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STABLE INTERACTIONS BETWEEN THE EXTENDED CHERN-SIMONS THEORY AND A CHARGED SCALAR FIELD WITH HIGHER DERIVATIVES: HAMILTONIAN FORMALISM

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The constrained Hamiltonian formalism for the extended higher derivative Chern–Simons theory of an arbitrary finite order is considered. It is shown that the n-th order theory admits an (n–1)-parametric series of conserved tensors. It is clarified that this theory admits a series of canonically non-equivalent Hamiltonian formulations, where a zero-zero component of any conserved tensor can be chosen as a Hamiltonian. The canonical Ostrogradski Hamiltonian is included into this series. An example of interactions with a charged scalar field is also given, which preserve the selected representative of the series of Hamiltonian formulations.

Keywords: higher-derivative theories, Hamiltonian formalism, extended Chern–Simons theory.

INTRODUCTION

The issue of constructing a Hamiltonian formalism in higher-derivative theories has been in the focus of investigations over a number of years, starting from the study by Ostrogradski [1]. For the first time a procedure for constructing the Hamiltonian formalism with interactions for degenerate higher-derivative theories was proposed in [2]. Based on this procedure and its numerous modifications, a number of different gauge theories, gravity models among them, have been investigated [3, 4]. The major feature of the Hamiltonian theory, constructed via the Ostrogradski method and its generalizations, is the unboundedness of the Hamiltonian, which gives rise to well-known troubles associated with the dynamics stability and a subsequent construction of the quantum theory [5–7].

The alternative procedures of a transition into the Hamiltonian formalism, not relying on the Ostrogradski construction, were first introduced in the Pais-Uhlenbeck oscillator theory [8, 9]. In a more recent work [10], it has been shown that every free higher-derivative theory admits a series of canonically non-equivalent Hamiltonians and Poisson brackets, which includes a canonical representative. An explicit construction of a series of non-canonical Hamiltonian formulations in the gauge field theory was for the first time performed in [11, 12], where we dealt with the extended 3-rd and 4-th order Chern–Simons theory (CS-theory) [13].

In the present paper a Hamiltonian formulation is considered for an arbitrary-order extended CS theory. It is shown that a free extended CS-theory admits an (n–1)-parametric series of Hamiltonian formulations such that a zero-zero component of an arbitrary representative of the series of conserved tensors [14] could be taken as a Hamiltonian. It is also demonstrated that the earlier constructed [15] non-Lagrangian interaction vertices of the extended CS-theory with the charged scalar field retain the selected representative in the series of Hamiltonians.

This work is organized as follows. Some basic facts on the CS-theory are presented in Section 1, where among other things the equation of motion and the conserved tensors are derived in explicit forms. A Hamiltonian formulation of the extended CS-theory is constructed in Section 2, where the calculations largely implement the techniques of using

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Hankel and Bezout matrices that are available, e.g., in [16]. In Section 3, a Hamiltonian formulation is discussed for the case of interactions with a charged scalar field.

1. EXTENDED *n*-TH ORDER CHERN–SIMONS THEORY

The extended Chern-Simons theory of the n-th order is a class of models of vector field $A = A_\mu(x)dx^\mu$, $\mu = 0, 1, 2$ in the three-dimensional Minkowski space with the following action functional:

$$S[A(x)] = \frac{1}{2}\int *A \wedge \left(\sum_{k=1}^{n} \alpha_k m^{2-k} (*d)^k A \right), \quad *dA = \varepsilon_{\mu\nu\rho}\partial^\mu A^\nu dx^\rho, \quad \varepsilon_{012} = 1, \tag{1}$$

where m is the constant with the dimension of mass, the real numbers $\alpha_1, \ldots, \alpha_n$ are the model parameters, such that α_n differs from zero, the symbols * and d indicate the Hodge operator and de Rham differential. The Minkowski metric signature is mostly negative. The Lagrange equation, following from the action functional (1), is given by

$$\frac{\delta S}{\delta A} = \left(\sum_{k=1}^{n} \alpha_k m^{2-k} (*d)^k \right) A = 0\,. \tag{2}$$

The action (1) and the equations of motion (2) are invariant with respect to the standard gradient gauge transformation for field A.

