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RIGID BODY INERTIA PROPERTIES

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Definitions are given and properties of the rigid body inertia characteristics are formulated. The influence of the geometrical symmetry of a rigid body on its characteristics is described. The geometrical approach to the material presentation is used.

Keywords: rigid body, inertia characteristics, symmetry.

INTRODUCTION

In all textbooks on theoretical mechanics, the inertia properties of a rigid body are considered (for example, see [1–5]), but accurate definitions and proofs of their properties are lacking. In the present work, precise geometrical definitions of centers of masses and moments of inertia about a point, an axis, and a plane as well as of the inertia center are given, and their properties are studied. The influence of the geometrical symmetry of the body on its characteristics is described.

1. MOTION IN E_3

Definition 1. The non-singular mapping $D : E_3 \to E_3$ is called space motion if

$$\rho(a,b) = \rho(D(a), D(b)), \quad \forall a,b \in E_3 .$$

From the definition, the existence follows of the orthogonal operator $\hat{d} : E_3 \to E_3$ for which the relation

$$\boldsymbol{D}(\boldsymbol{a})\boldsymbol{D}(\boldsymbol{b}) = \hat{d}\boldsymbol{ab}; \quad \forall a,b \in E_3 , \tag{1}$$

holds true. Hereinafter, the vector is designated by $\boldsymbol{ab}$.

2. INERTIA CHARACTERISTICS OF A BODY

Let $a(i)$ and $m(i)$ denote the location and mass of the i th particle of a body, and m denotes the mass of the body.

Definition 2. The point $c \in E_3$ the radius-vector of which relative to a certain point $o \in E_3$ has the form

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$$moc = m(i)oa(i) \qquad (2)$$

is called the center of mass of the body. Hereinafter, summation is performed over the indices repeated twice in the monomial.

Definition 3. The scalar

$$I_o = m(i)oa(i)^2 \qquad (3)$$

is called the moment of inertia relative to the point o.

Definition 4. The scalar

$$I_o(l) = m(i)[oa(i), l]^2 \qquad (4)$$

is called the moment of inertia relative to the straight line passing through the point o parallel to the unit vector l.

Definition 5. The scalar

$$\Pi_o(l) = m(i)(oa(i), l)^2 \qquad (5)$$

is called the moment of inertia relative to the plane with the normal l passing through the point o.

Definition 6. The quantity

$$\hat{I}_o = m(i)(oa(i)^2 \hat{E} - oa(i) \circ oa(i)) \qquad (6)$$

is called the inertia tensor relative to the point o, where $\hat{E}$ is the unit tensor, and the Kronecker product of vectors [6] is defined as follows:

$$(a \circ b)x = a(b, x), \quad \forall a, b, x \in E_3 .$$

3. PROPERTIES OF THE INERTIA CHARACTERISTICS OF A RIGID BODY

3.1. Relationships between the inertia characteristics

3.1.1. $I_o(l) = (l, \hat{I}_o l)$. 3.1.2. $2I_o = sp(\hat{I}_o)$.

3.1.3. $2I_o = \sum_\alpha I_o(e_\alpha)$, $(e_\alpha, e_\beta) = \delta_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$.

3.1.4. $I_o = \sum_\alpha \Pi_o(e_\alpha)$. 3.1.5. $I_o = I_o(l) + \Pi_o(l)$.

Proofs of the properties are omitted here.

3.2. General properties

3.2.1. Location of the center of masses is independent of the pole choice, that is, definition 2 is constructive in character (there is only one center of masses for a body).

3.2.2. The distance between the center of mass and any point of the body in motion is constant.

3.2.3. If a body is in the external homogeneous gravity field, the center of masses in the equilibrium position lies on the straight line passing through the suspension point and parallel to the free fall acceleration.

3.2.4. Steiner's theorem. The moment of inertia of a system relative to an axis is equal to the sum of the moment of inertia relative to the parallel axis passing through the center of masses and of the product of the system mass by the squared distance between the axes: $I_o(\boldsymbol{l}) = I_c(\boldsymbol{l}) + ms^2$.

3.2.5. The inertia tensor is positively defined: $(\boldsymbol{u}, \hat{I}_o \boldsymbol{u}) > 0, \quad \forall \boldsymbol{u} \neq 0$.

