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QUANTUM ELECTRONICS

REGARDING NONSTATIONARY QUADRATIC QUANTUM SYSTEMS

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With the help of the evolution operator method, we have established unitary connection between quadratic systems, namely between a free particle with variable mass $M(t)$, a particle with variable mass $M(t)$ in a variable homogeneous field, and a harmonic oscillator with variable mass $M(t)$ and frequency $\omega(t)$, on which a variable force $F(t)$ acts. Knowledge of the unitary connection allowed us to express easily in general form the propagators, invariants, wave functions, and other functions of a linear potential and a harmonic oscillator in terms of the corresponding quantities for a free particle. We have analyzed the linear and quadratic invariants in detail. Results known in the literature follow as particular cases from the general results obtained here.

Keywords: nonstationary quadratic systems, evolution operator, invariants, wave functions, unitary connection.

INTRODUCTION

Nonstationary quadratic quantum systems – a free particle with variable mass [1–3], a particle with variable mass in a variable homogeneous field [2–7], a nonstationary harmonic oscillator (with a driving force and without a driving force) [8–16] – like their stationary analogs, play an important role in many branches of physics. They find wide application in statistical physics, the theory of superconductivity, atomic and nuclear physics, molecular spectroscopy, quantum field theory, etc. (see the References to [14]).

Quadratic quantum systems are among exactly solvable quantum-mechanical problems. Other exactly solvable nonstationary systems include a singular oscillator with variable frequency [17] and a relativistic particle in a variable homogeneous field [18, 19]. Many examples of exact nonstationary solutions of the Klein–Gordon and Dirac relativistic wave equations are contained in [20, 21]. Questions of the generation of exactly solvable potentials (stationary and nonstationary) by the method of *dressing* the differential operators are expounded in [22, 23].

Exact solutions of the equation of motion are always of interest from a physical as well as a mathematical point of view since they can model real physical phenomena and allow the most complete tracking of changes in physical quantities or lead to the establishment of new mathematical relations between special functions. Exact analytical solutions can also be useful in grounding approximate solution methods, in particular when verifying numerical methods for solving the equation of motion. Examples: 1) the time-dependent harmonic oscillator models the behavior of a charged particle in a variable magnetic field whereas a time-dependent linear potential models the motion of a charged particle in a variable homogeneous electric field (for example, see [11, 24]); 2) an ion in a Pauli trap is

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described by a parametric oscillator model [25]. The Hamiltonian of a parametric oscillator with arbitrary time dependence of its frequency has the form $H = \hat{p}^2 / 2m + m\omega^2(t)\hat{x}^2 / 2$. For an ion in a trap, the time dependence of the frequency has the form $\omega^2(t) = 1 + k^2 \sin^2 \Omega t$; 3) induced Raman scattering can also be described with the help of a quadratic Hamiltonian (for example, see [26]).

To investigate nonstationary quantum systems, the following methods are commonly used: the method of invariants [11, 12], the path integral method [9], the method of spacetime transformations (for example, see [6]), the method of generating functions [17, 27], and the evolution operator method [2, 7, 28, 29].

The aim of the present work is to construct first- and second-order invariants and various wave functions for a free particle with variable mass, a particle with variable mass in a variable homogeneous field, and a harmonic oscillator with variable mass and frequency under the action of a variable force. In this paper, we use the evolution operator method. As is well known, for a quantum system described by the Hamiltonian $H(t)$, the evolution operator is defined by the formula

$$U(t,t_0) = T\exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t} H(t')dt'\right\}, \tag{1}$$

where T is the time-ordering operator. Since the Hamiltonian of the system is a Hermitian operator, the evolution operator is unitary: $U^+U = UU^+ = 1$. The variation of the states of a quantum system in time is described by the time-dependent Schrödinger equation $\hat{S}(t)\,\psi(t) = 0$ or the evolution operator $U(t,t_0)$ if the wave function of the system at the initial time, $\psi(t_0)$ is known, i.e.,

$$\psi(t) = U(t,t_0)\,\psi(t_0)\,. \tag{2}$$

Here the Schrödinger operator has the form $\hat{S}(t) = i\hbar\partial_t - H(t)$. The evolution operator also satisfies the Schrödinger equation

$$\hat{S}(t)U(t,t_0) = 0 \tag{3}$$

with the obvious initial condition $U(t_0,t_0) = 1$. The kernel of the evolution operator is called the Green's function or the propagator of the quantum system and contains all the information about the system.

This paper is organized as follows: Section 1 presents in general, chronologically disentangled form expressions for the evolution operators, and also the limiting and unitary connections between them; Section 2 considers construction of linear and quadratic invariants in general form for quadratic quantum systems, and also these same invariants in general form for the given quantum systems, as well as particular cases of linear and quadratic invariants; Section 3 establishes unitary connections between the considered systems. The main results are summed up in the Conclusions.

1. EVOLUTION OPERATORS

In what follows, evolution operators of the considered systems will play an important role. Therefore, we present the explicit form of each of them. We will work in coordinate space.

1.1. A free quantum particle with variable mass $M(t)$. In this case, the Schrödinger equation and the evolution operator have the form

$$\hat{S}_F(x,t)\psi_F(x,t)=0\,,\ \hat{S}_F=i\hbar\partial_t+\frac{\hbar^2}{2M(t)}\partial_x^2+V_0(t)\,, \tag{1.1}$$

$$U_F(x,t)=e^{i\Lambda_0(t)+i\hbar S_2(t)\partial_x^2}\,, \tag{1.2}$$

where $\Lambda_0(t)=\hbar^{-1}\int_{t_0}^{t}V_0(t')dt'$, $S_2(t)=\int_{t_0}^{t}\frac{dt'}{2M(t')}$, and $V_0(t)$ is a potential well (or barrier), whose depth (height) varies with time.

