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ONE-DIMENSIONAL FOKKER–PLANCK EQUATION WITH QUADRATICALLY NONLINEAR QUASILOCAL DRIFT

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The Fokker–Planck equation in one-dimensional spacetime with quadratically nonlinear nonlocal drift in the quasilocal approximation is reduced with the help of scaling of the coordinates and time to a partial differential equation with a third derivative in the spatial variable. Determining equations for the symmetries of the reduced equation are derived and the Lie symmetries are found. A group invariant solution having the form of a traveling wave is found. Within the framework of Adomian's iterative method, the first iterations of an approximate solution of the Cauchy problem are obtained. Two illustrative examples of exact solutions are found.

Keywords: nonlinear Fokker–Planck equation, quasilocal approximation, Lie symmetries, traveling waves, Adomian decomposition method, exact solutions.

INTRODUCTION

In the nonequilibrium statistical mechanics of complex systems, phenomenological approaches have been developed, based on nonlinear generalizations of the linear Fokker–Planck (FP) equation, well known in the theory of Brownian motion [1], and describing ensembles of interacting particles (subsystems) with different potential interactions. The particles of an ensemble are found in a surrounding medium and experience random actions from the chaotically moving particles of the surrounding medium (thermostat).

A wide range of works have been dedicated to nonlinear FP equations (for example, see [2–6] and the literature cited therein). In the one-dimensional case, the nonlinear FP equation describing the evolution in time t of the one-particle distribution function $p(x,t)$ of the variable *x*, characterizing the state of an individual particle (subsystem) of the ensemble, can be written in the form [2]

$$\partial_t p(x,t) = -\partial_x D_1(x,t,p)p(x,t) + \partial_{xx} D_2(x,t,p)p(x,t) . \tag{1}$$

Here we have made use of the notation $\partial_t = \partial / \partial t$, $\partial_x = \partial / \partial x$, and $\partial_{xx} = \partial^2 / \partial x^2$ for partial derivatives; $D_1(x,t,p)$ and $D_2(x,t,p)$ are the drift and diffusion coefficients, respectively, and they depend on p. The nonlinearity in Eq. (1) reflects the influence on an individual particle (subsystem) of the remaining ensembles (subsystems) in the mean field approximation [2, 3]. In the case of long-range action in the ensemble of particles of the system, the nonlinear FP equation becomes nonlocal. As an example, we cite the nonlocal FP equation for systems with self-organization [2, 3]

$$\partial_t p(x,t) = -\partial_x D_1(x,t,m)p(x,t) + \partial_{xx} D_2(x,t,m)p(x,t),$$

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with coefficients D_1 and D_2 which depend on the mean value of x, $m=\langle x\rangle$, over the distribution $p(x,t)$. In the more general case, the coefficients D_1 and D_2 depend on the weighted mean

$$m = m(x,t,p) = \int_{-\infty}^{\infty} b(x,x')p(x',t)dx' \tag{2}$$

with weight function $b(x,x')$. All of the functions encountered here are assumed to be smooth, so that integrals of the type appearing in Eq. (2) exist.

In [7, 8] for the one-dimensional FP equation with quadratic nonlocal drift

$$\partial_t p(x,t) = -\partial_x \left[k_1(t) + k_2(t)m(x,t,p)\right] p(x,t) + D\partial_{xx} p(x,t)\,, \tag{3}$$

where $k_1(t)$ and $k_2(t)$ are prescribed smooth functions of time t and D is a constant diffusion coefficient, approximate solutions were constructed in the semiclassical approximation in the small parameter D for the function $b(x,x')$ of general form.

In the present work, we consider the problem of integrating Eq. (3) in the quasilocal approximation, i.e., under the condition that the function $b(x,x')$ has the form

$$b(x,x') = \frac{1}{2\varepsilon}\begin{cases} 1, & x' \in [x-\varepsilon, x+\varepsilon], \\ 0, & x' \notin [x-\varepsilon, x+\varepsilon], \end{cases} \tag{4}$$

where ε is a small parameter characterizing the size of the localization region of the interaction. The quasilocal approximation defined by the function assigned in Eq. (4) represents in simplified form the long-range actions among the particles of the considered ensemble as the average action on an individual particle, located at the point x, of other particles of the ensemble found in the interval $[x-\varepsilon, x+\varepsilon]$. The function $b(x,x')$ assigned by Eq. (4) to within the factor $(2\varepsilon)^{-1}$ is the indicator function of the interval $[x-\varepsilon, x+\varepsilon]$.

