DOI 10.1007/s11182-018-1434-2

Russian Physics Journal, Vol. 61, No. 3, July, 2018 (Russian Original No. 3, March, 2018)

NONCOMMUTATIVE REDUCTION OF THE BLOCH EQUATION IN THE HEISENBERG–WEYL GROUP

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UDC 536.77, 517.986.66

The Bloch equation in the Heisenberg–Weyl group is considered. A λ-representation of the Lie algebra of a Heisenberg–Weyl group of arbitrary dimensionality is constructed, and an expression for the statistical sum in the Heisenberg–Weyl group is obtained. Expressions for the statistical sum of the Heisenberg–Weyl group and other thermodynamic quantities are analyzed.

Keywords: noncommutative integration method, thermodynamics of homogeneous spaces, Heisenberg–Weyl group, statistical sum.

INTRODUCTION

In quantum field theory in an arbitrary gravitational background the heat kernel (density matrix) of the wave equation operator of the given theory is an important means. Knowledge of the heat kernel allows one, for example, to investigate single-loop divergences, which contribute to the statistical sum (trace of the density matrix). The heat kernel also carries information about the propagator and anomalies of the given quantum theory [1]. Problems that arise in the construction of the heat kernel for a number of field theories are considered in [2–4]. In these works, the heat kernel is constructed with the help of the eigenfunctions of the wave equation operator. However, there is another way to construct this object – namely by solving the Bloch equation, which contains within itself the wave equation operator of the given quantum theory.

On the other hand, the heat kernel is useful in a study of the geometry of manifolds. In this case, it is defined by the eigenfunctions of the Laplace–Beltrami operator of the given manifold. As is well known, there exists a connection between the invariants of a compact manifold and the statistical sum, which is defined by the heat kernel [5, 6]. In the case of noncompact manifolds, the connection with the geometry is more complex, but nevertheless exists (see, for example, [7]).

In this paper, we consider applying the noncommutative integration technique to reduce the Bloch equation and search for a statistical sum in the instance of a Heisenberg–Weyl group of arbitrary dimensionality. In [8] Micheev and Shirokov showed that in the case of noncompact manifolds it is easily possible with the help of noncommutative integration to separate the integration over the infinite volume of the manifold and transform to the statistical sum per unit volume. Note that Micheev and Shirokov [9] and Calin *et al*. [10] consider a particular case of the given problem for the 3-dimensional Heisenberg–Weyl group.

1. HARMONIC ANALYSIS IN THE HEISENBERG–WEYL GROUP

Let G_n be a $(2n+1)$-dimensional Heisenberg–Weyl group, and let W_n be its Lie algebra. The commutation relations of W_n in the basis $\{e_A\} = \{e_i, t_\alpha, E\}$ have the form

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Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 3, pp. 143–150, March, 2018. Original article submitted December 12, 2017.

$$[e_i, t_\alpha] = \delta_{i,(\alpha-n)} E, \;\; [e_i, E] = 0, \;\; [t_\alpha, E] = 0,$$

where $A = 1, ..., 2n+1$, $i = 1, ..., n$, $\alpha = n+1, ..., 2n$, and δ_{AB} is the Kronecker delta symbol. We choose canonical coordinates of the second kind in this group:

$$g(x, y, z) = \exp(zE) \prod_{\alpha=n+1}^{2n} \exp(y^\alpha t_\alpha) \prod_{i=1}^{n} \exp(x^i e_i).$$

The left-invariant fields ξ and the Maurer–Cartan one-forms dual to them, ω, and also the right-invariant fields η and the Maurer–Cartan one-forms dual to them, σ, can be found with the help of a procedure described in [11]:

$$\xi_i = \partial_{x^i}, \;\; \xi_\alpha = \partial_{y^\alpha} + x^{\alpha-n}\partial_z, \;\; \xi_{2n+1} = \partial_z; \;\; \omega^i = dx^i, \;\; \omega^\alpha = dy^\alpha, \;\; \omega^{2n+1} = -\sum_{\alpha=n+1}^{2n} x^{\alpha-n} dy^\alpha + dz,$$
$$\eta_i = -\partial_{x^i} - y^{i+n}\partial_z, \;\; \eta_\alpha = -\partial_{y^\alpha}, \;\; \eta_{2n+1} = -\partial_z; \;\; \sigma^i = -dx^i, \;\; \sigma^\alpha = -dy^\alpha, \;\; \sigma^{2n+1} = \sum_{i=1}^{n} y^{n+i} dx^i - dz. \tag{1}$$