The series of the second-rank conserved tensors in the theory (1) was constructed in [14]. The most common representative of this series is written as follows:

$$T_{\mu\nu}(\alpha,\beta) = \frac{1}{2} m^2 \sum_{r,s=1}^{n-1} C_{r,s}(\alpha,\beta)\left(F_\mu^{(r)} F_\nu^{(s)} + F_\nu^{(r)} F_\mu^{(s)} - \eta_{\mu\nu}\eta^{\rho\sigma} F_\rho^{(r)} F_\sigma^{(s)} \right), \tag{3}$$

where numbers $\alpha = (\alpha_1, \ldots, \alpha_n)$ are the parameters in the Lagrangian and quantities $\beta = (\beta_1, \ldots, \beta_n)$ are the parameters of the series; the following notation is used:

$$F^{(r)} = (m^{-1} * d)^r A, \quad r = 0, \ldots, n-1, \tag{4}$$

where $F^{(0)}{}_\mu \equiv A_\mu$. The quadratic matrix $C_{r,s}(\alpha, \beta)$, $r, s = 1, \ldots, n$, is determined by the generating relation

$$\sum_{r,s=1}^{n-1} C_{r,s}(\alpha,\beta) z^{r-1} u^{s-1} = \frac{M(z)N(u) - M(u)N(z)}{z-u}, \tag{5}$$

where the polynomials of a single variable $M(z)$, $N(z)$ of degree $n-1$ are given by

$$M(z) = \sum_{r=0}^{n-1} \alpha_{r+1} z^r, \quad N(z) = \sum_{r=0}^{n-1} \beta_{r+1} z^r\,. \tag{6}$$

Quantity $C_{r,s}(\alpha, \beta)$ (5) is known as the Bezout matrix of the polynomials $M(z)$, $N(u)$ [16]. Also note that the polynomial

$$M'(z) \equiv zM(z) = \sum_{r=1}^{n} \alpha_r z^r\,, \tag{7}$$

derived by a formal substitution of variable z for the CS-operator $*d$ in the Lagrange equations, is termed as a characteristic polynomial of the theory (1) [14].

The conserved tensors in the model (1) are determined as the coefficients at the parameters $\beta_1, \ldots, \beta_n$ in the series (3)

$$T^{(r)}_{\mu\nu}(\alpha) = \frac{\partial T_{\mu\nu}(\alpha,\beta)}{\partial\beta_r}, \quad r = 1, ..., n\,. \tag{8}$$

By construction, $T^{(1)}{}_{\mu\nu}(\alpha)$ coincides with the canonical energy-momentum tensor of the theory (1), while $T^{(r)}{}_{\mu\nu}(\alpha)$, $r = 2, \ldots, n-1$, are new independent conserved quantities. Quantity $T^{(n)}{}_{\mu\nu}(\alpha)$ (7) is a linear combination of other conserved tensors due to the identity

$$\sum_{r=1}^{n} \alpha_r T^{(r)}_{\mu\nu}(\alpha) = 0\,. \tag{9}$$

We keep $T^{(n)}{}_{\mu\nu}(\alpha)$ for the sake of a convenient inclusion of interactions. The expansion of an arbitrary representative of the series (3) in terms of the basis of independent operators $T^{(r)}{}_{\mu\nu}(\alpha)$, $r = 1, \ldots, n-1$ (7) is given by

$$T_{\mu\nu}(\alpha,\beta) = \frac{1}{\alpha_n}\sum_{r=1}^{n-1}(\beta_r\alpha_n - \beta_n\alpha_r)T^{(r)}_{\mu\nu}(\alpha)\,. \tag{10}$$

The canonical energy-momentum tensor is always included into the series (3); it corresponds to the parameter values

$$\beta_1 = 1, \quad \beta_2 = \beta_3 = \ldots = \beta_n = 0\,, \tag{11}$$

while other values of the parameters $\beta_1, \ldots, \beta_n$ in the series (3) determine non-canonical conserved quantities.

The zero-zero component $T_{00}(\alpha, \beta)$ of the conserved tensor (3) is given by

$$T_{00}(\alpha,\beta) = \sum_{r,s=1}^{n-1} C_{r,s}(\alpha,\beta)\left(F^{(r)}_i F^{(s)}_i + F^{(r)}_0 F^{(s)}_0\right), \tag{12}$$

where $i = 1, 2$, and summation over the recurrent index is assumed. This quantity is a quadratic form of the variables (4); it is positive definite if the matrix $C_{r,s}(\alpha, \beta)$ (5) is positive definite. A possibile existence of a bounded representative in the series (12) is determined by the structure of the roots of the characteristic polynomial (7): there is a bounded conserved tensor, if all non-zero roots of the characteristic polynomial are real and different, and the non-zero root is a multiple of one or two [14]. In terms of the polynomial $M(z)$ (6) it is sufficient to require all its roots to be real and different. The canonical energy of the theory (1) is included into the series (12) at the parameter values (11); it is an inevitably unbounded quantity, if $n > 2$.