3.2.6. The inertia tensor is symmetric and real: $I_o = I_o, \quad I_o = I_o$.

3.3. Relationships between the inertia characteristics about the pole and the center of masses

$$3.3.1.\ I_o = I_c + m\,\boldsymbol{oc}^2 .\ 3.3.2.\ I_o(\boldsymbol{l}) = I_c(\boldsymbol{l}) + m[\boldsymbol{oc}, \boldsymbol{l}]^2 .$$

$$3.3.3.\ \Pi_o(\boldsymbol{l}) = \Pi_c(\boldsymbol{l}) + m(\boldsymbol{oc}, \boldsymbol{l})^2 .\ 3.3.4.\ \hat{I}_o = \hat{I}_c + m(\boldsymbol{oc}^2 \hat{E} - \boldsymbol{oc} \circ \boldsymbol{oc}) .$$

3.4. Inertia characteristics of a compound rigid body

Let us subdivide a body into parts σ as follows:

$$(V_\alpha) \neq \varnothing,\ (V_\alpha) \cap (V_\alpha) = \varnothing,\ \bigcup (V_\alpha) = (V), \quad \alpha = 1, \ldots, \sigma ,$$

where (V) and (V_α) are regions occupied by the body and its part number α. Here $m(\alpha)$ is the mass of the part of the body, $n(\alpha)$ is the number of particles in the part, $m(\alpha i)$ is the mass of the i th particle from region number α, and $c(\alpha)$ is the center of masses of region number α. Then

$$3.4.1.\ m\boldsymbol{oc} = m(\alpha)\boldsymbol{oc}(\alpha) ,\ 3.4.2.\ I_c = I_{c(\alpha)}(\alpha) + m(\alpha)\boldsymbol{cc}(\alpha)^2 . \tag{7}$$

Here $m(\alpha) = \sum_{\alpha=1}^{n(\alpha)} m(\alpha i), \quad m(\alpha)\boldsymbol{oc}(\alpha) = \sum_{\alpha=1}^{n(\alpha)} m(\alpha i)\boldsymbol{oa}(\alpha i)$.

$$3.4.3.\ I_c(\boldsymbol{l}) = I_{c(\alpha)}(\boldsymbol{l}) + m(\alpha)[\boldsymbol{l}, \boldsymbol{cc}(\alpha)]^2 .$$

$$3.4.4.\ \Pi_c(\boldsymbol{l}) = \Pi_{c(\alpha)}(\boldsymbol{l}) + m(\alpha)(\boldsymbol{l}, \boldsymbol{cc}(\alpha))^2 .$$

$$3.4.5.\ \hat{I}_c = \hat{I}_{c(\alpha)}(\alpha) + m(\alpha)(\boldsymbol{cc}(\alpha)^2 \hat{E} - \boldsymbol{cc}(\alpha) \circ \boldsymbol{cc}(\alpha)) . \tag{8}$$

4. SYMMETRY OF A RIGID BODY

Let $N = \{1, 2.3 \ldots n\}$, where n is the number of particles of the rigid body.

Definition 7. Motion D is called the symmetry element (or the symmetry) of the rigid body if the non-singular mapping $\delta : N \to N$, such that

$$D(a(i)) = a(\delta(i)), \quad m(\delta(i)) = m(i) \tag{9}$$

can be specified.

4.1. Symmetry of the inertia characteristics of a rigid body

If D is the symmetry of a rigid body, then:

4.1.1. The equality

$$D(c) = c \tag{10}$$

holds true.

Comment 1. From this property it follows that:

1) If a body possesses a symmetry axis, the center of masses is on the symmetry axis; if a body possesses several symmetry axes, the center of masses lies at the intersection of these axes.

2) If a body possesses an inversion center, this center coincides with the center of masses.

3) If a body possesses a plane of symmetry (reflection), the center of masses lies in this plane.

4.1.2. For the center of inertia with respect to the point o, the equality

$$\hat{d}\,\hat{I}_o = \hat{I}_{D(o)}\hat{d} \tag{11}$$

is true.

Comment 2. From Eqs. (10) and (11), we obtain

$$\hat{d}\,\hat{I}_{\tilde{n}} = \hat{I}_{\tilde{n}}\hat{d}\,.$$

Analogous equality is true for any point o belonging to the symmetry axis/plane.