Let W_n^* be the space that is dual to W_n (its coalgebra). The group G_n acts in the coalgebra via the co-adjoint representation, which is defined by the rule $\left\langle \mathrm{Ad}_g^* f, X \right\rangle = \left\langle f, \mathrm{Ad}_{g^{-1}} X \right\rangle$, where $\langle . , . \rangle$ is the action of the covector on the vector, $f \in W_n^*$, $X \in W_n$, and Ad_g is the adjoint representation of the group. The co-adjoint action of the group exfoliates the coalgebra into orbits of the co-adjoint representation (*K*-orbits).

Let C_{IJ}^K be the structure constants of the Lie algebra W_n, where $I, J, K = 1, ..., 2n+1$. The rank of the matrix $C_{IJ}(f) = C_{IJ}^K f_K$, where f_K are the coordinates of the covector relative to the dual basis $\{e^K\}$, is constant on a *K*-orbit [12] and defines subspaces in W_n^* which are invariant with respect to co-adjoint action. In what follows, we will retain the notation introduced in [12].

Below, we will have need of a description of *K*-orbits of the Heisenberg–Weyl group. Each nondegenerate *K*-orbit (orbit of maximum dimensionality) passes through a parameterized covector $\lambda(j) = (0, ..., 0, \pm j)$, where $j > 0$, and is defined by the value of the Casimir function $K^{(\pm)}(f) = \pm f_{2n+1}$. All nondegenerate *K*-orbits can be distinguished by the conditions

$$O_{\lambda(j)}^{(\pm)} = \{f \in M_{(\pm)} \mid K^{(\pm)}(f) = \pm j\}, \quad M_{(\pm)} = \{f \in W_n^* \mid \pm f_{2n+1} > 0\}, \quad \dim O_{\lambda(j)}^{(\pm)} = 2n,$$

where $M_{(\pm)}$ are subspaces that are invariant with respect to the co-adjoint action of the group and consist of orbits of dimensionality $2n$. Each orbit represents a $2n$-dimensional Euclidean space.

In turn, the degenerate *K*-orbits pass through the covector $\lambda(j) = (j_1, ..., j_{2n}, 0)$ and are defined by the value of the Casimir functions $K_P^{(n)}(f) = f_P$, where $P = 1, ..., 2n$. Any degenerate *K*-orbit can be assigned by the condition

$$O_{\lambda(j)}^{(n)} = \{f \in M_{(n)} \mid K_1^{(n)}(f) = j_1, ..., K_{2n}^{(n)}(f) = j_{2n}\}, \quad M_{(n)} = \{f \in W_n^* \mid f_{2n+1} = 0\}, \quad \dim O_{\lambda(j)}^{(n)} = 0,$$

where $M_{(n)}$ is a subspace that is invariant with respect to the co-adjoint action of the group and consists of orbits of dimensionality 0. We will not consider degenerate orbits further.

Each orbit is a symplectic manifold [12]. On nondegenerate K-orbits, let us consider a canonical transformation to Darbu coordinates of the following form: $f_A(p,q,\lambda(j)) = p_a\alpha_A^a(q) + \chi_A(q,\lambda(j))$, where a = 1, ..., n. For such a transformation to exist, it is necessary and sufficient that a polarization exist for the covector $\lambda(j) = (0,...,0,\pm j)$. The subalgebra $H \subset W_n$ is called a polarization of the covector λ if $\langle \lambda, [H,H] \rangle = 0$. The subalgebra $H = \mathrm{span}\{t_{n+1}, ..., t_{2n}, E\}$ is a polarization of the covector $\lambda(j) = (0, ..., 0, \pm j)$, the supplementary subalgebra has the form $\mathrm{span}\{e_1, ..., e_n\}$. Here $\mathrm{span}\{e_1, ..., e_n\}$ is a vector space spanning the vectors $e_1, ..., e_n$. The corresponding transformation to canonical variables, linear in the parameters p, is given by the relations

$$f_i(p,q,\lambda(j)) = p_i,\ f_\alpha(p,q,\lambda(j)) = \pm jq^{\alpha-n},\ f_{2n+1}(p,q,\lambda(j)) = \pm j;\ q = (q^1, \dots, q^n) \in Q = \mathbb{R}^n. \tag{2}$$

