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CHARACTERISTICS OF A DEGENERATE NEUTRON GAS IN A MAGNETIC FIELD WITH ALLOWANCE FOR THE ANOMALOUS MAGNETIC MOMENT OF THE NEUTRON

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General expressions for the dependence of the Fermi energy, pressure, and total energy of a degenerate neutron gas in a magnetic field on the magnitude of the field and the neutron concentration with allowance for the anomalous magnetic moment of the neutron have been obtained in implicit form, and the dependence of these quantities on the field is presented in graphical form for the neutron concentration $C = 10^{38}$ cm^{-3}, which is typical for neutron stars. Analytical estimates of the pressure have been made for the magnitude of the fields possible in neutron stars ~10^{17}–10^{19} G and this neutron concentration ~10^{38} cm^{-3}, including when the neutron gas is close to its saturated state with preferred orientation of the anomalous magnetic moment of all the neutrons in alignment with the field. It is found that even such fields ~10^{17} G have practically no effect on the pressure in comparison with the case when the field is absent, an effect being possible only for $B \sim 10^{18}$–10^{19} G. The analytical dependence on the neutron concentration of the corresponding field B_S at which the neutron gas transitions to the saturated state has been found in explicit form. It is established that for $B > B_S$ the indicated characteristics of the neutron gas, and likewise its state, no longer change.

Keywords: neutron, magnetic field, anomalous magnetic moment, Fermi energy, pressure.

The question of the properties of a neutron gas in a magnetic field is very topical in connection with the existence of magnetars – a special kind of neutron star possessing superstrong surface magnetic fields up to 10^{15} G [1], and in their interior $\sim 10^{17}$ G (and possibly higher – up to 10^{19} G), which can radically influence the properties and evolution of these astrophysical objects, including, for example, their equilibrium radius [2].

In this regard, a calculation of such typically quantum characteristics of a degenerate neutron gas as its Fermi energy, pressure, etc., with allowance for the interaction of the anomalous magnetic moment (AMM) of the neutrons with the magnetic field, can be of fundamental importance, even though the AMM of the neutron p_N is usually neglected due to its smallness, having the order of magnitude of the nuclear magneton $p_n = \dfrac{e\hbar}{2Mc}$, and having the value $p_N = \sigma_N p_n$ where $\sigma_N \approx -1.9$. By M here and below, we understand the mass of the neutron since for our estimates it is possible to assume that $M_P \approx M_N \equiv M$; however, in a superstrong magnetic field $B \sim 10^{17}$–10^{19} G, the corresponding energy

$$E_B = \xi \varepsilon_B, \; \varepsilon_B = |p_N| B, \tag{1}$$

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which depends on the orientation of the neutron spin: $\xi = \pm 1$ (the value $\xi = 1$ corresponds to the AMM aligned opposite the field, and the spin, correspondingly, aligned with the field, and $\xi = -1$ corresponds to the reverse situation) can, in principle, be quite large and comparable, for example, to the Fermi energy without this interaction taken into account.

These aspects of the problem have already been partly touched upon in the literature. Thus, Avancini *et al.* [3] examined the influence of a magnetic field on the quark component of the interior of a neutron star and on the QCD phase diagrams, Landers and Jones [4] examined its effect on the structure of a neutron star, Chiu *et al.* [5] took into account the contribution of the interaction of the AMM of the electron with a magnetic field, and Strickland *et al.* [6] examined the influence of the magnetic field induced anisotropy of space.

In the present paper we have extended the standard methods of quantum statistics laid out in the classical work by Landau and Lifshitz [7] to the case of the presence of an AMM on neutral particles with spin 1/2 (neutrons), which enabled us to arrive at conclusions which are more transparent in comparison to the above-mentioned works, regarding the influence of the magnetic field on the characteristics of a degenerate neutron gas in a magnetic field, with a graphical illustration of their dependence on the magnitude of the field.