In Section 1, the FP equation (Eq. (3) together with Eq. (4)) with the smallness of the parameter ε and transformations of the variables x and t taken into account reduces to an equation with a local nonlinearity and partial derivatives of higher order. By introducing a potential function for $p(x,t)$ and transforming to *fast variables*, we are able to reduce the obtained equation to an equation with third-order partial derivatives with respect to the spatial variable with accuracy to $o(\varepsilon^2)$. Next we consider methods for constructing solutions for the reduced equation. In Section 2, we apply symmetry analysis of the differential equations [9–12] to the reduced equation obtained in Section 1. The determining equation for the symmetries is written down and Lie symmetries are found. Within the context of the Lie symmetries, we consider a group invariant solution of traveling wave type. In Section 3, we apply Adomian's iterative decomposition method to the reduced equation, allowing us to construct approximate solutions of the Cauchy problem in the form of an iterative series. Expressions for the first terms of the series are found for an initial function of general form. For two initial functions of special form, we find examples of exact solutions.

1. REDUCTION OF THE QUASILOCAL FOKKER–PLANCK EQUATION

We write out the expansion of Eqs. (2) and (3) with the function $b(x,x')$ having the form given by formula (4) in a power series in ε with accuracy to $o(\varepsilon^2)$. We expand the integral in a power series in ε:

$$\int_{x-\varepsilon}^{x+\varepsilon} p(x',t)\,dx' = p(x,t)\cdot 2\varepsilon + \frac{\varepsilon^3}{3}\partial_{xx} p(x,t) + o(\varepsilon^3)\,. \tag{5}$$

Substituting expressions (2), (4), and (5) into Eq. (3), we obtain an equation with accuracy to $o(\varepsilon^2)$:

$$\partial_t p(x,t) = -k_1(t)\,\partial_x p(x,t) - k_2(t)\partial_x \left[p^2(x,t) + \frac{\varepsilon^2}{6} p(x,t)\partial_{xx} p(x,t) \right] + D\partial_{xx} p(x,t)\,. \tag{6}$$

Transforming the coordinates according to the formulas

$$\tau = t\,,\ \xi = x - r(t)\,,$$

where the function $r(t)$ is chosen from the condition $dr(t)/dt = k_1(t)$, we reduce Eq. (6) to the form

$$-\partial_\tau \tilde{p}(\xi,\tau) - k_2(\tau)\partial_\xi \left(\tilde{p}^2(\xi,\tau) + \frac{\varepsilon^2}{6}\tilde{p}(\xi,\tau)\partial_{\xi\xi}\tilde{p}(\xi,\tau) \right) + D\partial_{\xi\xi}\tilde{p}(\xi,\tau) = 0\,. \tag{7}$$

Here we have introduced the notation $p(x,t) = p(\xi + r(\tau),\tau) = \tilde{p}(\xi,\tau)$.

We introduce a potential $w(\xi,\tau)$ for $\tilde{p}(\xi,\tau)$ by setting $w(\xi,\tau) = \partial_\xi^{-1}\tilde{p}(\xi,\tau)$. Making the substitution $\tilde{p}(\xi,\tau) = \partial_\xi w(\xi,\tau)$ in Eq. (7) and integrating the obtained equation over ξ, we obtain

$$-w_\tau(\xi,\tau) - k_2(\tau)\left(w_\xi^2(\xi,\tau) + \frac{\varepsilon^2}{6} w_\xi(\xi,\tau) w_{\xi\xi\xi}(\xi,\tau) \right) + D w_{\xi\xi}(\xi,\tau) + \varphi(\tau) = 0\,, \tag{8}$$

where $\varphi(\tau)$ is an arbitrary function arising as a result of the integration. Here and below, we employ the following notation for the partial derivatives in the form of subscripts: $w_\tau(\xi,\tau) = \partial_\tau w(\xi,\tau)$, $w_\xi(\xi,\tau) = \partial_\xi w(\xi,\tau)$, $w_{\xi\xi}(\xi,\tau) = \partial_{\xi\xi} w(\xi,\tau)$, etc. By redefining the potential: $w(\xi,\tau) \to w(\xi,\tau) + \int \varphi(\tau)d\tau$, we can eliminate the function $\varphi(\tau)$ from the equation without loss of generality.