1.2. A quantum particle with variable mass $M(t)$ in a variable homogeneous field. In this case, we have

$$\hat{S}_L(x,t)\psi_L(x,t)=0,\ \hat{S}_L=i\hbar\partial_t+\frac{\hbar^2}{2M(t)}\partial_x^2+F(t)x+V_0(t)\,, \tag{1.3}$$

$$U_L(x,t)=V_L(x,t)U_F(x,t)\,, \tag{1.4}$$

where we have introduced the following notation:

$$V_L(x,t)=e^{i\varphi_0(x,t)}e^{-S_1(t)\partial_x}\,,\ \varphi_0=\hbar^{-1}\left[x\delta(t)-S_0(t)\right], \tag{1.5}$$

$$\delta(t)=\int_{t_0}^{t}F(t')dt'\,,\ S_0(t)=\int_{t_0}^{t}\frac{\delta^2(t')}{2M(t')}dt'\,,\quad S_1(t)=\int_{t_0}^{t}\frac{\delta(t')dt'}{M(t')}\,. \tag{1.6}$$

The operator V_L (Eqs. (1.5)) can be rewritten in the form

$$V_L(x,t)=e^{-S_1(t)\partial_x}e^{i\hbar^{-1}[M(t)\dot{S}_1(t)x+\sigma_L(t)]}\,, \tag{1.7}$$

where the function σ_L is the classical action for a particle in a variable homogeneous field, i.e.,

$$\sigma_L(t)=\int_{t_0}^{t}\left[\frac{1}{2}M(t')\dot{S}_1^2(t')+F(t')S_1(t')\right]dt'\,. \tag{1.8}$$

For $M(t)=m=$ const we have

$$S_0(t)=\frac{\delta_2(t)}{2m},\quad S_1(t)=\frac{\delta_1(t)}{m},\quad S_2(t)=\frac{\tau}{2m}\,, \tag{1.9}$$

where $\delta_1(t)=\int_{t_0}^{t}\delta(t')dt'$, $\delta_2(t)=\int_{t_0}^{t}\delta^2(t')dt'$, and $\tau=t-t_0$. Formulas (1.2) and (1.4) set up a unitary connection between a free quantum particle and a quantum particle with variable mass in a variable homogeneous field [2], i.e.,

$$\hat{S}_L=V_L\hat{S}_FV_L^{-1}\,,\ \hat{S}_F=V_L^{-1}\hat{S}_LV_L\,. \tag{1.10}$$

1.3. A harmonic oscillator with variable mass $M(t)$ and variable frequency $\omega(t)$, on which a variable force

$F(t)$ *acts*. Such an oscillator is described by the Schrödinger equation

$$\hat{S}_H(x,t)\,\psi_H(x,t)=0\,,$$

$$\hat{S}_H(x,t)=i\hbar\partial_t+\frac{\hbar^2}{2M(t)}\partial_x^2-\frac{1}{2}M(t)\omega^2(t)x^2+F(t)x+V_0(t)\,, \tag{1.11}$$

and the corresponding evolution operator is given by the expression

$$U_H(x,t)=U_1(x,t)U_H^{(0)}(x,t)\;, \tag{1.12}$$

in which we have introduced the following notation:

$$U_H^{(0)}(x,t)=e^{\frac{1}{2}b(t)+i\Lambda_0(t)}e^{i\alpha(t)x^2}e^{b(t)x\partial_x}e^{iS(t)\partial_x^2}\,,$$

$$U_1(x,t)=e^{-\xi(t)\partial_x}e^{i\hbar^{-1}\left[M(t)\dot{\xi}(t)x+\sigma_H(t)\right]}\,. \tag{1.13}$$

Here $U_H^{(0)}(x,t)$ is the evolution operator of a harmonic oscillator when a force is not acting on it, and $U_1(x,t)$ is the operator *generating* the action of the force on the oscillator. The function $\alpha(t)$ is the solution of the Ricatti equation

$$\dot{\alpha}(t)+\frac{2\hbar}{M(t)}\alpha^2(t)=-\frac{1}{2\hbar}M(t)\omega^2(t)\,, \tag{1.14}$$

the function $\xi(t)$ satisfies the equation

$$\frac{d}{dt}\left[M(t)\dot{\xi}(t)\right]+M(t)\omega^2(t)\xi(t)=F(t) \tag{1.15}$$

with natural initial conditions $\xi(t_0)=0, \dot{\xi}(t_0)=0$, and the function $\sigma_H(t)$ is the classical action for a harmonic oscillator in the presence of an external force:

$$\sigma_H(t)=\int_{t_0}^{t}\left[\frac{1}{2}M(t')\dot{\xi}^2(t')-\frac{1}{2}M(t')\,\omega^2(t')\xi^2(t')+F(t')\,\xi(t')\right]dt'\,. \tag{1.16}$$

It is well known that the Ricatti equation (Eq. (1.14)) can be reduced to a second-order linear homogeneous differential equation

$$\frac{d}{dt}[M(t)\,\dot{\eta}(t)]+M(t)\omega^2(t)\eta(t)=0 \tag{1.17}$$

by introducing the new function $\eta(t)$ using the formula

$$\alpha(t)=\frac{M(t)\dot{\eta}(t)}{2\hbar\eta(t)}\,. \tag{1.18}$$

The functions $b(t)$ and $S(t)$ in formula (1.13) are defined as follows:

$$b(t) = -2\hbar\int_{t_0}^{t}\frac{\alpha(t')}{M(t')}dt' = \ln\left(\frac{\eta(t_0)}{\eta(t)}\right),$$

$$S(t) = \hbar\int_{t_0}^{t}\frac{e^{2b(t')}}{2M(t')}dt' = \hbar\eta^2(t_0)\int_{t_0}^{t}\frac{dt'}{2M(t')\eta^2(t')}. \tag{1.19}$$

We emphasize that the following initial conditions for the functions α and η: $\alpha(t_0) = 0$, $\eta(t_0) = \text{const}$, and $\dot{\eta}(t_0) = 0$ follow from the requirement $U_H^{(0)}(x, t_0) = 1$ on the evolution operator (see formula (1.13)). Note that the operator V_L generates the action of the force $F(t)$ on a free particle, and that the operator U_1 generates the action of the force $F(t)$ on an oscillator.