To calculate the Fermi energy in this situation, we write the usual expression of Fermi statistics for the concentration of the Fermi gas (formula (56.5) in [7]), replacing the spin statistical weight g by the sum $\sum\limits_{\xi=\pm 1}$ over orientations of the AMM, and the kinetic energy ε in the argument of the exponential of this formula by the total energy (see formula (3) below):

$$C = \frac{M^{3/2}}{\sqrt{2}\pi^2\hbar^3} \sum_{\xi=\pm 1} \int_0^\infty \frac{\sqrt{\varepsilon}\, d\varepsilon}{\exp\left[\dfrac{E-\mu}{T}\right]+1}\,, \quad \varepsilon = \frac{p^2}{2M}\,, \tag{2}$$

where $\varepsilon = \dfrac{p^2}{2M}$ is the kinetic energy of the neutron and

$$E = \varepsilon + E_B \tag{3}$$

is the total energy, including the above-mentioned energy E_B (Eq. (1)) of the interaction with the magnetic field.

In the case of a degenerate gas, i.e., in the limit $T \to 0$, $\mu \to \mu_{\mathrm{F}}$, expression (2) takes the form

$$C = \frac{M^{3/2}}{\sqrt{2}\pi^2\hbar^3} \sum_{\xi=\pm 1} \int_0^{\varepsilon_{\mathrm{F}\xi}} \sqrt{\varepsilon}\, d\varepsilon = \frac{\sqrt{2}M^{3/2}}{3\pi^2\hbar^3} \sum_{\xi=\pm 1} {\varepsilon_{\mathrm{F}\xi}}^{3/2}\,, \tag{4}$$

where

$$\varepsilon_{\mathrm{F}\xi} = \mu_{\mathrm{F}} - \xi\varepsilon_B \tag{5}$$

is the kinetic energy at the Fermi level for given ξ, and μ_{F} is the value of the chemical potential at $T = 0$, i.e., the Fermi energy, being in our case a function of *B* and *C*: $\mu_{\mathrm{F}} \equiv \mu_{\mathrm{F}}(B, C)$. The form of this function with Eqs. (4) and (5) taken into account can, in principle, be found from the equation

$$C = \frac{\sqrt{2}M^{3/2}}{3\pi^2\hbar^3}\left[(\mu_{\mathrm{F}} + \varepsilon_B)^{3/2} + (\mu_{\mathrm{F}} - \varepsilon_B)^{3/2}\right], \tag{6}$$

in which it is useful, for convenience, to transform to dimensionless variables, writing it in the form

$$\frac{3\pi^2}{\sqrt{2}}\tilde{C} = \left(\tilde{\mu} + \frac{|\sigma_N|}{2}\tilde{B}\right)^{3/2} + \left(\tilde{\mu} - \frac{|\sigma_N|}{2}\tilde{B}\right)^{3/2} . \tag{6a}$$

The dimensionless variables introduced here are defined as follows:

$$\tilde{\mu} = \frac{\mu_F}{Mc^2},\ \tilde{B} = \frac{B}{B_0},\ \tilde{C} = C\lambda_C{}^3 , \tag{7}$$

$$\lambda_C = \frac{\hbar}{Mc} \approx 2.1\cdot 10^{-14}\ \text{cm},\ B_0 = \frac{M^2c^3}{e\hbar} \approx 1.5\cdot 10^{20}\ \text{G}. \tag{7a}$$

In what follows, we use a typical value of the neutron concentration in a neutron star [8]:

$$C \approx 10^{38}\,\text{cm}^{-3} . \tag{8a}$$

The corresponding dimensionless parameter is equal to

$$\tilde{C} \approx 9.3\cdot 10^{-4} . \tag{8b}$$

Taking the above-stated value of σ_N into account, we thus write Eq. (6a) in the following form, convenient for numerical calculations:

$$2.0\cdot 10^{-2} \approx (\tilde{\mu} + 0.95\times \tilde{B})^{3/2} + (\tilde{\mu} - 0.95\times \tilde{B})^{3/2} , \tag{8c}$$

which determines in implicit form the value of $\tilde{\mu}$ in the function of the dimensionless parameter $\tilde{B}$ at a given concentration: $\tilde{\mu} \equiv \tilde{\mu}(\tilde{B}) \to \tilde{\mu}_{\tilde{C}}(\tilde{B})$ (i.e., according to the definitions given by Eqs. (7)) and the function $\mu_F \to \mu_{FC}(B)$.