4.1.3. For the moment of inertia about the plane with the normal $\boldsymbol{l}$, the equalities

$$\Pi_c(\boldsymbol{l}) = \Pi_c(-\boldsymbol{l}), \quad \Pi_o(\boldsymbol{l}) = \Pi_{D(o)}(\hat{d}\,\boldsymbol{l})$$

are true.

4.1.4. For the moment of inertia about a straight line, the equality

$$I_c(\boldsymbol{l}) = I_c(-\boldsymbol{l}), \quad I_o(\boldsymbol{l}) = I_{D(o)}(\hat{d}\,\boldsymbol{l})$$

is true.

4.1.5. For the moment of inertia about the point o, the equality

$$I_o = I_{D(o)}$$

is true.

Proofs

4.1.1. We now put $b = b(i)$ in Eq. (1) and consider Eq. (9). As a result, we obtain $\boldsymbol{D}(\boldsymbol{a}(\boldsymbol{i}))\boldsymbol{b}(\boldsymbol{\delta}(\boldsymbol{i}))$ $= \hat{d}\boldsymbol{a}(\delta(i))\boldsymbol{b}(\boldsymbol{i})$. After multiplication of this equality by $m(\delta(i)) = m(i)$ and summation over i, we obtain

$$\boldsymbol{D}(\boldsymbol{a})\boldsymbol{c} = \hat{d}\boldsymbol{a}\boldsymbol{c}\,.$$

On the other hand, from Eq. (1) we have

$$\boldsymbol{D}(\boldsymbol{a})\boldsymbol{D}(\boldsymbol{c}) = \hat{d}\boldsymbol{a}\boldsymbol{c}\,.$$

Comparing two last equalities, we obtain $\boldsymbol{c} = \boldsymbol{D}(\boldsymbol{c})$, which was to be proven.

4.1.2. It is easy to prove the property of the Kronecker products

$$\hat{d}\boldsymbol{x} \circ \hat{d}\boldsymbol{y} = \hat{d}(\boldsymbol{x} \circ \boldsymbol{y})\hat{d}^T . \quad (12)$$

Now we prove that Eq. (11) with allowance for Eqs. (1), (6), and (8) assumes the form

$$\hat{d}(\hat{I}_o)\hat{d}^T = m(i)(\boldsymbol{oa}(\boldsymbol{i})^2 \hat{E} - \hat{d}\{\boldsymbol{oa}(\boldsymbol{i}) \circ \boldsymbol{oa}(\boldsymbol{i})\}\hat{d}^T) = m(i)(\boldsymbol{D}(\boldsymbol{o})\boldsymbol{D}(\boldsymbol{a}(\boldsymbol{i}))^2 \hat{E} - \hat{d}\boldsymbol{oa}(\boldsymbol{i}) \circ \hat{d}\boldsymbol{oa}(\boldsymbol{i}))$$

$$= m(i)(\boldsymbol{D}(\boldsymbol{o})\boldsymbol{D}(\boldsymbol{a}(\boldsymbol{i}))^2 \hat{E} - \boldsymbol{D}(\boldsymbol{o})\boldsymbol{D}(\boldsymbol{a}(\boldsymbol{i})) \circ \boldsymbol{D}(\boldsymbol{o})\boldsymbol{D}(\boldsymbol{a}(\boldsymbol{i})) = \hat{I}_{D(o)} .$$

This just proves it.

4.1.3. The validity of the first formula follows from Eq. (5). For the second formula, we have

$$m(i)(\boldsymbol{oa}(\boldsymbol{i}),\boldsymbol{l})^2 = m(i)(\hat{d}\boldsymbol{oa}(\boldsymbol{i}),\hat{d}\boldsymbol{l})^2 = m(i)(\boldsymbol{D}(\boldsymbol{o})\boldsymbol{D}(\boldsymbol{a}(\boldsymbol{i}),\hat{d}\boldsymbol{l})^2 = m(\delta(i))(\boldsymbol{D}(\boldsymbol{o})\boldsymbol{a}(\boldsymbol{\delta}(\boldsymbol{i})),\hat{d}\boldsymbol{l})^2 \Pi_{D(o)}(\hat{d}\boldsymbol{l}) .$$

This just proves it.