Let us consider three particular cases of oscillator model (1.11) and find an explicit form of the functions η, α, S, and ξ for them (see formulas (1.14)–(1.19)).

1) Stationary oscillator with a driving force. This model is described by the Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2}{2}\hat{x}^2 - F(t)\hat{x} \tag{1.20}$$

and the functions

$$\eta(t) = \omega_0\cos\omega_0\tau,\ \alpha(t) = -\frac{m\omega_0}{2\hbar}\tan\omega_0\tau,$$

$$S(t) = \frac{\hbar}{2m\omega_0}\tan\omega_0\tau,\ \xi(t) = \frac{1}{m\omega_0}\int_{t_0}^{t}F(t')\sin\omega_0(t-t')dt'. \tag{1.21}$$

Hence, in the limit $\omega_0 \to 0$ we have $\eta(t) \to 0$, $\alpha(t) \to 0$, $S(t) \to \hbar S_2(t)$, and $\xi(t) \to S_1(t)$, i.e., we obtain the functions $S_1(t)$ and $S_2(t)$ corresponding to linear potential (1.3).

2) Parametric oscillator for $F(t) = 0$. This model is described by the Hamiltonian [14]

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\left[\omega_0^2 + \frac{2\lambda^2}{\cosh^2(\lambda t)}\right]\hat{x}^2. \tag{1.22}$$

For this model we have

$$\eta(t) = \omega_0\cos(\omega_0\tau) - \lambda\sin(\omega_0\tau)\tanh(\lambda\tau),$$

$$\alpha(t) = -\frac{m[\omega_0^2\sin(\omega_0\tau) + \lambda\omega_0\cos(\omega_0\tau)\tanh(\lambda\tau) + \lambda^2\sin(\omega_0\tau)\cosh^{-2}(\lambda\tau)]}{2\hbar[\omega_0\cos(\omega_0\tau) - \lambda\sin(\omega_0\tau)\tanh(\lambda\tau)]},$$

$$S(t) = \frac{\hbar\omega_0[\omega_0\sin(\omega_0\tau) + \lambda\cos(\omega_0\tau)\tanh(\lambda\tau)]}{2m(\omega_0^2 + \lambda^2)[\omega_0\cos(\omega_0\tau) - \lambda\sin(\omega_0\tau)\tanh(\lambda\tau)]}. \tag{1.23}$$

If $\lambda = 0$, these formulas coincide with formulas (1.21) for the stationary oscillator for $F = 0$. In order to take the limit $\omega_0 \to 0$ in expressions (1.23), we take $\eta(t)$ in the form $\eta(t) = \cos(\omega_0\tau) - \frac{\lambda}{\omega_0}\sin(\omega_0\tau)\tanh(\lambda\tau)$. This is possible since the function $\eta(t)$ is defined to within a constant factor. Thus, in the limit $\omega_0 \to 0$ from expressions (1.23) we obtain expressions for an oscillator with frequency $\omega^2(t) = \frac{2\lambda^2}{\cosh^2(\lambda t)}$, namely

$$\eta(t) = 1 - \lambda\tau\tanh(\lambda\tau),\ \alpha(t) = -\frac{\lambda[\tanh(\lambda\tau) + \lambda\tau\cosh^{-2}(\lambda\tau)]}{1 - \lambda\tau\tanh(\lambda\tau)},$$

$$S(t) = \frac{\hbar\tanh(\lambda\tau)}{2m\lambda[1 - \lambda\tau\tanh(\lambda\tau)]}. \tag{1.24}$$

From a comparison of integral representation (1.19) with expression (1.24) for $S(t)$, we obtain the integral formula

$$\int\frac{dx}{(1 - x\tanh x)^2} = \frac{\tanh x}{1 - x\tanh x} + C. \tag{1.25}$$

It is easy to convince oneself of its validity by checking it directly.

3) The Caldirola–Kanai oscillator with a driving force. The Hamiltonian of this model has the form [14]

$$H = \frac{\hat{p}^2}{2m}e^{-2\lambda t} + \frac{m\omega_0^2}{2}\hat{x}^2 e^{2\lambda t} - F(t)\hat{x}. \tag{1.26}$$

The functions η, α, S, and ξ in the given case are assigned by the formulas

$$\eta(t) = e^{-\lambda\tau}Q(t),\ \alpha(t) = -\frac{m\omega_0^2\sin(\omega\tau)}{2\hbar Q(t)}e^{2\lambda t},\ S(t) = \frac{\hbar\sin(\omega\tau)}{2mQ(t)}e^{-2\lambda t_0},$$

$$\xi(t) = \frac{1}{m\omega}\int_{t_0}^{t} e^{-\lambda(t+t')}F(t')\sin\omega(t - t')dt', \tag{1.27}$$

where $Q(t) = \omega\cos(\omega\tau) + \lambda\sin(\omega\tau)$ and $\omega = \sqrt{\omega_0^2 - \lambda^2} > 0$. For $\lambda = 0$ functions (1.27) transform to the functions assigned by formulas (1.21) for a stationary oscillator with the driving force $F(t)$ (see Eq. (1.20)), whereas for $\omega_0 = 0$ they reproduce the corresponding formulas for a linear potential when the mass $M(t) = me^{2\lambda t}$, i.e.,

$$\eta(t) = i\lambda,\ \alpha(t) = 0,\ S(t) = \frac{1}{4m\lambda}\left(e^{-2\lambda t_0} - e^{-2\lambda t}\right) = S_2(t),$$

$$\xi(t) = \frac{1}{m}\int_{t_0}^{t} e^{-2\lambda t'}\delta(t')dt' = S_1(t). \tag{1.28}$$