The special representation of the Lie algebra in the space $L_2(Q, d\mu(q))$, whose action is defined by the operators $l_A(q,\lambda(j)) = \frac{i}{\hbar} f_A(-i\hbar\partial_q, q, \lambda(j))$, is called the λ-representation. The explicit form of the operators of the λ-representation follows from relations (2):

$$l_i(q,\lambda(j)) = \partial_{q^i},\ l_\alpha(q,\lambda(j)) = \pm\frac{i}{\hbar} jq^{\alpha-n},\ l_{2n+1}(q,\lambda(j)) = \pm\frac{i}{\hbar} j. \tag{3}$$

Let us consider the space $L_2(Q, d^nq)$, where $d^nq = dq^1 \cdot \ldots \cdot dq^n$, with scalar product $(\varphi,\psi) = \int_Q \overline{\varphi(q)}\psi(q)d^nq,\ \varphi,\psi \in L_2(Q,d^nq)$. The operators defined by Eqs. (4) are symmetric with respect to the given scalar product.

We now introduce the raising (boost) of the λ-representation of the Lie algebra to the λ-representation of the Lie group by the operators T^λ:

$$\frac{d}{dt}(T^\lambda(\exp tX)\varphi)(q)\bigg|_{t=0} = l_X(q,\lambda)\varphi(q), (T^\lambda(g)\varphi)(q) = \int_Q D^\lambda_{qq'}(g)\varphi(q')d^nq', \tag{4}$$

where $\varphi \in L_2(Q,d^nq)$. The family of generalized functions $D^\lambda_{qq'}(g)$, defined in Eqs. (4), obey the system of equations

$$[\eta_A(g) + l_A(q,\lambda)]D^\lambda_{qq'}(g) = 0,\ [\xi_A(g) - \overline{l_A^\dagger(q',\lambda)}]D^\lambda_{qq'}(g) = 0, \tag{5}$$

with initial conditions $D^\lambda_{qq'}(e) = \delta(q,q')$, where $\delta(q,q')$ is the delta function in Q. Solution of Eqs. (5), using Eqs. (1) and (3), gives

$$D^{\lambda(\pm)}_{qq'}(g) = \exp\left(\mp\frac{ij}{\hbar}\left[z + \sum_{\alpha=n+1}^{2n} y^\alpha q'^{\alpha-n}\right]\right)\delta(x+q'-q),\ D^{\lambda(-)}_{qq'}(g) = \overline{D^{\lambda(+)}_{qq'}(g)}, \tag{6}$$

where we have introduced the notation $\delta(x+q'-q) = \prod_{i=1}^n \delta(x^i + q'^i - q^i)$. Since $D^{\lambda(-)}_{qq'}(g)$ changes over to $D^{\lambda(+)}_{qq'}(g)$ as a result of replacing j by $-j$, it is possible to alter the domain of definition $j \in (0,\infty) \to j \in (-\infty,0) \cup (0,\infty)$ and set $D^\lambda_{qq'}(g) \equiv D^{\lambda(+)}_{qq'}(g)$ for all j except for j = 0, and set $D^\lambda_{qq'}(g) \equiv 0$ for

$j = 0$. The functions $D^{\lambda}_{qq'}(g)$, as can be seen from Eqs. (6), are globally defined for $z \in \left(0, 2\pi\right)$ and $j \in \mathrm{Z} \setminus \{0\}$, which corresponds to the case of the compact subgroup $\exp(zE)$. Moreover, $D^{\lambda}_{qq'}(g)$ are globally defined for $z \in \mathbb{R}$ and $j \in \mathbb{R} \setminus \{0\}$. The last case corresponds to the case of the noncompact subgroup $\exp(zE)$.

The family of functions $D^{\lambda}_{qq'}(g)$ satisfies the orthogonality and completeness conditions

$$\int_G \overline{D^{\tilde{\lambda}}_{\tilde{q}\tilde{q}'}(g)} D^{\lambda}_{qq'}(g) d\mu(g) = \delta(q, \tilde{q}) \delta(q', \tilde{q}') \delta(\lambda, \tilde{\lambda}), \tag{7}$$

$$\int_{Q \times Q \times J} \overline{D^{\lambda}_{qq'}(\tilde{g})} D^{\lambda}_{qq'}(g) d^n q d^n q' d\mu(\lambda) = \delta(g, \tilde{g}), \tag{8}$$

