in Fig. 1*d*, highlights the stress field associated with the hole in the plate. This additional stress field is characterized by the stress tensor component:

$$\sigma_y = \frac{\sigma a}{a-b}\left[-1+\frac{b}{a-b}+\frac{x+a}{c}\left[-1-\frac{b}{a-b}+\frac{b^2}{c^2}\right]\right]. \tag{4}$$

The maximum stress concentration occurs at the end of the semi-major axis of the ellipse. According to Eq. (1), the stress-concentration factor can be calculated as

$$k = \sigma_{y=0} / \sigma = 1 + 2a/b\,. \tag{5}$$

Equation (5) is used to calculate the stress concentration [4]. For example, according to Eq. (5), the stress concentration factor is 3 for a circular hole ($a = b$). The stress concentration factor of the additional stress field (Eq. (4)) is written as

$$k = 2a/b. \tag{6}$$

A comparison shows that the difference between Eq. (5) and Eq. (6) is unity. According to Eq. (3), the stress concentration factor is 2 for a circular hole ($a = b$). This difference is insignificant at a higher a/b ratio.

Substitution of $b = (a/\eta)^{1/2}$ into Eq. 3 leads to the equation for the stress concentration factor of the crack with η curvature:

$$k = \sigma_y / \sigma = 2\sqrt{a\eta}\,. \tag{7}$$

Equation (7) indicates a parabolic dependence between the crack curvature η and the stress concentration factor k at a given crack length a:

$$\eta = k^2/4a. \tag{8}$$

The stress distribution in the a–η coordinate is presented in Fig. 2. As can be seen, the stress concentration rises with decreasing both the crack length and the curvature. According to Eq. (8), the dependence between the crack curvature η and the stress concentration factor k is $\eta = k^2/4a$ at the crack length a. The similar dependence is observed between the stress concentration factor k and the crack length a at η = const, *viz*. $a = k^2/4\eta$.

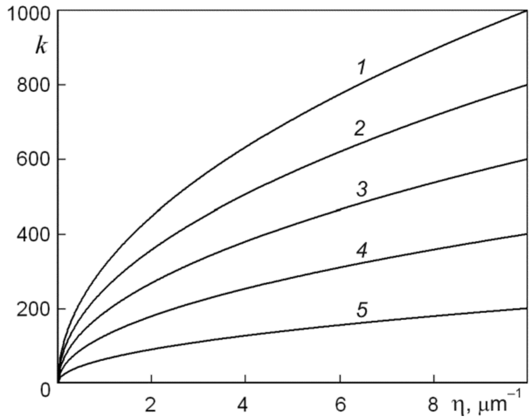

Fig. 3

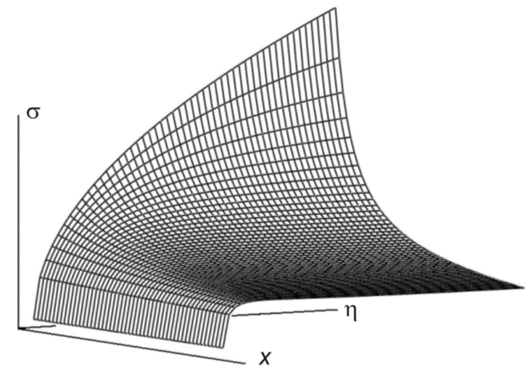

Fig. 4

Fig. 3. k-η dependences at different crack length: *1* – 25 mm, *2* – 16 mm, *3* – 9 mm, *4* – 4 mm, *5* – 1 mm.

Fig. 4. Stress distribution as $\sigma_y(x, \eta)$ function.

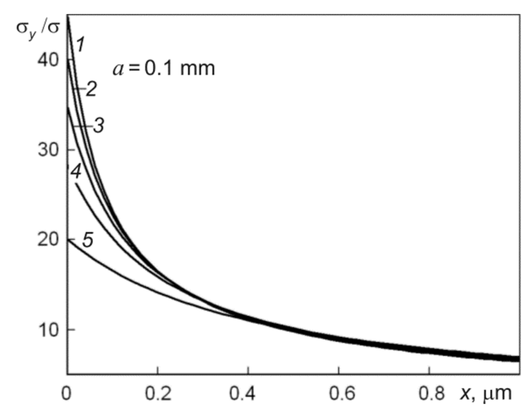

Fig. 5. σ_y/σ dependence on x for different crack curvature: *1* – 5 μm^{-1}, *2* – 4 μm^{-1}, *3* – 3 μm^{-1}, *4* – 2 μm^{-1}, *5* – 1 μm^{-1}.

The crack curvature $\eta = 10\ \mu m^{-1}$ presented in Fig. 3, matches the curvature $r = 0.1\ \mu m$. The stress σ_y nearby the crack having a 1 mm length, is 200 times higher than the external stress σ (curve *5*). The increase in the crack length leads to a rapid growth in the stress concentration. The stress σ_y nearby the crack with a 25 mm length is 1000 times higher than the applied stress σ (curve *1*).