Formulas (1.12) and (1.13) set up a unitary connection, on the one hand, between nonstationary harmonic oscillators for $F=0$ and $F\neq 0$, i.e.,

$$\hat{S}_H = U_1 \hat{S}_H^{(0)} U_1^{-1}\,,\ \hat{S}_H^{(0)} = U_1^{-1}\hat{S}_H U_1\,, \tag{1.29}$$

and, on the other, between a nonstationary harmonic oscillator and a free particle under the condition that the mass $M(t)$ of the free particle has been renormalized, i.e., it has been replaced by $M_{\mathrm{Re}n}(t)=M(t)e^{-2b(t)}$. The Schrödinger operator and the evolution operator for a free quantum particle with renormalized mass are obtained from Eqs. (1.1) and (1.2) by replacing $M(t)$ in them by $M_{\mathrm{Re}n}(t)$, and $\hbar S_2(t)$ by $S(t)$, respectively, i.e., they are equal to

$$\hat{S}_F^{\mathrm{Re}n} = i\hbar\partial_t + \frac{\hbar^2}{2M_{\mathrm{Re}n}(t)}\partial_x^2 + V_0(t)\,,\ U_F^{\mathrm{Re}n}(x,t) = e^{i\Lambda_0(t)+iS(t)\partial_x^2}\,. \tag{1.30}$$

Thus, we have

$$\hat{S}_H = V_{FH}\,\hat{S}_F^{\mathrm{Re}n}\,V_{FH}^{-1}\,,\ \hat{S}_F^{\mathrm{Re}n} = V_{FH}^{-1}\,\hat{S}_H\,V_{FH}\,, \tag{1.31}$$

where

$$V_{FH} = U_1 V_{FH}^{(0)}\,,\ V_{FH}^{(0)} = e^{\frac{1}{2}b(t)}\,e^{i\alpha(t)\,x^2}\,e^{b(t)x\partial_x}\,. \tag{1.32}$$

In what follows, we will find invariants and wave functions for the considered quadratic functions with the help of the operators considered here.

2. INVARIANTS

An invariant is defined as a time-dependent operator $I(t)$, whose mean value does not depend on time, i.e., $d\bar{I}(t)/dt=0$. In other words, an invariant $I(t)$ is an operator that commutes with the Schrödinger operator: $[\hat{S}(t),I(t)]=0$. As is well known (for example, see [24]), if there exists in the quantum system an evolution operator $U(t)$, then it is possible to construct $2N$ independent (basis) invariants $\hat{x}_0(t)$ and $\hat{p}_0(t)$ according to the formulas

$$\hat{x}_0(t) = U(t)\hat{x}U^{-1}(t)\,,\ \hat{p}_0(t) = U(t)\hat{p}U^{-1}(t)\,, \tag{2.1}$$

where N is the number of degrees of freedom of the system. They correspond to the initial points in the phase space of a classical system. Generally, operators (2.1) are linear combinations of the operators $\hat{x}$ and $\hat{p}$ with time-dependent coefficients, i.e.,

$$\hat{x}_0(t) = e_1(t)\hat{x} + e_2(t)\hat{p} + e_3(t)\,,\ \hat{p}_0(t) = d_1(t)\hat{x} + d_2(t)\hat{p} + d_3(t)\,. \tag{2.2}$$

If the Hamiltonian of the system has the form $H=\alpha_2(t)\hat{p}^2+\beta_2(t)\hat{x}^2-F(t)\hat{x}$, then these coefficients satisfy the equations

$$\dot{e}_1 = 2\beta_2 e_2\,,\ \dot{e}_2 = -2\alpha_2 e_1\,,\ \dot{e}_3 = -Fe_2\,,\ \dot{d}_1 = 2\beta_2 d_2\,,\ \dot{d}_2 = -2\alpha_2 d_1\,,\ \dot{d}_3 = -Fd_2\,,\ d_2e_1 - d_1e_2 = 1 \tag{2.3}$$

with initial conditions $e_1(t_0) = d_2(t_0) = 1$ and $e_2(t_0) = e_3(t_0) = d_1(t_0) = d_3(t_0) = 0$. Equations (2.3) follow from the commutation relations $[\hat{S}, \hat{x}_0] = 0$, $[\hat{S}, \hat{p}_0] = 0$, and $\left[\hat{p}_0, \hat{x}_0\right] = -i\hbar$. All remaining invariants can be expressed in terms of the basis invariants. For example, invariants that are linear and quadratic in powers of $\hat{x}_0$ and $\hat{p}_0$ can be expressed, generally, as

$$I_1(t) = A_{10}\hat{p}_0 + B_{10}\hat{x}_0 + C_{10}, \tag{2.4}$$

$$I_2(t) = A_{20}p_0^2 + B_{20}\hat{x}_0^2 + C_{20}\hat{p}_0\hat{x}_0 + \tilde{C}_{20}\hat{x}_0\hat{p}_0 + D_{20}\hat{p}_0 + E_{20}\hat{x}_0 + F_{20}, \tag{2.5}$$

where the coefficients $A_{10}, B_{10,}, \ldots$ are arbitrary constants, in general complex.

We write out the explicit form of invariants (2.2), (2.4), and (2.5) for the quadratic quantum systems considered in Section 1.

1) For a free quantum particle with variable mass they are equal to

$$\hat{x}_{F0}(t) = \hat{x} - 2S_2(t)\hat{p}, \; \hat{p}_{F0}(t) = \hat{p}, \tag{2.6}$$

$$I_{F1}(t) = A_{F1}(t)\hat{p} + B_{10}\hat{x} + C_{10}, \tag{2.7}$$

$$I_{F2}(t) = A_{F2}(t)\hat{p}^2 + B_{20}\hat{x}^2 + C_{F2}(t)\hat{p}\hat{x} + \tilde{C}_{F2}(t)\hat{x}\hat{p} + D_{F2}(t)\hat{p} + E_{20}\hat{x} + F_{20}, \tag{2.8}$$

where

$$A_1 = A_{10} - 2B_{10}S_2,$$

$$A_{F2} = A_{20} + 4B_{20}S_2^2 - 2(C_{20} + \tilde{C}_{20})S_2, \; C_{F2} = C_{20} - 2B_{20}S_2,$$

$$\tilde{C}_{F2} = \tilde{C}_{20} - 2B_{20}S_2, \; D_{F2} = D_{20} - 2E_{20}S_2(t).$$