2. HAMILTONIAN FORMULATION OF AN EXTENDED CHERN–SIMONS THEORY

Let us show that the theory (1) admits an $n-1$-parametric series of canonical non-equivalent Hamiltonian formulations such that almost any representative of the series of conserved quantities (12) can be chosen as a Hamiltonian. In order to achieve the desired result, first we are going to reduce the order of equations (2) to the first order in the derivative by time $t = x^0$, then we are going to find a Poisson bracket and a Hamiltonian reducing these equations to a Hamiltonian form.

Let us introduce new variables absorbing the time derivatives of the initial vector field A_μ, using spatial components of one-form $F^{(r)}{}_i$, $i = 1, 2$, $r = 1, ..., n-1$ (4). Then the first-order formulation in time for the theory (2) would be written as

$$\partial_0 F_i^{(0)} = \partial_i A_0 - m\varepsilon_{ij} F_j^{(1)}, \tag{13}$$

$$\partial_0 F_i^{(r)} = \frac{1}{m}\varepsilon_{ij}\partial_k\left(\partial_k F_j^{(r-1)} - \partial_j F_k^{(r-1)}\right) - m\varepsilon_{ij}F_j^{(r+1)}, \quad r = 2, ..., n-2, \tag{14}$$

$$\partial_0 F_i^{(n-1)} = \frac{1}{m}\varepsilon_{ij}\partial_k\left(\partial_k F_j^{(n-2)} - \partial_j F_k^{(n-2)}\right) + \frac{1}{\alpha_n} m\varepsilon_{ij}\sum_{r=1}^{n-1}\alpha_r F_j^{(r)}, \tag{15}$$

$$\Theta \equiv m\sum_{r=1}^{n}\alpha_k \varepsilon_{ij}\partial_i F_j^{(r-1)} = 0. \tag{16}$$

The equivalence of equations (2) and (13)–(16) can be observed as follows. Formulas (13) and (14) express the auxiliary derivatives $F^{(r)}{}_i$, $i = 1, 2$, $r = 1, ..., n-1$ in terms of the derivatives of A. As soon as all auxiliary derivatives are excluded, (15) and (16) reproduce the spatial and temporal members of the equations of motion (2). Note also that within the first-order formalism quantity Θ (16) does not involve any time derivatives and can be treated as a constraint.

The system of first-order equations (13)–(15) is a Hamiltonian one, provided that there is a Hamiltonian $H(\alpha, \beta)$ and a Poisson bracket $\{\ ,\ \}_{\alpha,\beta}$, such that

$$\partial_0 F_i^{(r)} \approx \{F_i^{(r)}, \textstyle\int H(\alpha,\beta)d\boldsymbol{x}\}_{\alpha,\beta}, \quad r = 0, ..., n-1. \tag{17}$$

The sign «$\approx$» denotes an equality of the left- and right-hand sides of the relation modulo constraint Θ (16). An introduction of the β parameters includes into consideration a possible existence of several different Hamiltonian formulations for the same equations. We choose a Hamiltonian of the theory in the form

$$H(\alpha,\beta) = T_{00}(\alpha,\beta) + \left(k_0 A_0 + \frac{1}{m}\sum_{r=1}^{n-1}k_r\varepsilon_{ij}\partial_i F_j^{(r-1)}\right)\Theta, \tag{18}$$

where $T_{00}(\alpha, \beta)$ is the zero-zero component (11) of the general conserved tensor (13), expressed in terms of the phase-space variables as follows:

$$T_{00}(\alpha,\beta) = \frac{1}{2}m^2\sum_{r,s=1}^{n-1}C_{r,s}(\alpha,\beta)\left(F_i^{(r)}F_i^{(s)} + \partial_i F_j^{(r-1)}\left(\partial_i F_j^{(s-1)} - \partial_j F_i^{(s-1)}\right)\right), \tag{19}$$

and Θ is the constraint (16). Constants k_0, k_1, …, k_{n-1} are introduced into the Hamiltonian (18) for the sake of convenience; they will be determined in what follows.

The Poisson bracket is determined from the condition that equations (13)–(15) are written as (17) with the Hamiltonian (18), which is equivalent to a system of equations with respect to an unknown Poisson bracket and parameters k_0, k_1, …, k_{n-1},

$$\{F_i^{(0)}, H(\alpha,\beta)\}_{\alpha,\beta} = \partial_i A_0 - m\varepsilon_{ij}F_j^{(1)}, \tag{20}$$

$$\{F_i^{(r)}, H(\alpha,\beta)\}_{\alpha,\beta} = \frac{1}{m}\varepsilon_{ij}\partial_k\left(\partial_k F_j^{(r-1)} - \partial_j F_k^{(r-1)}\right) - m\varepsilon_{ij}F_j^{(r+1)}, \quad r = 2, ..., n-2, \tag{21}$$

$$\{F_i^{(n-1)}, H(\alpha,\beta)\}_{\alpha,\beta} = \frac{1}{m}\varepsilon_{ij}\partial_k\left(\partial_k F_j^{(n-2)} - \partial_j F_k^{(n-2)}\right) + \frac{1}{\alpha_n} m\varepsilon_{ij}\sum_{r=1}^{n-1}\alpha_r F_j^{(r)} . \tag{22}$$