We transform in Eq. (8) from the variables (ξ,τ) to the *fast variables* (x',t') with the help of a scaling transformation prescribed by the formulas $t' = \varepsilon^{-2}\tau$ and $x' = \varepsilon^{-1}\xi$; thus, $\partial_\tau = \varepsilon^{-2}\partial_{t'}$ and $\partial_\xi = \varepsilon^{-1}\partial_{x'}$. Introducing the notations $u(x',t') = w(\varepsilon x', \varepsilon^2 t')$ and $b(t') = -k_2(\varepsilon^2 t')/6$, we reduce Eq. (8) to the following form:

$$-u_{t'}(x',t') + D u_{x'x'}(x',t') + b(t')\left(6u_{x'}^2(x',t') + u_{x'}(x',t') u_{x'x'x'}(x',t') \right) = 0\,. \tag{9}$$

It can be seen from Eq. (9) that transforming to the variables (x',t') eliminates the parameter ε from Eq. (8). We shall call Eq. (9) the reduced equation of the nonlocal FP equation (Eq. (3) together with Eqs. (2) and (4)). Note that taking into account higher terms of the expansion in Eq. (5) and correspondingly in the quasilocal FP equation (Eq. (3) together with Eq. (4)) according to the scheme laid out above adds terms with higher derivatives (of fourth order) to the reduced equation (Eq. (9)), where the parameter ε, as before, does not enter into the resulting equation.

A further goal of the given work is to address the problem of integration (both exact and approximate) of the reduced equation (Eq. (9)). To simplify the notation in what follows, we will omit the prime on the variables (x',t') in Eq. (9), going over to the notation $x' \to x$, $t' \to t$.

2. LIE SYMMETRIES

We present here the minimally necessary concepts and definitions elucidating how to find the symmetries of Eq. (9) on the basis of the determining equations. A detailed exposition of the theory of group (symmetry) analysis of differential equations can be found in numerous publications (for example, see [9–12]). In what follows, we will follow [9–12]. We introduce the jet space $J^{(r)}$ of order r, $r \in Z_+$, where Z_+ is the set of positive integers. The point z_r of the space $J^{(r)}$, by definition, is assigned by the set of real independent variables $t, x, u, u_1, u_2, \dots, u_r \in R^1$, $z_r = (t, x, u, u_1, u_2, \dots, u_r) \in J^{(r)}$. Inclusion of the space $J^{(r)}$ in $J^{(r+m)}$, $r, m \in Z_+$ is defined in a natural way: the point z_r of the space $J^{(r)}$ is considered as a point of the space $J^{(r+m)}$ if we augment the coordinate list with zeroes, i.e., $z_r = (t, x, u, u_1, u_2, \dots, u_r, \underbrace{0, 0, \dots, 0}_{m}) \in J^{(r+m)}$. We also introduce the expanded space $J^{(r,1)}$, $z_{r,1} = (t, x, u, u_{(1)}, u_1, u_2, \dots, u_r) \in J^{(r,1)}$, where $u_{(1)}$ is a real variable, $u_{(1)} \in R^1$.

Let us consider the space $\mathcal{F}$ of smooth functions $f(z_r) = f(t, x, u, u_1, u_2, \dots, u_r)$ on the jet space $J^{(r)}$. We introduce the total differentiation operator acting on the functions $f(z_r)$:

$$\hat{D}_x = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_2} + \dots \tag{10}$$

It is immediately verified that $u_1 = \hat{D}_x u$, $u_2 = \hat{D}_x u_1$, … $u_{i+1} = \hat{D}_x u_i$. We write Eq. (9) in the following form:

$$-u_t(x,t) + \hat{H}(u)(x,t) = 0, \tag{11}$$

where the operator $\hat{H}$ is defined by the expression

$$\hat{H}(u)(x,t) = Du_{xx}(x,t) + b(t)\left(6u_x^2(x,t) + u_x(x,t)u_{xxx}(x,t)\right). \tag{12}$$