If in expression (2.7) we choose $A_{10} = \dfrac{i\lambda_2}{\sqrt{2\hbar}}$, $B_{10} = \dfrac{\lambda_1}{\sqrt{2\hbar}}$, and $C_{10} = 0$, then we obtain the invariant (annihilation operator) of [2]

$$A_F(t) = \frac{1}{\sqrt{2\hbar}}[\lambda_1\hat{x} + i\varepsilon_F(t)\hat{p}], \quad \varepsilon_F = \lambda_2 + 2i\lambda_1 S_2, \tag{2.9}$$

where λ_1 and λ_2 are complex numbers satisfying the condition $\mathrm{Re}(\lambda_1^{\bullet}\lambda_2) = 1$.

2) For a particle with variable mass in a variable homogeneous field we have

$$\hat{x}_{L0}(t) = \hat{x} - 2S_2(t)\hat{p}, \; \hat{p}_{L0}(t) = \hat{p} - \delta(t), \tag{2.10}$$

$$I_{L1}(t) = A_{L1}(t)\hat{p} + B_{10}\hat{x} + C_{L1}(t), \tag{2.11}$$

$$I_{L2}(t) = A_{L2}(t)\hat{p}^2 + B_{20}\hat{x}^2 + C_{L2}(t)\hat{p}\hat{x} + \tilde{C}_{L2}(t)\hat{x}\hat{p} + D_{L2}(t)\hat{p} + E_{L2}(t)\hat{x} + F_{L2}, \tag{2.12}$$

where

$$A_{L1} = A_{10} - 2B_{10}S_2\,,\ C_{L1} = C_{10} - A_{10}\delta + B_{10}\nu\,,$$

$$A_{L2} = A_{20} + 4B_{20}S_2^2 - 2(C_{20} + \tilde{C}_{20})S_2\,,\ C_{L2} = C_{20} - 2B_{20}S_2\,,$$

$$\tilde{C}_{L2} = \tilde{C}_{20} - 2B_{20}S_2\,,\ D_{L2} = D_{20} - 2A_{20}\delta - 4B_{20}S_2\nu + (C_{20} + \tilde{C}_{20})(2\nu + S_1) - 2E_{20}S_2\,,$$

$$E_{L2} = 2B_{20}\nu - (C_{20} + \tilde{C}_{20})\delta + E_{20}\,,\ F_{L2} = A_{20}\delta^2 + B_{20}\nu^2 - (C_{20} + \tilde{C}_{20})\delta\nu - D_{20}\delta + E_{20}\nu + F_{20}\,, \tag{2.13}$$

and $\nu = 2\delta(t)S_2(t) - S_1(t)$. If we set $A_{10} = \frac{i\lambda_2}{\sqrt{2\hbar}}$, $B_{10} = \frac{\lambda_1}{\sqrt{2\hbar}}$, and $C_{10} = 0$ in expressions (2.11), we obtain the invariant (annihilation operator) of [2]

$$A_L(t) = \frac{1}{\sqrt{2\hbar}}\left[\lambda_1\left(\hat{x} - S_1\right) + i\varepsilon_F(t)\left(\hat{p} - \delta\right)\right]. \tag{2.14}$$

3) For a harmonic oscillator with variable mass and frequency, acted on by a variable force, we have

$$\hat{x}_{H0}(t) = -M(t)\dot{a}_2(t)\hat{x} + a_2(t)\hat{p} + M(t)\Delta_2(t),$$
$$\hat{p}_{H0}(t) = -M(t)\dot{a}_1(t)\hat{x} + a_1(t)\hat{p} + M(t)\Delta_1(t), \tag{2.15}$$

$$I_{H1}(t) = A_{H1}(t)\hat{p} + B_{H1}(t)\hat{x} + C_{H1}(t)\,, \tag{2.16}$$

$$I_{H2}(t) = A_{H2}(t)\hat{p}^2 + B_{H2}(t)\hat{x}^2 + C_{H2}(t)\hat{p}\hat{x} + \tilde{C}_{H2}(t)\hat{x}\hat{p} + D_{H2}(t)\hat{p} + E_{H2}(t)\hat{x} + F_{H2}(t)\,, \tag{2.17}$$

$$A_{H1} = A_{10}a_1 + B_{10}a_2\,,\ B_{H1} = -M(A_{10}\dot{a}_1 + B_{10}\dot{a}_2) = -M\dot{A}_{H1}\,,$$

$$C_{H1} = C_{10} + M(A_{10}\Delta_1 + B_{10}\Delta_2)\,,$$

$$A_{H2} = A_{20}a_1^2 + B_{20}a_2^2 + (C_{20} + \tilde{C}_{20})a_1a_2\,,$$

$$B_{H2} = M^2[A_{20}\dot{a}_1^2 + B_{20}\dot{a}_2^2 + (C_{20} + \tilde{C}_{20})\dot{a}_1\dot{a}_2]\,,$$

$$C_{H2} = -M(A_{20}a_1\dot{a}_1 + B_{20}a_2\dot{a}_2 + C_{20}a_1\dot{a}_2 + \tilde{C}_{20}a_2\dot{a}_1)\,, \tag{2.18}$$

$$\tilde{C}_{H2} = -M(A_{20}a_1\dot{a}_1 + B_{20}a_2\dot{a}_2 + C_{20}a_2\dot{a}_1 + \tilde{C}_{20}a_1\dot{a}_2)\,,$$

$$D_{H2} = D_{20}a_1 + E_{20}a_2 + M[2(A_{20}a_1\Delta_1 + B_{20}a_2\Delta_2) + (C_{20} + \tilde{C}_{20})(a_1\Delta_2 + a_2\Delta_1)]\,,$$

$$E_{H2} = -M(D_{20}\dot{a}_1 + E_{20}\dot{a}_2) - M^2[2(A_{20}\dot{a}_1\Delta_1 + B_{20}\dot{a}_2\Delta_2) + (C_{20} + \tilde{C}_{20})(\dot{a}_1\Delta_2 + \dot{a}_2\Delta_1)]\,,$$

$$F_{H2} = M^2[A_{20}\Delta_1^2 + B_{20}\Delta_2^2 + (C_{20} + \tilde{C}_{20})\Delta_1\Delta_2] + M(D_{20}\Delta_1 + E_{20}\Delta_2) + F_{20}\,,$$

where

$$a_1 = e^{-b(t)} = \eta(t)/\eta(0)\,,\ a_2 = -2\hbar^{-1}S(t)\,a_1(t)\,,$$

$$\Delta_i \equiv \Delta_i(t) = \dot a_i(t)\,\xi(t) - a_i(t)\dot\xi(t)\,,\ i = 1, 2\,.$$