Relations (20)–(22) represent a system of linear equations for the unknown matrix elements of the Poisson brackets of the phase-space variables $F^{(r)}{}_i$, $i = 1, 2$, $r = 0, ..., n-1$. In the class of field-independent Poincare-invariant Poisson brackets a solution to this system is written as follows:

$$\{F_i^{(n-1)}(\boldsymbol{x}), F_j^{(n-1)}(\boldsymbol{y})\}_{\alpha,\beta} = \frac{1}{\alpha_n m \det C(\alpha,\beta)}\left(\sum_{r=1}^{n-1}\alpha_r M^{r,n-1}(\alpha,\beta)\right)\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}) , \tag{23}$$

$$\{F_i^{(r)}(\boldsymbol{x}), F_j^{(s)}(\boldsymbol{y})\}_{\alpha,\beta} = -\frac{M^{r,s+1}(\alpha,\beta)}{m\det C(\alpha,\beta)}\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}), \quad r, s+1 = 1, ..., n-1 , \tag{24}$$

$$\{A_i(\boldsymbol{x}), A_j(\boldsymbol{y})\}_{\alpha,\beta} = -\frac{\gamma}{m\det C(\alpha,\beta)}\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}) . \tag{25}$$

Parameters $k_0, k_1, ..., k_{n-1}$ are determined by the formulas

$$k_0 = -\frac{\det C(\alpha,\beta)}{\alpha_1\gamma + \sum_{s=2}^{n}\alpha_s M^{s-1,1}(\alpha,\beta)},$$
$$k_r = \frac{\gamma C_{r,1}(\alpha,\beta) + \sum_{s=2}^{n-1} C_{r,s}(\alpha,\beta) M^{s-1,1}(\alpha,\beta)}{\alpha_1\gamma + \sum_{s=2}^{n}\alpha_s M^{s-1,1}(\alpha,\beta)}, \quad r = 1, ..., n-1. \tag{26}$$

In (23)–(26), quantity γ is a free parameter, $M^{r,s}(\alpha, \beta)$ denotes a matrix coupled with the Bezout matrix $C_{r,s}(\alpha, \beta)$ (5):

$$\sum_{k=1}^{n-1} C_{r,k}(\alpha,\beta) M^{k,s}(\alpha,\beta) = \det C(\alpha,\beta)\delta_r^s . \tag{27}$$

The solution (23)–(26) of equations (20)–(22) is well-defined if

$$\det C(\alpha,\beta) \neq 0, \quad \alpha_1\gamma + \sum_{s=2}^{n}\alpha_s M^{s-1,1}(\alpha,\beta) \neq 0 . \tag{28}$$

The compatibility conditions (28) for equations (20)–(22) have a straightforward physical interpretation. The first relation is equivalent to the non-degeneracy of the quadratic form of the Hamiltonian (18). The second relation guarantees that the constraint Θ (16) generates gauge symmetries for the vector potential A.

The solution check (23)–(26) of equations (20)–(22) uses the following relations:

$$M^{r,s}(\alpha,\beta) - M^{k,l}(\alpha,\beta) = 0, \quad s + r = k + l , \tag{29}$$

$$\sum_{k=1}^{n-1}\alpha_k M^{r,k}(\alpha,\beta)+\alpha_n M^{r+1,n}(\alpha,\beta)=0,\quad r=1,\dots,n-2\,. \tag{30}$$

These conditions are satisfied, since the matrix added to the Bezout matrix $C_{r,s}(\alpha, \beta)$ (5) is a Hankel matrix constructed using the polynomials (6). The proofs of these statements can be found in [16].

Formulas (17), (18), (23)–(26) determine the series of Hamiltonian formulations for the extended CS-theory (1). An arbitrary representative of this series is determined by the $2n+1$ parameters $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma$. Quantities $\alpha_1, \dots, \alpha_n$ determine the model parameters (1), numbers $\beta_1, \dots, \beta_{n-1}$ select a representative in the series of conservation laws (3), which will be prescribed by a Hamiltonian. Quantities β_n, γ are auxilliary: β_n can always be absorbed by redefinition of the parameters $\beta_1, \dots, \beta_{n-1}$, and constant γ determines a specific representative in the class of equivalence of Poisson brackets (23)–(25). The Poisson brackets of the physical observables are independent of the value of γ. The total number of the parameters resulting in non-equivalent Hamiltonian formulations in the model (1) is equal to $n-1$, thus an extended CS-theory admits an $(n-1)$-parametric series of Hamiltonian formulations.