where J is a set of values of the parameters j, $\delta(\lambda, \tilde{\lambda})$ is the generalized delta-function in J, $d\mu(\lambda)$ is a measure on J, and $d\mu(g) = dx^1 \ldots dx^n dy^{n+1} \ldots dy^{2n} dz$ is the two-sided invariant Haar measure on G_n. Relation (7) can be taken as the definition of the delta function $\delta(\lambda, \tilde{\lambda})$. Thus we obtain in the case of the noncompact subgroup $\exp(zE)$

$$\delta(\lambda, \tilde{\lambda}) = \frac{(2\pi\hbar)^{n+1}}{j^n} \delta(j - \tilde{j}), \ \int_J (\bullet) d\mu(\lambda) = \int_{-\infty}^{\infty} (\bullet) \frac{j^n dj}{(2\pi\hbar)^{n+1}},$$

where $\delta(j - \tilde{j})$ is the delta function in $\mathbb{R}^n$, and the integration over J is taken from $-\infty$ from $+\infty$. This is possible due to the fact that the point 0 is the measure of the null set and its inclusion in the integration interval does not change the value of the integral.

In the case of the compact subgroup $\exp(zE)$ the parameter $j \in \mathrm{Z} \setminus \{0\}$ and the integration over j is replaced by the sum

$$\delta(\lambda, \tilde{\lambda}) = \frac{(2\pi\hbar)^{n+1}}{j^n} \delta_{j\tilde{j}}, \ \int_J (\bullet) d\mu(\lambda) = \sum_{j \in \mathrm{Z} \setminus \{0\}} (\bullet) \frac{j^n}{(2\pi\hbar)^{n+1}},$$

where $\delta_{j\tilde{j}}$ is the Kronecker delta symbol. By virtue of properties (7) and (8) for functions from $L_2(G, d\mu(g))$ with scalar product $(\varphi, \psi) = \int_G \overline{\varphi(g)} \psi(g) d\mu(g)$, the generalized Fourier transform is defined as follows:

$$\hat{\varphi}^{\lambda}(q, q') = \int_G \varphi(g) \overline{D^{\lambda}_{qq'}(g)} d\mu(g), \varphi(g) = \int_{Q \times Q \times J} \hat{\varphi}^{\lambda}(q, q') D^{\lambda}_{qq'}(g) d^n q d^n q' d\mu(\lambda). \tag{9}$$

We write equality (9) explicitly as

$$\hat{\varphi}^{\lambda}(q, q') = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \varphi(q - q', y, z) \exp\left(\frac{ij}{\hbar}\left[z + \sum_{\alpha=n+1}^{2n} y^{\alpha} q'^{\alpha-n}\right]\right) \prod_{\alpha=n+1}^{2n} dy^{\alpha} dz,$$

$$\varphi(x, y, z) = \int_{Q \times J} \hat{\varphi}^{\lambda}(q, q - x) \exp\left(-\frac{ij}{\hbar}\left[z + \sum_{\alpha=n+1}^{2n} y^{\alpha} (q^{\alpha-n} - x^{\alpha-n})\right]\right) d^n q d\mu(\lambda),$$

where $\varphi(x, y, z)$ is the coordinate notation for the function $\varphi(g)$. Note that the usual Fourier transform is realized over the variables z and j. Transformation (9) converts the action of the operators $\xi_A(g)$ and $\eta_A(g)$ into the action of the operators $\overline{l_A(q,\lambda)}$ and $l_A(q',\lambda)$, respectively.

As was shown in [13], the functions $D^{\lambda}_{qq'}(g)$ can be represented in the form

$$D^{\lambda}_{qq'}(g) = \frac{1}{(2\pi)^n} \int_P \exp\left(\sum_{i=1}^{n} p_i q^i{}' - S^{\lambda}(g, q, p) \right) dp,$$

where $dp = dp_1 \cdot \ldots \cdot dp_n$ and $P = \mathbb{R}^n$. In our case, we write $S(g, q, p)$ as

$$S^{\lambda}(x, y, z; q, p') = \sum_{i=1}^{n} (q^i + x^i) p'^i + \sum_{\alpha=n+1}^{2n} j q^{\alpha-n} y^{\alpha} + \frac{j}{\hbar} z, \tag{10}$$

$$dS^{\lambda}(x, y, z; q, p') = \pi_i dx^i + \pi_{\alpha} dy^{\alpha} + \pi_{2n+1} dz - p_i(x,\pi) dq^i - q'^i(x,\pi) dp'_i + \tau(x,\pi) dj.$$