The dependence $\tilde{\mu}(\tilde{B})$ is represented in graphical form in Fig. 1. Some important conclusions can be drawn without detailed numerical calculations, namely: the last term in Eq. (8c) describes the contribution of the neutrons with their AMM aligned against the field and has physical meaning only for $\tilde{\mu} \geq 0.95 \times \tilde{B}$, and the value $\tilde{\mu} = 0.95 \times \tilde{B}$, with the notation in this particular case $\tilde{\mu} \to \tilde{\mu}_{\tilde{C}S}$, $\tilde{B} \to \tilde{B}_S$, corresponds to the saturated state for given C (see Eqs. (8a) and (8b)), when the AMMs of all the neutrons are aligned with the field; in this latter case, the first term on the right-hand side of Eq. (8c) is equal to $(2\tilde{\mu}_{\tilde{C}S})^{3/2}$ or, what is the same thing, $(1.9\tilde{B}_S)^{3/2}$.

Thus, it is not hard to obtain from Eq. (8c) the result

$$\tilde{\mu}_{\tilde{C}S} \approx 3.7\cdot 10^{-2} ,\ \tilde{B}_S \approx 3.8\cdot 10^{-2} . \tag{9}$$

This value of $\tilde{B}_S$ corresponds to the field $B_S = \tilde{B}_S \times B_0 \approx 5.7\cdot 10^{18}$ G, which is very close to possible values in the interior of neutron stars and in principle is not ruled out. These values (equalities (9)) coincide approximately with the result of numerical calculation (the *bottom* point of the graph in Fig. 1).

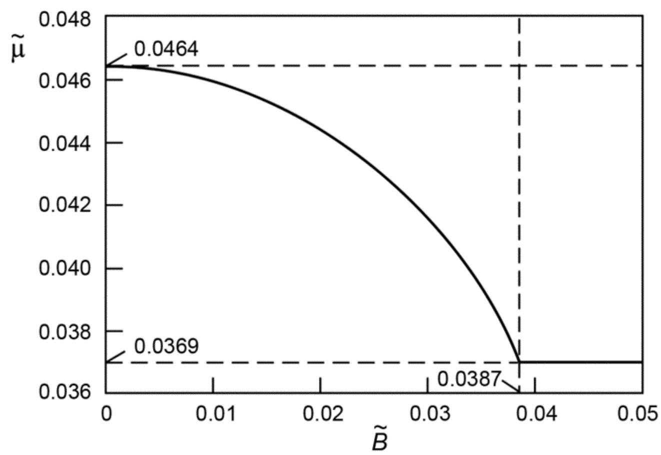

Fig. 1. Dependence of the dimensionless Fermi energy $\tilde{\mu}$ (Eq. (7)) of a degenerate neutron gas on the dimensionless field $\tilde{B}$ (Eq. (7)) according to formula (8c) for a typical neutron concentration $C \sim 10^{38}$ cm^{-3} in neutron stars. The horizontal line in the graph is *drawn by hand* and corresponds to the saturated state.

On the other hand, the value $\tilde{\mu} \equiv \tilde{\mu}_0$ in the absence of a field can be found from Eq. (6a) after first setting $\tilde{B} = 0$, upon which the right-hand side of Eq. (6a) is now equal to $2\tilde{\mu}^{3/2} \to 2\tilde{\mu}_0{}^{3/2}$. Thus, it is easy to obtain the result

$$\tilde{\mu}_{\tilde{C}S} = \frac{\tilde{\mu}_0}{2^{1/3}} \approx 0.8\tilde{\mu}_0 \,. \tag{9a}$$

Taking the value of $\tilde{\mu}_{\tilde{C}S}$ given by Eq. (9) into account, this is likewise in agreement with the value $\tilde{\mu} \to \tilde{\mu}_0$ at the *upper* point of this graph. Since we thus have $\tilde{\mu}_{\tilde{C}S} < \tilde{\mu}_0$, the Fermi energy at a given concentration is a monotonically decreasing function of the field in the interval $0 - B_S$, as is clear from the result of numerical calculation.