For all admissible parameter values in the Hamiltonian the Poisson bracket represents a non-degenerate tensor

$$\det\left\{F_i^{(r)}(\boldsymbol{x}),F_j^{(s)}(\boldsymbol{y})\right\}_{\alpha,\beta}=-\frac{1}{\alpha_n\det^2 C(\alpha,\beta)}\left(\alpha_1\gamma+\sum_{r=2}^{n}\alpha_r M^{r-1,1}\right)\neq 0\,. \tag{31}$$

In this case, the Hamilton eqiations (13)–(16) result from the least-action principle for the functional

$$S(\alpha,\beta)=\int\left(m\sum_{r,s=0}^{n-1}\Omega_{r,s}(\alpha,\beta)\varepsilon_{ij}F_i^{(r)}\partial_0 F_i^{(s)}-H(\alpha,\beta)\right)d^3x\,. \tag{32}$$

The symplectic form $\Omega_{r,s}(\alpha, \beta)$ is determined by the generating relation

$$\sum_{r,s=0}^{n-1}\Omega_{r,s}(\alpha,\beta)z^r u^s=-\frac{\det C(\alpha,\beta)}{\alpha_1\gamma+\sum\limits_{s=2}^{n}\alpha_s M^{s-1,1}(\alpha,\beta)}\frac{M'(z)N'(u)-M'(u)N'(z)}{z-u}\,, \tag{33}$$

where $M'(z)$ is the characteristic polynomial (7) of the theory (1) n, while $N'(z)$ is prescribed by the formula

$$N'(z)=\beta_1+\sum_{r=1}^{n-1}\left(\beta_{r+1}-\frac{1}{\det C(\alpha,\beta)}\left(\beta_1\gamma+\sum_{k=2}^{n}\beta_k M^{k-1,1}(\alpha,\beta)\right)C_{1,r}(\alpha,\beta)\right)z^r\,. \tag{34}$$

In order to derive relations (33), (34), it is necessary to use the formula of inversion of the Hankel matrix of the Poisson brackets from [16]. The sum total of formulas (32)–(34) makes it possible to systematically reconstruct the symplectic structure, provided that the Hamiltonian of the theory has been prescribed.

The Ostrogradski canonical Hamiltonian formulation [1] is reproduced by the formulas (18), (32)–(34) for the following parameter values in the Hamiltonian:

$$\beta_1=1,\quad \beta_2=\beta_3=\dots=\beta_n=0,\quad \gamma=0\,. \tag{35}$$

The first-order action in this case is given by

$$S(\alpha,\beta)=\int\left(m\sum_{r,s=0}^{n-1}\alpha_{s+r+1}\varepsilon_{ij}F_i^{(r)}\partial_0 F_i^{(s)}-T_{00}^{(0)}(\alpha)-A_0\Theta\right)d^3x\,, \tag{36}$$

where $T^{(0)}{}_{00}(\alpha)$ is the zero-zero component of the canonical energy-momentum tensor, and by construction it is assumed that $\alpha_r = 0$ for all $r > n$. Obviously, the canonical Hamiltonian action (36) is not equivalent to the common representative of the series (32), since the canonical Hamiltonian is always unbounded from below, while in the general case bounded representatives are admissible.

3. INTERACTIONS WITH A CHARGED SCALAR FIELD

In [16], we proposed the following interaction vertices between the extended CS-theory (1) and the scalar field $\varphi = \mathrm{Re}\,\varphi(x) + i\,\mathrm{Im}\,\varphi(x)$ with higher derivatives

$$\left(\sum_{k=1}^{n}\alpha_k m^{2-k}(*d)^k\right)A - \sum_{a=1}^{N} ie_a j^{(a)}(\beta;\varphi,A) = 0, \quad \left(\prod_{a=1}^{N}(D_\mu D^\mu + \rho^a m^2)\right)\varphi = 0\,. \tag{37}$$

Here $\beta_1, \ldots, \beta_n, e_1, \ldots, e_N$ are the coupling constants and the following notation is used:

$$j^{(a)}(\beta;\varphi,A) = i\left(\varphi^{(a)}(D_\mu\varphi^{(a)})^* - \varphi^{(a)*}(D_\mu\varphi^{(a)})\right), \quad \varphi^{(a)} = \left(\prod_{b\neq a}\frac{D_\mu D^\mu + \rho^b m^2}{(\rho^a - \rho^b)m^2}\right)\varphi\,. \tag{38}$$