The functions a_1, a_2, and $\Delta_{1,2}$ satisfy the following initial conditions $a_1(t_0)=1$, $\dot a_1(t_0)=0$, $a_2(t_0)=0$, $M(t_0)\,\dot a_2(t_0)=-1$, and $\Delta_1(t_0)=\Delta_2(t_0)=0$.

1) Let $A_{10}=il/\sqrt{2}\hbar$, $B_{10}=1/\sqrt{2}l$, and $C_{10}=0$, where $m=M(0)$ and $\omega_0=\omega(0)$, and $l=(\hbar/m\omega_0)^{1/2}$ is the amplitude of the zero oscillations of the oscillator. In this case, the linear invariant (2.16) is written in the form

$$I_{H1}(t) = a(t) - \frac{i}{\sqrt{2\hbar}}\int_{t_0}^{t} F(t')\,\varepsilon_H(t')\,dt'\,, \tag{2.19}$$

where the operator

$$a(t) = \frac{i}{\sqrt{2\hbar}}\left[\varepsilon_H(t)\hat p - M(t)\dot\varepsilon_H(t)\hat x\right] \tag{2.20}$$

and its Hermitian conjugate $a^+(t)$ are annihilation and creation operators for the nonstationary oscillator given by Eq. (1.11) for $F(t)=0$, i.e., $[a, a^+]=1$. Here the function ε_H has the form

$$\varepsilon_H(t) = (m\omega_0)^{-1/2}[1 + 2i\hbar^{-1}m\omega_0 S(t)]\,a_1(t) = (m\omega_0)^{-1/2}(a_1 - im\omega_0 a_2)\,. \tag{2.21}$$

It satisfies Eq. (1.17) and the condition $M(\dot\varepsilon_H\varepsilon_H^* - \varepsilon_H\dot\varepsilon_H^*) = 2i$. For $M(t)=m=\text{const}$ and $\omega(t)=\omega_0=\text{const}$, Eq. (2.16) yields the linear invariant for the stationary oscillator with a driving force obtained in [12]:

$$A(t) = \frac{e^{i\omega_0(t-t_0)}}{\sqrt{2}}\left(\frac{\hat x}{l} + \frac{il}{\hbar}\hat p\right) - \frac{il}{\sqrt{2}\hbar}\int_{t_0}^{t} F(t')e^{i\omega_0(t'-t_0)}dt'\,. \tag{2.22}$$

If we set $M(t)=e^{2\Gamma(t)}$ and $F(t)=f(t)e^{2\Gamma(t)}$ in Eq. (2.19), we obtain a linear invariant for the Caldirola–Kanai oscillator with the driving force that was obtained in [12]. Following [11, 15, 24], we set $\varepsilon_H=\rho(t)\exp[i\gamma(t)]$, where

$$\rho(t) = |\varepsilon_H(t)| = (m\omega_0)^{-1}\sqrt{a_1^2 + m^2\omega_0^2 a_2^2} \ \text{ and } \ \gamma(t) = \int_{t_0}^{t}\frac{dt'}{M(t')\rho^2(t')}\,.$$

Here the function $\rho(t)$ will be the solution of the nonlinear equation

$$\ddot\rho + \frac{\dot M}{M}\dot\rho + \omega^2(t)\rho = \frac{1}{M^2\rho^3}\,. \tag{2.23}$$

Let us express the invariants $a(t)$ and $a^+(t)$ in terms of ρ:

$$a(t)=\frac{1}{\sqrt{2\hbar}}\left[\frac{\hat{x}}{\rho}+i(\rho\hat{p}-M\dot{\rho}\hat{x})\right]e^{i\gamma},\ a^{+}(t)=\frac{1}{\sqrt{2\hbar}}\left[\frac{\hat{x}}{\rho}-i(\rho\hat{p}-M\dot{\rho}\hat{x})\right]e^{-i\gamma}. \qquad (2.24)$$

Operators analogous to the two represented in Eqs. (2.24) were obtained in [13, 15, 24] by another technique, but they, in contrast to operators (2.24), are not linear invariants of a nonstationary oscillator. With the help of operator (2.19) and its Hermitian conjugate, we can construct the quadratic invariant

$$I_{H2}(t)=I_{H1}^{+}I_{H1}+1/2=\frac{1}{2\hbar}\left[\frac{\hat{x}^2}{\rho^2}+(\rho\hat{p}-M\dot{\rho}\hat{x})^2\right]-\frac{\hat{x}}{\hbar\rho}\int_{t_0}^{t}F(t')\rho'\sin(\gamma-\gamma')dt'$$

$$-\frac{1}{\hbar}(\rho\hat{p}-M\dot{\rho}\hat{x})\int_{t_0}^{t}F(t')\rho'\cos(\gamma-\gamma')dt'+\frac{1}{2\hbar}\int_{t_0}^{t}F(t')F(t'')\rho'\rho''e^{i(\gamma'-\gamma'')}dt'dt'', \qquad (2.25)$$

where $\rho\equiv\rho(t),\ \rho'\equiv\rho(t')$, etc. This invariant is a generalization of the Lewis–Riesenfeld invariant [11] to the case of a variable mass and a variable force. Its particular case when $F(t)=0$ was obtained in [15] by another technique. Invariant (2.25) is a particular case of quadratic invariant (2.17), corresponding to the following values of the coefficients assigned in Eqs. (2.18):

$$A_{20}=(2m\omega_0\hbar)^{-1},\ C_{20}=\tilde{C}_{20}=0,\ D_{20}=0,\ E_{20}=0\ \text{and}\ F_{20}=0.$$