In Fig. 4, the stress σ_y distribution nearby the crack is presented as a function of two variables, namely coordinate x and crack curvature η. The certain value of the crack curvature determines the stress distribution σ_y in front of the crack. Such stress distributions are presented in Fig. 5, where the increase in the crack curvature results in the growth in the stress concentration at the tip of the crack. The higher the crack curvature the more rapid is the stress drop on the x-axis. The influence of the curvature is significant only at the tip of the crack. According to Fig. 5, this influence is insignificant already at a distance of $x > 0.4\ \mu m$ for the crack with $a = 0.1$ mm.

Figure 6 shows the dependences between the ratio σ_y/σ and the crack curvature η at a different distance x to the tip of the crack. At $x = 0$ (curve *1*), the ratio σ_y/σ defines stresses at the semi-minor axis a, which is governed by the parabolic dependence (Eq. (8), curve *1*). The condition in Eq. (8) is satisfied only at $x = 0$. As can be seen from Fig. 6, this condition is not satisfied at $x > 0$.

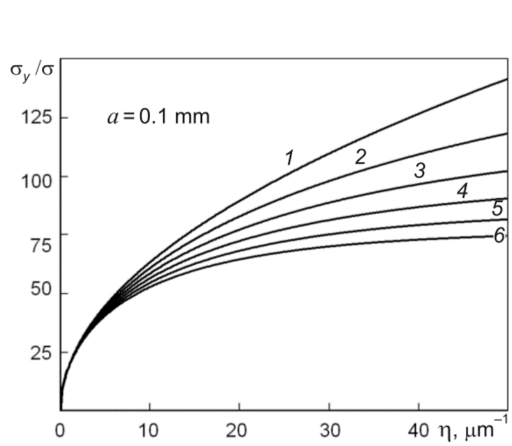

Fig. 6

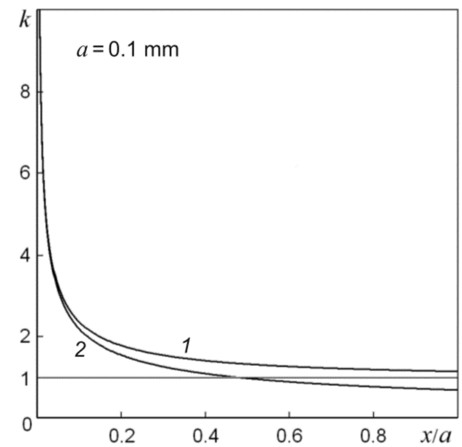

Fig. 7

Fig. 6. σ_y/σ dependences on crack curvature η for different distance: *1* – 0 mm, *2* – 0.1 mm, *3* – 0.2 mm, *4* – 0.3 mm, *5* – 0.4 mm, *6* – 0.5 mm.

Fig. 7. Stress distribution σ_y according to Eq. (10), curve *1* and Eq. (9), curve *2*.

GRIFFITH CRACK

Fracture mechanics of solids with cracks is based on quantitative relations proposed by Griffith [7], who considered a crack with a zero thickness. It was accepted that the classical Griffith crack had a significant drawback such as stress singularity at the tip of the crack. Approaching to the tip of the crack caused an unlimited stress growth. Let us consider the problem of stress singularity in a solid with a Griffith crack in more detail.

In [8], Irwin considered the stress distribution in the vicinity of the Griffith crack. He showed that in the case of the opening mode fracture, when the crack edges separated without shear, the ratio σ_y/σ along the tensile axis at the tip of the crack was

$$\sigma_y / \sigma = \sqrt{0.5a/x}\ . \tag{9}$$

On the other hand, based on Eq. (1), at $b = 0$ we have

$$\sigma_y / \sigma = \frac{x+a}{\sqrt{x^2 + 2xa^2}}\,. \tag{10}$$

In Fig. 7, curve *1* describes the stress distribution σ_y according to Eq. (10). At greater distances, the stress distribution σ_y approaches to the external stress level of $\sigma_y/\sigma = 1$. Curve *2* describes the ratio σ_y/σ according to Eq. (9), which is usually used to analyze the stress-strain state at the tip of the crack [9]. The comparative analysis shows that at a small distance to the tip of the crack, when the distance x does not exceed a twentieth of a half-length, the values of the stress distribution do not differ from each other. However, with increasing distance from the tip of the crack, curve *2* tends to zero, rather than the external stress σ as curve *1* does, which is in contradiction to the facts. At a greater distance to the crack, the stress in the plate cannot be lower than the external stress. The use of Eq. (9) is therefore limited by the condition of $x \ll a$.