For $B > B_S$ the Fermi energy and other characteristics of the system of neutrons (including, for example, the pressure and the magnitude of the intrinsic field B' (see below), which is negligibly small in comparison with the field B) no longer vary since the state of the system with the AMMs of all its neutrons aligned with the field does not vary. The total pressure in a neutron star will grow with growth of B only on account of the pressure of the magnetic field (in this regard, see Skobelev [9]).

The dimensionless kinetic energy $\tilde{\varepsilon}_{F\xi} = \varepsilon_{F\xi} / Mc^2$ at the Fermi level in the saturated state at $\xi = -1$, as given by Eq. (5), taking the value $\tilde{\varepsilon}_B \approx 0.95 \times \tilde{B}_S \approx 3.8 \cdot 10^{-2}$ into account, is equal to $\tilde{\varepsilon}_{F\xi} \approx 2\, \tilde{\mu}_{\tilde{C}S} \approx 7.4 \cdot 10^{-2}$, and for $\xi = 1$ is of course equal to zero since in this state neutrons with their AMM aligned against the field are generally absent. This value of $\tilde{\varepsilon}_{F\xi}$ is in line with the nonrelativistic character of the approximation, according to which the inequalities $\tilde{\varepsilon}_{F\xi}$, $\tilde{\mu}_{\tilde{C}S} << 1$ should be fulfilled.

Note also that for the magnitude of the field in the interior of those neutron stars known as magnetars [1] $B_{\max} \equiv B_{17} \sim 10^{17}$ G (the value most often figuring in the estimates) and correspondingly $\tilde{B}_{\max} \equiv \tilde{B}_{17} \approx 6.6 \cdot 10^{-4}$, the value of $\tilde{\mu}$, equal to $\tilde{\mu} \equiv \tilde{\mu}_{17} \approx 4.6 \cdot 10^{-6}$, as is clear from the graph in Fig. 1, is practically the same as its value $\tilde{\mu}_0$ in the absence of a field.

We next write out an expression for the Ω-potential, for which in formula (56.6) of [7] it is necessary to make the same substitutions as those made in the foregoing case to obtain formula (2). This gives

$$\Omega = -\frac{2}{3}\frac{M^{3/2}}{\sqrt{2}\pi^2\hbar^3} V \sum_{\xi=\pm 1}\int_0^\infty \frac{\varepsilon^{3/2} d\varepsilon}{\exp\left[\frac{E-\mu}{T}\right]+1}. \quad (10)$$

Next, starting out from the well-known thermodynamic relation $\Omega = - PV$, it is now possible to find the pressure P of a degenerate neutron gas. Specifically, using formula (10), we first write the general formula for the pressure, analogous to formula (2):

$$P = \frac{\sqrt{2}M^{3/2}}{3\pi^2\hbar^3} \sum_{\xi=\pm 1}\int_0^\infty \frac{\varepsilon^{3/2} d\varepsilon}{\exp\left[\frac{E-\mu}{T}\right]+1}. \quad (11)$$

For a degenerate gas we hence obtain an expression that is analogous to formula (6):

$$P = \frac{2^{3/2}M^{3/2}}{15\pi^2\hbar^3} [(\mu_F + \varepsilon_B)^{5/2} + (\mu_F - \varepsilon_B)^{5/2}], \quad (12)$$

and in dimensionless variables[1]

$$\tilde{P} = \frac{2^{3/2}}{15\pi^2}\left[\left(\tilde{\mu} + \frac{|\sigma_N|}{2}\tilde{B}\right)^{5/2} + \left(\tilde{\mu} - \frac{|\sigma_N|}{2}\tilde{B}\right)^{5/2}\right], \quad (12a)$$

$$\tilde{P} = \frac{P}{P_0}, \; P_0 = \frac{Mc^2}{\lambda_C{}^3} \approx 1.5\cdot 10^{38} \text{ dyn/cm}^2. \quad (12b)$$

Hence we find the dimensionless pressure in the saturated state, employing the notation introduced earlier:

$$\tilde{P}_{\tilde{C}S} = \frac{2^{3/2}}{15\ \pi^2}(2\tilde{\mu}_{\tilde{C}S})^{5/2}, \quad (13a)$$

and the dimensionless pressure in the absence of a magnetic field

$$\tilde{P}\Big|_{B=0} = \frac{2^{3/2}}{15\pi^2}\ 2(\tilde{\mu}_0)^{5/2}. \quad (13b)$$

Thus we obtain

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[1] To avoid misunderstandings, we remark that in the paper by Skobelev [9] which analyzed the influence of inhomogeneity of the field on the magnitude of the neutron pressure, the quantity P_0 (Eq. (12b) in [9]) was scaled to the electron.