The given function defines a special canonical transformation preserving the symplectic form $\Omega = d\pi_A \wedge dg^A = d\pi_i \wedge dx^i + d\pi_{\alpha} \wedge dy^{\alpha} + d\pi_{2n+1} \wedge dz = dp_i \wedge dq^i + dq'^i \wedge dp'_i + dj \wedge d\tau$ in the space $T^* G_n$. The canonical transformation is defined implicitly by the relations

$$p_i(x,\pi) = -\pi_i - \pi_{2n+1} y^{i+n}, \quad p'_i(x,\pi) = \pi_i, \; j(x,\pi) = \pi_{2n+1},$$

$$q^i(x,\pi) = \frac{\pi_{i+n}}{\pi_{2n+1}}, \; q'^i(x,\pi) = -\frac{\pi_{i+n}}{\pi_{2n+1}} - x^i, \tau(x,\pi) = -\sum_{\alpha=n+1}^{2n} \frac{\pi_{\alpha}}{\pi_{2n+1}} y^{\alpha} - z.$$

2. THERMODYNAMICS IN THE HEISENBERG–WEYL GROUP

Let us now turn to the solution of the fundamental problem of the thermodynamics of homogeneous spaces on the Heisenberg–Weyl group. As is well known [5], the statistical sum is the trace of the density matrix

$$Z(\beta) = \int \rho_{\beta}(g, g) d\mu(g), \tag{11}$$

where $\rho_{\beta}(g, g')$ is the density matrix, which is defined by the Bloch equation

$$\frac{\partial \rho_{\beta}(g, g')}{\partial \beta} + H \rho_{\beta}(g, g') = 0, \; \rho_{\beta}(g, g')|_{\beta=0} = \delta(g, g'), \tag{12}$$

where H is the Hamiltonian and $\delta(g, g')$ is the delta function on the group G. By virtue of the non-compactness of the Heisenberg–Weyl group, the integral in Eq. (11) will diverge, but in [8] Mikheev and Shirokov showed that with the help of the noncommutative integration method it is possible to separate the volume of the group and find the statistical sum per unit volume.

Let us consider the operator H, which is a quadratic function of left-invariant vector fields:

$$H(-i\hbar\eta) = -\hbar^2 G^{AB} \eta_A \eta_B.$$

Here $A, B = 1, ..., 2n+1$, $G^{AB} = \mathrm{diag}(A_1^2, ..., A_n^2, B_1^2, ..., B_n^2, C_1 + ... + C_n)$, and $A_i, B_i, C_i > 0$. The matrix G^{AB} is chosen in this form since any matrix in the Heisenberg–Weyl algebra can be reduced to the given matrix by internal automorphisms. Taking the explicit form of G^{AB} into account, it is possible to calculate the scalar curvature in the Heisenberg–Weyl group, which has the form $R = \left(A_1^2 B_1^2 + ... + A_n^2 B_n^2\right) / \left(2C_1 + ... + 2C_n\right)$. Integration of Eq. (12), as was shown in [8], reduces with the help of the method of orbits to integration of the equation

$$\frac{\partial R_\beta(q, q', \lambda)}{\partial \beta} + H(-i\hbar l(q', \lambda)) R_\beta(q, q', \lambda) = 0, \;\; R_\beta(q, q', \lambda)|_{\beta=0} = \delta(q, q'). \tag{13}$$

Thus, the statistical sum per unit volume of the group is defined as follows:

$$z(\beta) = \int_{Q\times J} R_\beta(q, q, \lambda) d^n q d\mu(\lambda). \tag{14}$$

We write Eq. (14) in explicit form:

$$\frac{\partial R_\beta(q, q', \lambda)}{\partial \beta} + \sum_{i=1}^{n} \left(-\hbar^2 A_i^2 \frac{\partial^2}{\partial q'^{i2}} + B_i^2 j^2 q'^{i2} + C_i j^2 \right) R_\beta(q, q', \lambda) = 0, \; R_\beta(q, q', \lambda)\Big|_{\beta=0} = \delta(q, q').$$