The parameters ρ^a, $a = 1, \ldots, N$, of the complex scalar field theory are non-negative and pairwise different. The covariant derivative is determined in a non-minimal way

$$D_\mu\varphi = \left(\partial_\mu - im\sum_{r=1}^{n}\beta_r F_\mu^{(r-1)}\right)\varphi\,. \tag{39}$$

Its action on the complex-conjugate scalar field is given by the complex conjugation of this expression. The gauge symmetry of the theory (37) is written as

$$\delta_\xi A_\mu(x) = \partial_\mu\xi(x), \quad \delta_\xi\varphi(x) = -i\beta_1\varphi^*(x)\,. \tag{40}$$

The conserved tensor in the theory (37) is determined by the relation

$$\Theta_{\mu\nu}(\alpha,\beta) = T_{\mu\nu}(\alpha,\beta) + \sum_{a=1}^{N} e_a(D_\mu\varphi(D^\mu\varphi)^* + \rho^a m^2\varphi\varphi^*)\,, \tag{41}$$

where $T_{\mu\nu}(\alpha, \beta)$ is determined by (3).

The first-order formulation for the theory (37) is given by

$$\partial_0 F_i^{(0)} = \partial_i A_0 - m\varepsilon_{ij}F_j^{(1)}\,, \tag{42}$$

$$\partial_0 F_i^{(r)} = \frac{1}{m}\varepsilon_{ij}\partial_k\left(\partial_k F_j^{(r-1)} - \partial_j F_k^{(r-1)}\right) - m\varepsilon_{ij}F_j^{(r+1)}, \quad r = 2, \ldots, n-2\,, \tag{43}$$

$$\partial_0 F_i^{(n-1)} = \frac{1}{m}\varepsilon_{ij}\partial_k\left(\partial_k F_j^{(n-2)} - \partial_j F_k^{(n-2)}\right) + \frac{1}{\alpha_n}m\varepsilon_{ij}\sum_{r=1}^{n-1}\alpha_r F_j^{(r)} - \frac{1}{\alpha_n m}\sum_{a=1}^{N} ie_a(j_a)_i(\beta;\varphi,A)\,, \tag{44}$$

$$\partial_0\varphi^{(a)} = \pi^{(a)}* + i\left(\beta_1 A_0 + \frac{1}{m}\sum_{r=2}^{n}\beta_r\varepsilon_{ij}\partial_i F_j^{(r-2)}\right)\varphi^{(a)}, \quad a = 1, ..., N \ + \text{c.c.}, \tag{45}$$

$$\partial_0\pi^{(a)} = \left(D_i D_i - \rho_a m^2\right)\varphi^{(a)} + i\left(\beta_1 A_0 + \frac{1}{m}\sum_{r=2}^{n}\beta_r\varepsilon_{ij}\partial_i F_j^{(r-2)}\right)\pi^{(a)}, \quad a = 1, ..., N \ + \text{c.c.}, \tag{46}$$

$$\Theta \equiv m\left(\sum_{r=1}^{n}\alpha_r\varepsilon_{ij}\partial_i F_j^{(r-1)} - \sum_{a=1}^{N} ie_a\left(\varphi^{(a)}\pi^{(a)} - (\varphi^{(a)})*(\pi^{(a)})*\right)\right) = 0\,. \tag{47}$$

Here the quantities $F^{(r)}_i$, $i = 1, 2$, $r = 1, ..., n-1$ (4), $\varphi^{(a)}$, $a = 1, ..., N$ (38), and $\pi^{(a)}$, $a = 1, ..., N$, are new auxiliary variables absorbing the derivatives of original vector and scalar fields. All the auxiliary variables are expressed from the equations (42)–(47), after which the resulting system coincides with the initial higher-derivative equations. Also note that in the first-order formalism, quantity Θ (47) does not involve any time derivatives and can be treated as a constraint.

Equations (42)–(46) are Hamiltonian in the sense of (17) with respect to the Hamiltonian

$$H(\alpha,\beta) = T_{00}(\alpha,\beta) + \sum_{a=1}^{N} e_a\left(\pi^{(a)}(\pi^{(a)})* + D_i\varphi(D_i\varphi)* + \rho^a m^2\varphi\varphi*\right) + \left(\beta_1 A_0 + \frac{1}{m}\sum_{p=1}^{n-1}\beta_{p+1}\varepsilon_{ij}\partial_i F_j^{(p-1)}\right)\Theta, \tag{48}$$

where $T_{00}(\alpha, \beta)$ was determined in (18). The Poisson bracket of the phase-space variables is determined by the relation

$$\left\{F_i^{(n-1)}(\boldsymbol{x}), F_j^{(n-1)}(\boldsymbol{y})\right\}_{\alpha,\beta} = \frac{1}{\alpha_n m \det C(\alpha,\beta)}\left(\sum_{r=1}^{n-1}\alpha_r M^{r-1,n-1}(\alpha,\beta)\right)\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y})\,, \tag{49}$$