4.1.4. The validity of the first formula follows from definition (6). In the second case, we have

$$I_o(\boldsymbol{l}) = (\boldsymbol{l},\hat{I}_o\boldsymbol{l}) = (\boldsymbol{l},\hat{d}^T \hat{I}_{D(o)}\hat{d}\boldsymbol{l}) = (\hat{d}\boldsymbol{l},\hat{I}_{D(o)}\hat{d}\boldsymbol{l}) = I_{D(o)}(\hat{d}\ \boldsymbol{l}) .$$

This just proves it.

4.1.5. From definitions (1) and (7), we have

$$I_o = m(i)\boldsymbol{oa}(\boldsymbol{i})^2 = m(i)\boldsymbol{D}(\boldsymbol{o})\boldsymbol{D}(\boldsymbol{a}(\boldsymbol{i}))^2 = m(\delta(i))\boldsymbol{D}(\boldsymbol{o})\boldsymbol{a}(\boldsymbol{\delta}(\boldsymbol{i}))^2 = I_{D(o)} .$$

This just proves it.

4.2. Explicit form of the inertia tensor of a symmetric body

In [7] it was shown that the geometrical image of the orthogonal operator in E_3 is either rotation about a certain axis by some angle or rotation through some angle about a certain axis with subsequent reflection from the plane perpendicular to this axis. Thus, any orthogonal operator in a proper system of coordinates is represented by the matrix with elements

$$d_{13} = d_{23} = d_{31} = d_{32} = 0, \quad d_{33} = \pm 1, \ d_{11} = d_{22} = a, \ d_{21} = -d_{12} = b .$$

Here $a = \cos\varphi$ and $b = \sin\varphi$.

Symmetry element – symmetry axis. We now choose the symmetry axis for the *OZ* axis; then $\hat{d}$ determines rotation about this axis; in this coordinate system, we obtain from condition (11)

$$I_{12}b = 0, \quad I_{13}b = 0, \ I_{23}b = 0, \ (I_{11} - I_{22})b = 0, \quad I_{13}(1-a) = 0, \quad I_{23}(1-a) = 0 . \quad (13)$$

This system has the following solutions:

1) If $b = 0$, then $a = -1$, and from Eq. (13) we obtain $I_{13} = I_{23} = 0$.

2) If $b \neq 0$, we obtain the diagonal tensor with elements $I_{11} = I_{22}$.

Symmetry element – rotation about a certain axis with subsequent reflection from the plane perpendicular to this axis. We choose the symmetry axis for the *OZ* axis and perform calculations analogous to the previous case:

$$I_{12}b = 0, \quad I_{13}b = 0, \quad I_{23}b = 0, \quad (I_{11} - I_{22})b = 0, \quad I_{13}(1+a) = 0, \quad I_{23}(1+a) = 0 . \tag{14}$$

This system has the following solutions:

1.1) For $b = 0$ and $a = -1$, this symmetry is inversion about the center of mass, and we obtain no information on the form of the inertia tensor.

1.2) For $b = 0$ and $a = 1$, from Eq. (14) we obtain $I_{13} = I_{23} = 0$.

2) For $b \neq 0$, from Eq. (14) we obtain a diagonal tensor; moreover, $I_{11} = I_{22}$.

5. INERTIA CHARACTERISTICS OF A RIGID BODY FOR THE MODEL OF A CONTINUOUS MEDIUM

If we represent an absolutely rigid body as a continuous medium occupying the region of space (V) with the substance density distribution $\rho(\boldsymbol{r})$, then definitions of the physical quantities will have explicit forms, for example, with designations

$$\boldsymbol{r_o} = \boldsymbol{oa}, \quad \rho_o(\boldsymbol{r}_o) = \rho(a), \quad a \in (V), \quad o \in E_3 .$$

For the radius-vector of the center of mass, we obtain

$$m\boldsymbol{oc} = \int_{(V)} \rho_o(\boldsymbol{r}_o)\boldsymbol{r}_o dv .$$

All properties that have been proved above for discrete bodies remain valid for continuous bodies if we accept the definition: the motion D is the symmetry of a body under the condition that

$$D(a) \in (V), \quad \rho(D(a)) = \rho(a), \quad \forall a \in (V) .$$

Comment on the geometrical characteristics of the region. Let us consider the region of space (V) with the volume V. Define the inertia tensor of the region relative to the point o as follows:

$$\hat{J}_o = V^{-1} \int_{(V)} (\boldsymbol{r}_o^2 \hat{E} - \boldsymbol{r}_o \circ \boldsymbol{r}_o) dv .$$

The center of the region, the moments of inertia relative to the point, axis, and plane can be determined analogously.