We shall seek the solution of the given equation in the form $R_\beta(q, q', \lambda) = \prod_{i=1}^{n} \exp(-C_i j^2 \beta) r_\beta^i(q^i, q'^i, \lambda)$, where each of the functions depends only on the *i*th variable. Thus, the variables in the equation are separated, and for the function $r_\beta^i(q^i, q'^i, \lambda)$, after the substitutions $q'^i = \hbar A_i \sqrt{2} x'^i$ and $q^i = \hbar A_i \sqrt{2} x^i$, we have an equation for the heat kernel of a one-dimensional harmonic oscillator:

$$\frac{\partial r_\beta^i(x^i, x'^i, \lambda)}{\partial \beta} + \left(-\frac{1}{2} \frac{\partial^2}{\partial x'^{i2}} + \frac{1}{2} a_i^2 x'^{i2} \right) r_\beta^i(x^i, x'^i, \lambda) = 0, \tag{15}$$

where $a_i^2 = 2\hbar^2 A_i^2 B_i^2 j^2$. Note that the dynamical symmetry group of the harmonic oscillator is a Heisenberg–Weyl group. For the heat kernel (Eq. (15)) the solution is well known [10]:

$$r_\beta^i(x^i, x'^i, \lambda) = \sqrt{\frac{a_i}{2\pi \sinh(a_i \beta)}} \exp\left(-\frac{a_i}{2\sinh(a_i \beta)} \left[(x^{i2} + x'^{i2}) \cosh(a_i \beta) - 2x^i x'^i \right] \right). \tag{16}$$

Thus, taking the introduced notation into account, the expression for the statistical sum of the heat kernel (given by Eq. (16)) takes the form

$$z_{ho}^1(\beta) = \frac{\exp(-C_i j^2 \beta)}{2\sinh(\hbar A_i B_i j \beta)}. \tag{17}$$

Taking Eq. (17) and the expressions for the spectral measure into account, we obtain an expression for the statistical sum in the Heisenberg–Weyl group. In the case of the noncompact subgroup $\exp(zE)$ we have

$$z_n(\beta) = \frac{1}{2^{n-1}(2\pi\hbar)^{n+1}} \int_0^{\infty} \exp\left(-\left(\sum_{i=1}^{n} C_i j^2 \beta\right)\right) j^n \left(\prod_{i=1}^{n} \sinh(\alpha_i j\hbar\beta)\right)^{-1} dj\,, \tag{18}$$

where $\alpha_i = A_i B_i$.

The integral entering into the expression for the statistical sum converges. Indeed, for arbitrary values of the parameters $j^n \left(\prod_{i=1}^{n} \sinh(\alpha_i j\hbar\beta)\right)^{-1} \leq \left(\prod_{i=1}^{n} \alpha_i j\hbar\beta\right)^{-1}$ the function $\exp\left(-\sum_{i=1}^{n} C_i j^2 \beta\right)$ is integrable on the given interval over j; consequently, the integral converges according to Abel's test. Moreover, for $n = 1$ it is possible to obtain the dependence of the statistical sum on the scalar curvature. Indeed, after making the substitution $j = y/\sqrt{2C}$ in Eq. (18), for $n = 1$ we have

$$z_1(\beta) = \frac{1}{2C(2\pi\hbar)^2} \int_0^{\infty} \exp\left(-(y^2\beta)/2\right) y^n \left(\sinh\left(\sqrt{R}\, y\hbar\beta\right)\right)^{-1}.$$

Similarly, for the case of the compact subgroup $\exp(zE)$ we have

$$z_n(\beta) = \frac{1}{2^{n-1}(2\pi\hbar)^{n+1}} \sum_{j=1}^{\infty} \exp\left(-\left(\sum_{i=1}^{n} C_i j^2 \beta\right)\right) j^n \left(\prod_{i=1}^{n} \sinh(\alpha_i j\hbar\beta)\right)^{-1}.$$

Applying Abel's test for the convergence of series, it is possible to shown that the given series converges. For $n = 1$ expression (18) takes the form

$$z_1(\beta) = \frac{1}{(2\pi\hbar)^2} \int_0^{\infty} \exp\left(-(Cj^2\beta)\right) j \left(\sinh(\alpha j\hbar\beta)\right)^{-1} dj\,. \tag{19}$$

In this case, it is possible in a simple way to find the high-temperature expansion of the statistical sum ($\beta \to 0$); specifically, we make the substitution $j^2\beta = x$ in the integral in Eq. (19). Then the integral in Eq. (19) transforms to

$$z_1(\beta) = \frac{1}{(2\pi\hbar\beta)^2} \int_0^{\infty} \exp\left(-x^2(C/\beta)\right) x \left(\sinh(\alpha\hbar x)\right)^{-1} dx\,. \tag{20}$$