$$\left\{F_i^{(r)}(\boldsymbol{x}), F_j^{(s)}(\boldsymbol{y})\right\}_{\alpha,\beta} = -\frac{M^{r,s-1}(\alpha,\beta)}{m\det C(\alpha,\beta)}\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}), \quad r, s-1 = 1, ..., n-1\,, \tag{50}$$

$$\left\{A_i(\boldsymbol{x}), A_j(\boldsymbol{y})\right\}_{\alpha,\beta} = -\frac{1}{\beta_1 m \det C(\alpha,\beta)}\left(\sum_{k=1}^{n-1}\beta_{k+1}M^{1,k}(\alpha,\beta)\right)\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y})\,, \tag{51}$$

$$\left\{\varphi^{(a)}(\boldsymbol{x}), \pi^{(b)}(\boldsymbol{y})\right\}_{\alpha,\beta} = \frac{1}{e_a}\delta^{ab}\delta(\boldsymbol{x}-\boldsymbol{y})\,. \tag{52}$$

Expressions (49)–(52) are well defined if

$$\det C(\alpha,\beta) \neq 0, \quad e_a \neq 0, \quad \beta_1 \neq 0\,. \tag{53}$$

The first pair of relations guarantees a non-degeneracy of the Hamiltonian, while the third condition ensures the presence of a gauging $U(1)$-transformation for the scalar field. A possibility where $\beta_1 = 0$ is considered as a special case and is not studied in what follows. Thus, it has been shown that almost all interactions constructed in [15] admit a Hamiltonian formulation. These Hamiltonian formulations are not canonically equivalent to the Ostrogradski one, since the original equations are non-Lagrangian.

Let us illustrate the overall structure of the Hamiltonian formulation using an extended third-order CS-theory as an example, which interacts with a charged massless scalar field. In this case, the equations of motion (37) are written as

$$\left(\frac{1}{m}\alpha_3(*d)^3+\alpha_2(*d)^2+m\alpha_1(*d)\right)A+ie\left(\varphi^*(D_\mu\varphi)-\varphi(D_\mu\varphi)^*\right)dx^\mu=0,\quad D_\mu D^\mu\varphi=0\,, \tag{54}$$

where $e = e_1$ is the coupling constant and the covariant derivative is determined by the relation

$$D_\mu\varphi=\left(\partial_\mu-i\left(\beta_1A_\mu+\beta_2F_\mu+\beta_3G_\mu\right)\right)\varphi,\quad F_\mu\equiv\frac{1}{m}(*dA)_\mu,\quad G_\mu\equiv\frac{1}{m^2}\left((*d)^2A\right)_\mu. \tag{55}$$

The first-order formulation (42)–(47) for equations (54) can be written as

$$\partial_0A_i=\partial_iA_0-m\varepsilon_{ij}F_j\,, \tag{56}$$

$$\partial_0F_i=\frac{1}{m}\varepsilon_{ij}\partial_k\left(\partial_kA_j-\partial_jA_k\right)-m\varepsilon_{ij}G_j\,, \tag{57}$$

$$\partial_0G_i=\frac{1}{m}\varepsilon_{ij}\partial_k\left(\partial_kF_j-\partial_jF_k\right)+\frac{1}{\alpha_n}\left(m\varepsilon_{ij}\left(\alpha_1F_j+\alpha_2G_j\right)+\frac{1}{m}ie\left(\varphi^*(D_i\varphi)-\varphi(D_i\varphi)^*\right)\right), \tag{58}$$

$$\partial_0\varphi=\pi^*+i\left(\beta_1A_0+\frac{1}{m}\left(\beta_2\varepsilon_{ij}\partial_iA_j+\beta_3\varepsilon_{ij}\partial_iF_j\right)\right)\varphi,\quad +\text{c.c.}\,, \tag{59}$$

$$\partial_0\pi=D_iD_i\varphi^*-i\left(\beta_1A_0+\frac{1}{m}\left(\beta_2\varepsilon_{ij}\partial_iA_j+\beta_3\varepsilon_{ij}\partial_iF_j\right)\right)\pi,\quad +\text{c.c.}, \tag{60}$$

$$\Theta\equiv m\varepsilon_{ij}\left(\alpha_1\partial_iA_j+\alpha_2\partial_iF_j+\alpha_3\partial_iF_j\right)+ie(\varphi\pi-\varphi^*\pi^*)=0\,. \tag{61}$$