2) Let $M(t)=me^{2\lambda t}$ and $\omega(t)=\omega_0=\text{const}$, which corresponds to the Caldirola–Kanai oscillator (Eqs. (2.26)). Note that in this case we have

$$a_1=\frac{\eta(t)}{\omega}=\frac{Q(t)}{\omega}e^{-\lambda\tau},\quad a_2=-\frac{\sin\omega\tau}{m\,\omega}e^{-\lambda(t+t_0)}. \qquad (2.26)$$

For the choice of coefficients (expressions (2.18)) in the form $A_{20}=1/2m$, $B_{20}=m\omega_0^2/2$, $C_{20}=\tilde{C}_{20}=\lambda/2$, $D_{20}=E_{20}=F_{20}=0$, and $t_0=0$, the quadratic invariant obtained in [14], namely

$$E=H+\frac{\lambda}{2}(\hat{p}\hat{x}+\hat{x}\hat{p}), \qquad (2.27)$$

follows from expression (2.17).

3) If we choose the coefficients in expressions (2.18) in the following way: $A_{20}=\omega_0^2/m^2$, $B_{20}=(\omega_0^2+\lambda^2)^2$, $C_{20}=\tilde{C}_{20}=0$, $D_{20}=0,\ E_{20}=0$, and $F_{20}=0$, we obtain the linear invariant for a parametric oscillator (Eq. (1.22)), which coincides with the result from [14]:

$$E=(\omega_0^2+\lambda^2\tanh^2\lambda t)\frac{\hat{p}^2}{m^2}+\frac{\lambda^3\sinh\lambda t}{m\cosh^3\lambda t}(\hat{p}\hat{x}+\hat{x}\hat{p})+\frac{\lambda^6\sinh^2\lambda t+\omega_0^2(\lambda^2+\omega_0^2)^2\cosh^6\lambda t}{\cosh^6\lambda t(\omega_0^2+\lambda^2\tanh^2\lambda t)}\hat{x}^2. \qquad (2.28)$$

3. UNITARY CONNECTION

As has already been stated, a quantum particle with variable mass in a variable homogeneous field is unitarily equivalent to a free quantum particle with variable mass, and a nonstationary harmonic oscillator with a driving force is unitarily equivalent to a free quantum particle with a variable, in-some-way renormalized mass. According to formulas (1.10) and (1.31), the indicated unitary connections between systems in the first case are realized by the operator V_L (Eqs. (1.5)), and in the second case, by the operator V_{FH} assigned in the first of Eqs. (1.32). Note also that the operator $V_{FH}^{(0)}$ assigned in the second of Eqs. (1.32) realizes the unitary connection between nonstationary harmonic oscillators for $F(t)=0$ and for $F(t)\neq 0$. In order to write out explicitly these connections between these invariants and between the wave functions of the indicated systems, let us first consider the operators

$$\hat{x}_1(t)=V_L\hat{x}V_L^{-1}=\hat{x}-S_1(t) \text{ and } \hat{p}_1(t)=V_L\hat{p}V_L^{-1}=\hat{p}-\delta(t), \tag{3.1}$$

and also the operators

$$\hat{x}_2(t)=V_{FH}\hat{x}V_{FH}^{-1}=\frac{\hat{x}-\xi(t)}{a_1(t)} \text{ and } \hat{p}_2(t)=V_{FH}\hat{p}V_{FH}^{-1}=a_1(t)\hat{p}-M(t)\dot{a}_1(t)\hat{x}+M(t)\Delta_1(t). \tag{3.2}$$

Recall that $\xi(t)$ is the solution of Eq. (1.15). Moreover, we introduce the notation $I_{nF}(t)\equiv I_{nF}(\hat{p},\hat{x},t)$ for an *n*th order invariant of a free particle. Then the unknown unitary connections between the invariants can be written in the form

$$I_{nL}(t)=V_L I_{nF}(\hat{p},\hat{x},t)V_L^{-1}=I_{nF}(\hat{p}_1,\hat{x}_1,t), \tag{3.3}$$

$$I_{nH}(t)\equiv V_{FH}I_{nF}^{\mathrm{Re}\,n}(\hat{p},\hat{x},t)V_{FH}^{-1}=I_{nF}^{\mathrm{Re}\,n}(\hat{p}_2,\hat{x}_2,t), \tag{3.4}$$

and the unitary connections between the wave functions, in the form

$$\psi_L(x,t)=V_L\psi_F(x,t)=e^{i\varphi_0(x,t)}\psi_F(x_1,t), \tag{3.5}$$

$$\psi_H(x,t)=V_{FH}\psi_F^{\mathrm{Re}\,n}(x,t)=e^{i\phi_0+2^{-1}b}e^{i(\nu_1x_2+\nu_2x_2^2)}\psi_F^{\mathrm{Re}\,n}(x_2,t), \tag{3.6}$$

where expressions of the form $I_{nF}^{\mathrm{Re}\,n}(\hat{p},\hat{x},t)$ and $\psi_F^{\mathrm{Re}\,n}(x,t)$, etc. are obtained from the corresponding expressions $I_{nF}(\hat{p},\hat{x},t)$ and $\psi_{nF}(x,t)$, etc. for a free particle by replacing $M(t)$ by $M^{\mathrm{Re}\,n}(t)$ and $S_2(t)$ by $\hbar^{-1}S(t)$, and the equalities:

$$x_1=x-S_1(t),\ p_1=p-\delta(t),$$

$$x_2=[x-\xi(t)]/a_1(t),\ p_2=a_1(t)p-M(t)\dot{a}_1(t)x+M(t)\Delta_1(t),$$

$$\phi_0=\hbar^{-1}\sigma_H+\Lambda_0,\ \nu_1=\hbar^{-1}M(t)a_1(t)\dot{\xi}(t),\quad \nu_2=\alpha(t)a_1^2(t). \tag{3.7}$$