Since the expansion

$$x(\sinh(\alpha\hbar x))^{-1} = \left(\frac{1}{\alpha\hbar} - \sum_{n=1}^{\infty} \frac{2(1-2^{2n-1})B_{2n}(\alpha\hbar)^{2n-1}}{(2n)!} x^{2n}\right) \tag{21}$$

is valid, where B_n are the Bernoulli numbers, and the integral in Eq. (20) converges, Watson's lemma is applicable to Eq. (20) [14]. Using the given lemma, we obtain the high-temperature expansion

$$z_1(\beta) = \frac{1}{(2\pi\beta\hbar)^{3/2}} \left(\frac{\sqrt{2}}{4\alpha\hbar\sqrt{C}} - \sum_{n=1}^{\infty} \frac{\sqrt{2}}{2} \frac{(1-2^{2n-1})B_{2n}(\alpha\hbar)^{2n-1}}{4^n\, n!\, C^{n+1/2}} \beta^n\right).$$

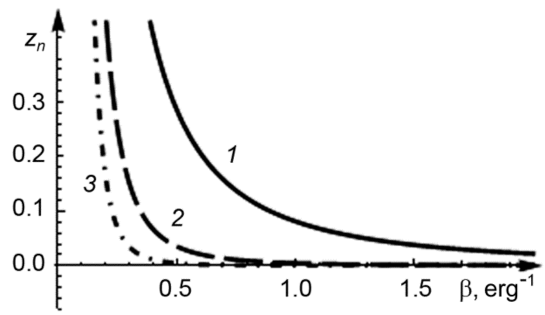

Fig. 1

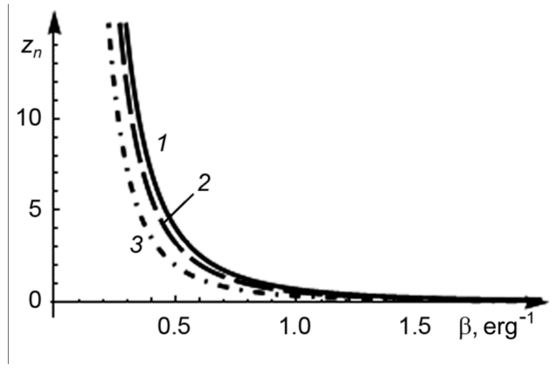

Fig. 2

Fig. 1. Dependence of the statistical sum z_n on the parameter β for identical values of the parameters A_i, B_i, and C_i: curve *1* corresponds to $n = 1$, curve *2* corresponds to $n = 2$, and curve *3* corresponds to $n = 3$.

Fig. 2. Dependence of the statistical sum z_n on the parameter β for different values of the parameter B_2: curve *1* corresponds to $B^2{}_2 = 1$, curve *2* corresponds to $B^2{}_2 = 1.25$, and curve *3* corresponds to $B^2{}_2 = 2$.

The statistical sum, as can be seen from Eq. (18), is a monotonic function of the variable β for all admissible values of the parameters. As $\beta\to\infty$, $z_n(\beta)\to 0$; as $\beta\to 0$, $z_n(\beta)\to\infty$, and only the velocity depends on the values of the parameters, and in such a way that $z_n(\beta)$ falls off at infinity.

Let us consider particular cases. Figure 1 plots graphs of the statistical sum for different values of n and identical values of the parameters A_i, B_i, and C_i. It can be seen that with growth of n the function $z_n(\beta)\to 0$ as $\beta\to\infty$ faster than the preceding function. Figure 2 displays the behavior of z_n as a function of the parameter B_2 for $n = 2$, $C_1 = 0.5$, $C_2 = 0.5$, $A^2{}_1 = 0.25$, $B^2{}_1 = 0.5$, and $A^2{}_2 = 0.25$.

Besides the statistical sum, there are other characteristics of the given manifold, for example, the average kinetic energy per particle and the specific heat, which are given by the formulas

$$u_n(T) = k_{\mathrm{B}} T^2 \frac{\partial \ln z_n(T)}{\partial T}, \quad c_v(T) = \frac{\partial u_n(T)}{\partial T},$$

where k_{B} is the Boltzmann constant and $T = 1/(k_{\mathrm{B}}\beta)$. These two quantities can be calculated explicitly in the case of the compact subgroup:

$$u_n(T) = \frac{\pi^2 F}{12} k_{\mathrm{B}}^2 T^2, \quad c_v(T) = \frac{\pi^2 F}{6} k_{\mathrm{B}}^2 T.$$

Here we have introduced the notation $F = \sum_{i=1}^{n} \frac{1}{\alpha_i}$. By virtue of the fact that the parameters A_i and B_i are positive, the average kinetic energy per particle and the specific heat are always increasing functions of the variable T and tend to infinity as $T\to\infty$, and the choice of parameters affects only their rate of growth.