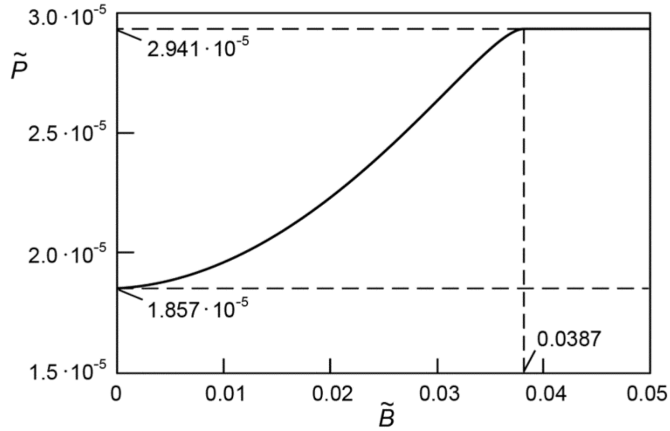

Fig. 2. Dependence of the dimensionless pressure $\tilde{P}$ (Eqs. (12a) and (12b)) of a degenerate neutron gas on the dimensionless field $\tilde{B}$ (Eq. (7)) according to formulas (8c) and (12a) for the neutron concentration $C \sim 10^{38}$ cm^{-3} that is typical in neutron stars. The horizontal line in the graph *was drawn by hand* and corresponds to the saturated state.

$$\frac{\tilde{P}_{\tilde{C}S}}{\tilde{P}\big|_{B=0}} = 2^{3/2}\left(\frac{\tilde{\mu}_{\tilde{C}S}}{\tilde{\mu}_0}\right)^{5/2}, \tag{14}$$

and, taking formula (9a) into account, we find

$$\frac{\tilde{P}_{\tilde{C}S}}{\tilde{P}\big|_{B=0}} = 2^{2/3}. \tag{15}$$

This value of the ratio essentially coincides with the numerically calculated dependence $\tilde{P}(\tilde{B})$ (Fig. 2).

One more important conclusion follows from this: the pressure of a degenerate neutron gas grows with increase of the strength of the field, reaching its maximum value

$$\tilde{P}_{\max} \equiv \tilde{P}_{\tilde{C}S} = 2^{2/3}\ \tilde{P}\big|_{B=0} \approx 1.6\ \tilde{P}\big|_{B=0} \tag{16}$$

in the saturated state for $\tilde{B} \equiv \tilde{B}_S$, so that in this state the pressure has increased by roughly a factor of 1.5. It is possible to arrive at the same conclusion from the form of the graph in Fig. 2.

Note also that the general expression for the dependence $\tilde{B}_S(\tilde{C})$ follows from Eq. (6a) under the condition that the second term on the right-hand side of Eq. (6a) vanish and has the form

$$\tilde{B}_S = \frac{1}{|\sigma_N|}\left(\frac{3\pi^2}{\sqrt{2}}\right)^{2/3}\tilde{C}^{2/3} \approx 4\ \tilde{C}^{2/3}, \tag{17a}$$

including the previous value obtained for $C \approx 10^{38}\,\mathrm{cm}^{-3}$ ($\tilde{C} \approx 9.3\cdot 10^{-4}$) as a particular case with the obvious dependence $B_S(C)$ in our notation:

$$B_S = \left(\frac{3\pi^2}{\sqrt{2}}\right)^{2/3} \frac{\hbar^2}{Me} C^{2/3} \approx 3.8 \frac{\hbar^2}{Me} C^{2/3} . \tag{17b}$$

The field values given by Eqs. (17a) and (17b), on the other hand, can also be interpreted as the minimum possible values for the given concentration at which the neutron gas is found in the saturated state.