Then the Hamiltonian (48) takes the following form:

$$\begin{aligned}H(\alpha,\beta)=\frac{m^2}{2}\Big[&(\beta_2\alpha_3-\beta_3\alpha_2)(G_iG_i+\partial_iF_j(\partial_iF_j-\partial_jF_i))+2(\beta_1\alpha_3-\beta_3\alpha_1)(G_iF_i+\partial_iF_j\\&\times(\partial_iA_j-\partial_jA_i))+(\beta_1\alpha_2-\beta_2\alpha_1)(F_iF_i+\partial_iA_j(\partial_iA_j-\partial_jA_i))\Big]+e\left(\pi\pi^*+D_i\varphi(D_i\varphi)^*\right)\\&+\left(\beta_1A_0+\frac{1}{m}\left(\beta_2\varepsilon_{ij}\partial_iA_j+\beta_3\varepsilon_{ij}\partial_iF_j\right)\right)\Theta\,.\end{aligned} \tag{62}$$

The Poisson brackets (49)–(52) of the phase-space variables are determined by the relations

$$\left\{G_i(\boldsymbol{x}),G_j(\boldsymbol{y})\right\}_{\alpha,\beta}=\frac{\beta_3\alpha_1^2-\beta_2\alpha_2\alpha_1+\beta_1(\alpha_2^2-\alpha_3\alpha_1)}{m\det C(\alpha,\beta)}\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y})\,, \tag{63}$$

$$\left\{G_i(\boldsymbol{x}),F_j(\boldsymbol{y})\right\}_{\alpha,\beta}=\frac{\beta_1\alpha_2-\beta_2\alpha_1}{m\det C(\alpha,\beta)}\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y})\,, \tag{64}$$

$$\left\{F_i(\boldsymbol{x}),F_j(\boldsymbol{y})\right\}_{\alpha,\beta}=\{G_i(\boldsymbol{x}),A_j(\boldsymbol{y})\}_{\alpha,\beta}=\frac{\beta_1\alpha_3-\beta_3\alpha_1}{m\det C(\alpha,\beta)}\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{65}$$

$$\left\{F_i(\boldsymbol{x}),A_j(\boldsymbol{y})\right\}_{\alpha,\beta}=\frac{\beta_3\alpha_2-\beta_2\alpha_3}{m\det C(\alpha,\beta)}\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{66}$$

$$\left\{A_i(\boldsymbol{x}),A_j(\boldsymbol{y})\right\}_{\alpha,\beta}=\frac{\beta_3^2\alpha_1-\beta_3\beta_2\alpha_2-\beta_3\beta_1\alpha_3+\beta_2^2\alpha_3}{m\beta_1\det C(\alpha,\beta)}\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{67}$$

$$\left\{\varphi(\boldsymbol{x}),\pi(\boldsymbol{y})\right\}_{\alpha,\beta}=\frac{1}{e}\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{68}$$

Here we used the notation

$$\det C(\alpha,\beta)\equiv\beta_3^2\alpha_1^2-\beta_3\beta_2\alpha_2\alpha_1+\beta_3\beta_1(2\alpha_2^2-2\alpha_3\alpha_1)+\beta_2^2\alpha_3\alpha_1-\beta_2\beta_1\alpha_3\alpha_1+\beta_1^2\alpha_3^2. \tag{69}$$

The Hamiltonian (62) and the Poisson brackets (63)–(66) are well defined whenever

$$\det C(\alpha,\beta)\neq 0,\quad \beta_1\neq 0,\quad e\neq 0. \tag{70}$$

At $\varphi=\pi=e=0$, the vector field dynamics is separated and the formulas (56)–(58), (62)–(67) reproduce one of the admissible representatives of the series of Hamiltonian formulations for the free extended third-order CS-theory [11]. Thus a correspondence is set with the results obtained earlier.

SUMMARY

In the present study it has been shown that the free extended n-th order Chern-Simons theory is a multi-Hamiltonian theory admitting an (n–1)-parametric series of canonically non-equivalent Hamiltonian formulations, in which case a zero-zero component of any representative of the series of conserved tensors could be selected as a Hamiltonian. For certain model parameters there are bounded Hamiltonians among the admissible ones, while in other cases the Hamiltonian is always unbounded. In a compact form, the Hamiltonian boundedness condition is equivalent to the requirement of a positive definiteness of the matrix $C_{r,s}(\alpha,\beta)$ (5). The canonical Hamiltonian formulation is included into the constructed series and its Hamiltonian is always unbounded. It has been shown at the interacting level that a class of non-Lagrangian vertices of interaction with the scalar field obtained in [15] preserves the unique representative of the series of symplectic structures, whose parameters are fixed by the values of the coupling constants. This allows preserving the dynamics stability and the possibility of a subsequent quantization of the theory at the interacting level.

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