Let us consider differences arising as a function of our choice of the subgroup $\exp(zE)$. Figure 3 plots graphs of the statistical sum for the case of the compact and the noncompact subgroup $\exp(zE)$ for identical values of the

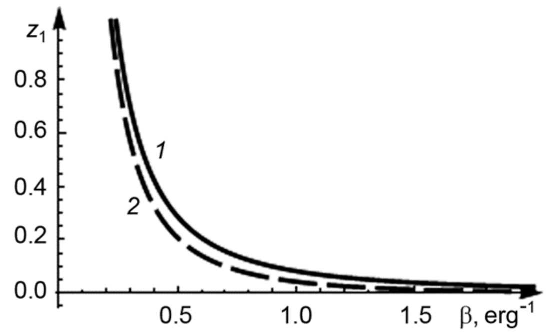

Fig. 3

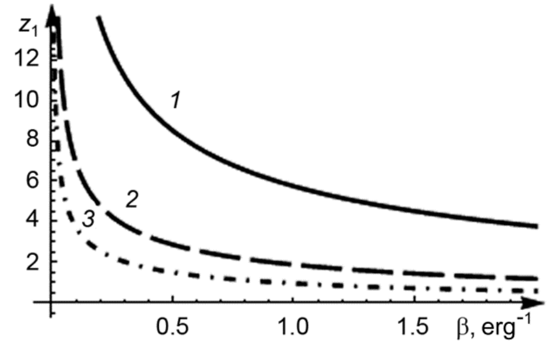

Fig. 4

Fig. 3. Behavior of the statistical sum z_1: curve *1* corresponds to the compact case, and curve *2* corresponds to the noncompact case.

Fig. 4. The statistical sum plotted as a function of the scalar curvature: curve *1* corresponds to $\beta = 0.1$, curve *2* corresponds to $\beta = 0.2$, and curve *3* corresponds to $\beta = 0.3$.

parameters. It can be seen that in the case of the compact subgroup, the statistical sum tends to zero faster with growth of β. Figure 4 plots graphs of the dependence of the statistical sum for $n = 1$ on the scalar curvature for different values of β with the parameter C fixed.

CONCLUSIONS

In this work, we have classified the orbits of the co-adjoint representation of the Heisenberg–Weyl group and obtained expressions for the operators of λ-representation of the Heisenberg–Weyl algebra and the kernel of the λ-representation of the Heisenberg–Weyl group. Knowledge of the operators of the λ-representation has allowed us to carry out a noncommutative reduction of linear differential equations with Heisenberg–Weyl symmetry groups of arbitrary dimensionality. We have shown that in the case of an arbitrary right-invariant metric on the $(2n+1)$-dimensional Heisenberg–Weyl group, the Bloch equation reduces to the Bloch equation of an n-dimensional harmonic oscillator. We obtained an expression for the statistical sum on the Heisenberg–Weyl group, and also the high-temperature expansion for the case $n = 1$. Analysis of these expressions for the statistical sum shows that $z_{n+1}(\beta)$ tends to zero faster than $z_n(\beta)$ with growth of β (see Fig. 1). In addition, we considered some particular cases of the dependence of the statistical sum on the parameters of the metric (see Fig. 2). It can be seen that the larger the parameter B, the faster the statistical sum tends to zero. A similar dependence also obtains for the parameters A.

We also considered the connection between the statistical sum and the topology of the Heisenberg–Weyl group. It can be seen (Fig. 3) that in the case of the compact subgroup $\exp(zE)$, the statistical sum tends to zero faster than in the opposite case. It can be seen from Fig. 4 that the statistical sum is a monotonically decreasing function of the scalar curvature of the Heisenberg–Weyl group. It is correspondingly not hard to observe that with growth of β, the statistical sum as a function of the scalar curvature falls to zero faster. We have obtained statistical quantities on the Heisenberg–Weyl group (the average kinetic energy per particle and the specific heat). The given quantities differ from the corresponding quantities for an ideal gas. This is due to the nontrivial nature of the topology of the considered group manifold.

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