Formulas (3.5) and (3.6) allow us to obtain, in turn, relations between the Wigner functions of the considered systems. To obtain these relations, we base ourselves on the fact that the Wigner function is a functional that depends on the wave function

$$W(p,x,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\psi^*\left(x-\frac{x'}{2},\,t\right)\psi\left(x+\frac{x'}{2},t\right)e^{-\frac{ipx'}{\hbar}}\,dx' . \tag{3.8}$$

As a result, we find

$$W_L(p,x,t) = e^{-\delta\partial_p}e^{-S_1\partial_x}W_F(p,x,t) = W_F(p_1,x_1,t)\,, \tag{3.9}$$

$$W_H(p,x,t) = e^{-bp\,\partial_p}e^{M(\Delta_1-\dot{a}_1x)\partial_p}e^{-\xi\partial_x}e^{-bx\partial_x}W_F^{\mathrm{Re}\,n}(p,x,t) = W_F^{\mathrm{Re}\,n}(p_2,x_2,t)\,. \tag{3.10}$$

Note that relations (3.5) and (3.9) were obtained in [2].

We present one example of the application of formulas (3.6) and (3.10). Toward this end, let us consider the plane wave [2]

$$\psi_{p_0}^F(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left\{\frac{i}{\hbar}\left[p_0x - p_0^2S_2(t)\right]\right\} \tag{3.11}$$

and the Wigner function

$$W_{Ai}^F(x,t) = 2^{-1/3}(\pi\hbar)^{-1}Ai(h(p,x,t))\,, \tag{3.12}$$

describing the motion of a free quantum particle with variable mass, where $Ai(x)$ is the Airy function, and $h(p,x,t) = \sqrt[3]{4}\left(Bx + b_0 - 2pBS_2(t) + p^2/\hbar^2B^2\right)$. Thus, the corresponding wave function of plane wave type and the corresponding Wigner function for a nonstationary harmonic oscillator with a driving force (Eq. (2.11)) will be given by the expressions

$$\psi_{p_0}^H(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left\{i\phi_0 + 2^{-1}b + i(\nu_1x_2 + \nu_2x_2^2) + \frac{i}{\hbar}\left[p_0x_2 - \hbar^{-1}p_0^2S(t)\right]\right\}, \tag{3.13}$$

$$W_{Ai}^H(p,x,t) = 2^{-1/3}(\pi\hbar)^{-1}Ai\left(h^{\mathrm{Re}\,n}(p_2,x_2,t)\right). \tag{3.14}$$

Naturally, Wigner function (3.12) satisfies the evolution equation for a free particle

$$\frac{\partial W}{\partial t} + \frac{p}{M(t)}\frac{\partial W}{\partial x} = 0\,, \tag{3.15}$$

and Wigner function (3.14) satisfies the evolution equation for a harmonic oscillator with a driving force

$$\frac{\partial W}{\partial t} + \frac{p}{M(t)}\frac{\partial W}{\partial x} - \left[M(t)\omega^2(t)x - F(t)\right]\frac{\partial W}{\partial p} = 0\,. \tag{3.16}$$

CONCLUSIONS

In this paper we have applied the evolution operator method to describe the properties of such simple nonstationary quadratic quantum systems as a free particle with variable mass, a particle with variable mass in a variable homogeneous field, and a nonstationary harmonic oscillator with variable driving force. Here, basing ourselves on the explicit, chronologically disentangled form of the evolution operators for these systems, we first constructed basis invariants, and then, with their help, constructed linear and quadratic invariants for an arbitrary time-variation law of the external parameters of the Hamiltonians.

Knowledge of the evolution operators allowed us to easily set up a unitary connection between the considered quadratic quantum systems. The given connection enabled us to obtain invariants, wave functions, and other functions, both for the particle in a homogeneous field and for a harmonic oscillator with a driving force from the corresponding expressions for a free particle with variable mass. By way of an example, using the evolution operator method we found the propagator in the *p*-representation for a nonstationary harmonic oscillator, acted on by a variable force. The evolution operator in the given case is equal to

$$U_H(p,t) = U_1(p,t)U_H^{(0)}(p,t),$$

where $U_H^{(0)}(p,t)$ is the evolution operator of a nonstationary oscillator $(F(t)=0)$

$$U_H^{(0)}(p,t) = e^{-\frac{1}{2}b(t)} e^{-i\alpha(t)\hbar^2\partial_p^2} e^{-i\hbar^{-2}S(t)e^{-2b(t)}p^2} e^{-b(t)p\partial_p},$$

and the operator U_1 is given by the formula

$$U_1(p,t) = e^{\frac{i}{\hbar}[\sigma_H(t)-p\xi(t)]} e^{-M(t)\dot{\xi}(t)\partial_p}.$$

Thus, we obtain

$$K_H(p_2,t;p_1,t_0) = \theta(t-t_0)\sqrt{\frac{i}{4\pi\mu_1(t)}}\exp\left\{i\left[\alpha_1(t)p_2^2 + \beta_1(t)p_2p_1 + \gamma_1(t)p_1^2\right]\right\}$$

$$= \sqrt{\frac{i\exp[b(t)]}{4\pi\hbar^2\alpha(t)}} e^{\frac{i}{\hbar}[\sigma(t)-\xi(t)p_2]} \exp\left\{-\frac{i}{4\hbar^2\alpha(t)}\left[p_2 - p_1e^{b(t)} - M(t)\dot{\xi}(t)\right]^2 - \frac{i}{\hbar^2}S(t)p_1^2\right\}.$$

This expression contains within itself the propagators of a free quantum particle with variable mass and a quantum particle with variable mass in a variable homogeneous field.

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