The inertia tensor of the body consisting of σ parts with density $\rho(\alpha)$, occupying the region $(V(\alpha))$ can be written in the form

$$\hat{I}_o = V(\alpha)\rho(\alpha)[(\boldsymbol{oc}(\alpha)^2 \hat{E} - \boldsymbol{oc}(\alpha) \circ \boldsymbol{oc}(\alpha)) + \hat{J}_{c(\alpha)}(\alpha)] . \tag{15}$$

The last equality simplifies considerably calculations of the inertia characteristics of different rigid bodies if their geometries are identical. Thus, to calculate the inertia tensor of the body obtained by removal of the media from a certain region, formula (15) can be used considering that the density in cavities is equal to zero. In addition, for two bodies with identical geometries and the known inertia tensor of one of them, the formula presented above allows one to determine automatically the inertia tensor of the second body.

Example

Let us consider the homogeneous sphere centered at the point o with the mass $m(o)$, density $\rho(o)$, and radius $R(o)$. The sphere has two spherical cavities with the centers, constant densities, and radii $c(\alpha)$, $\rho(\alpha)$, and $R(\alpha)$ $(\alpha = 1, 2)$. The points $o, c(\alpha)$ lie on one straight line whose orientation is determined by the unit vector $\boldsymbol{e}$. For convenience, we introduce the dimensionless quantities:

$$R(\alpha) = k(\alpha)R(o), \quad \boldsymbol{oc}(\alpha) = n(\alpha)R(o)\boldsymbol{e}.$$

Using formulas (7) and (15) and taking into account the spherical symmetry of the sphere and cavities, for the radius-vector of the center of mass of the obtained body c and the inertia tensor we obtain

$$(1 + (\rho(\nu)/\rho(o) - 1)k(\nu)^3)\boldsymbol{oc} = R(o)n(\alpha)k(\alpha)^3(\rho(\alpha)/\rho(o) - 1)\boldsymbol{e},$$

$$\hat{I}_o = m(o)R(o)^2[(2/5)\hat{E} + k(\alpha)^3(\rho(\alpha)/\rho(o) - 1)\{(n(\alpha)^2 + (2/5)k(\alpha)^2)\hat{E} - n(\alpha)^2\boldsymbol{e} \circ \boldsymbol{e}\}].$$

From here, in particular, it follows that:

1) If $\rho(\alpha) = \rho(o)$, we obtain the inertia tensor of the continuous sphere.

2) If $\rho(\alpha) = 0$, we obtain the inertia tensor of the sphere with empty cavities.

3) Obviously, with the proper choice of the density $\rho(\alpha)$, the center of mass of the body (point c) can coincide with the geometrical center of mass of the continuous sphere (point o).

4) Though the system possesses only the axial symmetry, nevertheless, for a definite relationship between the densities of the cavities, the inertia tensor will be scalar.

CONCLUSIONS

The definitions are given and the properties of the inertia characteristics of an absolutely rigid body are formulated. The relationship of the symmetry of the body with the properties of its inertia characteristics has been established. We consider that the material of this work can be included in textbooks on theoretical mechanics.

REFERENCES

1. H. Goldstein, Classical Mechanics, Addison-Wesley (1951).
2. Yu. F. Golubev, Fundamentals of Theoretical Mechanics [in Russian], Moscow State University Publishing House, Moscow (2000).
3. L. D. Landau and E. M. Lifshits, Theoretical Mechanics. Vol. 1. Mechanics [in Russian], Fizmatlit, Moscow (1958).
4. N. N. Nikitin, Course in Theoretical Mechanics [in Russian], Vysshaya Shkola, Moscow (2003).
5. I. I. Ol'khovskii, Theoretical Mechanics for Physicists [in Russian], Nauka, Moscow (1970).
6. L. I. Golovina, Linear Algebra and Some of Its Applications [in Russian], Nauka, Moscow (1985).
7. F. R. Gantmakher, Theory of Matrices [in Russian], Fizmatlit, Moscow (2004).