The relation of $y = b[1 - (x/a)^2]^{1/2}$ is derived from the canonical equation of ellipse $(x/a)^2 + (y/b)^2 = 1$. Substituting this in Eq. (2), we obtain the relation for the displacement of the crack periphery along the y-axis:

$$u_y = \sigma(b+2a)[1 - (x/a)^2]^{1/2}/E. \tag{11}$$

When $b \to 0$, the displacement of the crack edges occurs:

$$u_y = 2\sigma a[1 - (x/a)^2]^{1/2}/E.$$

Hence, the crack in the plate takes the form of an ellipse with the semi-minor axis $b = 2a\sigma/E$. As $b \neq 0$, the Griffith crack has no stress singularity at the end, and the stress concentration ($k = E/\sigma$) is observed in the loaded system. For a hypothetical material (steel) with a crack 1 mm long and 210 GPa Yong's modulus, the maximum opening of the Griffith crack is 4 µm at 210 MPa external tension. This is 250 times shorter than the half-length of the crack. The stress at the crack mouth is 1000 times higher than the external stress.

Therefore, there is no stress singularity in the Griffith crack. Nevertheless, this does not facilitate the solution of practical problems. Under the low stress applied, the stress concentration occurs at the tip of the crack, far greater than the yield point of the material. Consequently, there is still a need for new crack models allowing to clarify causes of the low strength of real materials. First of all, it is necessary to take into consideration the influence of plastic deformation on the stress-strain state of a solid with a crack.

ELASTIC STRAIN ENERGY OF CRACKED PLATE

Let us calculate the elastic strain energy of the plate with the elliptical hole under the uniaxial load. The identity of the stress-strain state of the plate with the elliptical hole and with the uniform plastic deformation zone allows us to determine the external stress effect on plastic deformations. According to the continuous defect theory [6, 10], the elastic strain energy is equal to the energy dissipated during the stress relaxation in the plastic deformation zone:

$$U = 0.5S(\varepsilon_{\delta}^{\mathrm{p}} + \varepsilon_{y}^{\mathrm{p}})\sigma = \sigma^2 \frac{\pi a^2}{E}, \tag{12}$$

where $S = \pi ab$ is the ellipse surface. A change in the crack length by $2da$, requires the energy consumption:

$$G = \frac{dU}{2da} = \frac{\pi\sigma^2 a}{E}. \tag{13}$$

In the physical sense, the energy consumption G determines the intensity of the elastic energy release during the crack propagation [11]. The energy consumption is also called the crack-driving force, which is the energy per new surface area, which appears during the crack propagation. The energy consumption G is the crack resistance (fracture toughness) parameter of the material.

The energy characteristic of the crack resistance of brittle materials with opening mode fracture is the stress intensity factor [10]:

$$K_I = \sqrt{GE} = \sigma\sqrt{\pi a}. \tag{14}$$

Equations (13) and (14) are usually used to determine the energy and force characteristics of the crack resistance of brittle materials with the Griffith crack. Our calculations show that these equations also hold for cracks with any curvature at the semi-major axis, not always being in the form of ellipse.

CONCLUSIONS

The focus of many publications concerning the stress-strain state of plates with a crack is on analytical methods of the problem solution. The main attention is paid to the maximum stress concentration or the elastic strain energy. The term stress intensity factor is introduced as a force characteristic of the Griffith crack with a zero curvature. In the literature, the influence of a nonzero curvature on the stress-strain state of a solid is not discussed.

This work studied the mechanical state of the plate with the elliptical shaped crack under the external load. The Griffith crack was a special case of such a system. The stress concentration factor, crack-driving force, and elastic strain energy were obtained for the stress-strain state of the plate under the uniaxial load. The following general laws of the mechanical state of the plate with the crack, not always in the form of an ellipse, with a curvature at the semi-major axis, were identified:

1. The stress concentration was governed by $k = 2(a\eta)^{1/2}$ dependence.

2. The influence of the curvature was significant only at the tip of the crack.

3. The Griffith crack had no stress singularity at the end. Under the external load, the Griffith crack in the plate took the form of an ellipse with the semi-minor axis $b = 2a\sigma/E$. The stress concentration in the loaded system was $k = E/\sigma$, where E is Young's modulus.

4. The stress distribution $\sigma_y = \sigma(0.5a/x)^{1/2}$, which is conventionally used for the stress-strain analysis of the plate with the Griffith crack, was incorrect already at a distance comparable with the half-length of the crack. The relation $\sigma_y = \sigma(x + a)/(x^2 + 2xa)^{1/2}$ was more correct.

5. The stress-strain state of the plate with the elliptical crack was similar to that with the uniform plastic deformation zone $\varepsilon^p = \sigma(1 + 2a/b)/E$.

6. The crack-driving force G and the stress intensity factor K_I did not depend on the crack curvature. Therefore, the known relations $G = \pi\sigma^2 a/E$ and $K_I = \sigma(\pi a)^{1/2}$ obtained for the Griffith crack, could be used as the crack resistance characterization of the material with a crack not always in the form of an ellipse and with the known curvature at the semi-major axis.

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