Similar to the value $\tilde{\mu}_{17}$, the value $\tilde{P}_{17} \approx 1.8574 \cdot 10^{-5}$ as can be seen from the graph in Fig. 2, is practically the same as the value of the dimensionless pressure $\tilde{P}\big|_{B=0} \approx 1.8570 \cdot 10^{-5}$ in the absence of a field. In other words, the generally accepted maximum value of the magnetic field $\sim 10^{17}$ G [1] in the interior of a neutron star has essentially no effect on the pressure of the neutron gas in comparison with the case of zero field, as was established above for the Fermi energy. However, for admissible field values $B \sim 10^{19}$ G in the interior of a neutron star the effect of an increase in the pressure or the Fermi energy, according to Eqs. (15), (16), and (9a), respectively, can be very significant, and this, like the influence of inhomogeneity of the field [9], must be taken into account, generally speaking, in theoretical models of neutron stars.

Note also that the formula for the energy U of a neutron gas in a magnetic field, analogous to formula (56.7) in [7], is

$$U = \frac{M^{3/2}}{\sqrt{2}\pi^2\hbar^3} V \sum_{\xi=\pm 1} \int_0^\infty \frac{\varepsilon^{3/2} d\varepsilon}{\exp\left[\frac{E-\mu}{T}\right]+1} \tag{18}$$

(in [7] this energy is denoted as E, but we use this symbol for the total energy of a neutron (Eq. (3)). Comparing expressions (11) and (18), we obtain the relation $PV = \frac{2}{3}U$ which coincides with formula (56.8) in [7] with the indicated change of notation, and the value of the dimensionless energy $\tilde{U} = U / Mc^2$ of a degenerate neutron gas following from Eq. (12a) and this relation:

$$\tilde{U} = \frac{\sqrt{2}}{5\pi^2} (V\lambda_C{}^{-3}) \left[\left(\tilde{\mu} + \frac{|\sigma_N|}{2} \tilde{B} \right)^{5/2} + \left(\tilde{\mu} - \frac{|\sigma_N|}{2} \tilde{B} \right)^{5/2} \right]. \tag{19}$$

As can be seen from formulas (12a) and (19), the dependence $\tilde{U}(\tilde{B})$ does not fundamentally differ from the dependence $\tilde{P}(\tilde{B})$ in Fig. 2.

Here we have not taken into account the intrinsic field of the neutrons B' equal to the total magnetic moment of the neutrons per unit volume of the gas and showing up when their AMMs are aligned with the field. For example, in the saturated state it is obviously equal to $B' = |p_N| C$, or, transforming to dimensionless quantities,

$$\tilde{B}' = \frac{|\sigma_N|}{2} \alpha \tilde{C} , \ \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137} . \tag{20}$$

In particular, for the standard value of the concentration (Eqs. (8a) and (8b)) we obtain the following estimate for the intrinsic field: $\tilde{B}' \approx 6.4 \cdot 10^{-6}$. This is many orders of magnitude less than the field value (Eq. (9)) needed for a degenerate neutron gas to transition to the saturated state. In this we note a radical difference from, say, ferromagnetism, B' being more similar in magnitude to the phenomenon of paramagnetism, this latter fact being explained by the smallness of the AMM of the neutron in comparison with the spin magnetic moment of the electron.

Thus, neglecting the intrinsic field of a neutron gas in the saturated state for $\tilde{B} = \tilde{B}_S$ (or $\tilde{B} > \tilde{B}_S$) is entirely justified, and values of $\tilde{B} < \tilde{B}_S$, as has been made clear, are generally of no importance since they have hardly any effect on the Fermi energy and pressure.

The theoretical fact, established here, of the insignificant effect of the magnetic field of magnetars with $B < (\sim)\, 10^{17}$ G on the Fermi energy and pressure of the degenerate neutron gas in their interiors is the main practical conclusion of this study. However, fields of 10^{17}–10^{19} G, comparable to B_S and entirely possible in the interior of neutron stars, can have a substantial effect.

To summarize, in this paper we have delineated to a significant extent the elements of the quantum statistics of a degenerate neutral Fermi gas with spin 1/2 in a magnetic field with allowance for the anomalous magnetic moment of its component particles (specifically, these can be neutrons), augmenting the results formulated in other works [3–6].

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