Differential equation (11) is represented in a natural way in geometric terms as a manifold in the space $J^{(r,1)}$. The relationship between a point of the space $J^{(r)}$ and the quantities entering into the differential equation is defined by the following condition: if the variable u is chosen as a function that depends on the independent variables t and x, i.e., $u = u(x,t)$, then the variables $u_{(1)}$ and u_i, $i = 1, \dots, r$, become derivatives:

$$u_{(1)} = u_t(x,t),\ u_1 = u_x(x,t),\ u_2 = u_{xx}(x,t),\ \dots,\ u_r = u_{\underbrace{xx\dots x}_{r}}(x,t). \tag{13}$$

We put a function $f(z)$ on $J^{(r,1)}$ in correspondence with the operator of Eq. (11) by setting

$$f(z_{r,1}) = -u_{(1)} + H(u), \tag{14}$$

where

$$H(u) = Du_2 + b(t)(6u_1^{\,2} + u_1 u_3). \tag{15}$$

The function $f(z_{r,1})$ assigned by Eq. (14) is the symbol of the operator of Eq. (11): by substituting expression (13) into Eq. (14) and setting $f(t,x,u(x,t),u_t(x,t),u_x(x,t),u_{xx}(x,t), u_{xxx}(x,t))=0$, we obtain Eq. (11). In the space $J^{(r,1)}$, Eq. (11) corresponds to the manifold prescribed by the condition

$$f(z_{r,1})=-u_{(1)}+H(u)=0. \tag{16}$$

Note that $H(u)$ is the symbol of the operator $\hat{H}$ assigned by Eq. (12).

The symmetry of Eq. (11) is equal to the real function $\sigma(z_{r,1})=\sigma(t,x,u,u_{(1)},u_1,u_2, \ldots)$ on $J^{(r,1)}$, with the help of which it is possible to construct the generator of the Lie group of transformations of the jet space leaving manifold (16) invariant. The symmetries are defined to within the function $f(z_{r,1})$ prescribing the equation. Therefore, without loss of generality, it is possible to eliminate the variable $u_{(1)}$ from $\sigma(t,x,u,u_{(1)},u_1,u_2, \ldots)$ with the help of Eq. (16) and consider only symmetries which are functions prescribed on $J^{(r)}$, i.e., $\sigma=\sigma(t,x,u,u_1,u_2, \ldots ,u_r)$. The determining equation for such symmetries has the following form:

$$\frac{\partial\sigma}{\partial t}+\sum_{k=0}\hat{D}_x^{\,k}(H)\frac{\partial\sigma}{\partial u_k}-\sum_{k=0}\hat{D}_x^{\,k}(\sigma)\frac{\partial H}{\partial u_k}=0, \tag{17}$$

where $H=H(u)$ is given by expression (15).

With the help of the symmetries, it is possible to find families of particular solutions of the equation, for example, group invariant solutions. If, using the generator of the group, it is possible to construct the corresponding transformation of the Lie group in explicit form, then to each such transformation there corresponds a symmetry operator of Eq. (11), mapping any solution of the equation onto some other solution of the equation. Finding symmetries and symmetry operators of an equation is one of the main tasks of symmetry analysis.

Symmetries by which transformations of the Lie group are found in explicit form are called Lie symmetries. To such belong symmetries that depend on the variables $t,x,u,u_{(1)},u_1$, i.e., $\sigma=\sigma(t,x,u,u_{(1)},u_1)$. In order to find such symmetries from the determining equation (Eq. (17)), it is necessary to seek solutions of this equation which include dependence on the variable u_3, $\sigma=\sigma(t,x,u,u_1,u_2,u_3)$. In such a solution it is necessary to substitute for u_3 using Eqs. (15) and (16). Note that, as a rule, dependence on u_2 also disappears upon substitution for u_3. As a result, we obtain the Lie symmetries $\sigma=\sigma(t,x,u,u_{(1)},u_1)$.

Finding $\sigma=\sigma(t,x,u,u_1,u_2,u_3)$ from Eq. (17) leads to the following result:

$$\sigma=\alpha H(u)+\beta u_1+\gamma u, \tag{18}$$

$$\alpha\dot{b}(t)=\gamma b(t). \tag{19}$$

Here $\alpha, \beta, \text{and}\, \gamma$ are constants, $H(u)$ has the form given by Eq. (15), and the dot above a symbol denotes the derivative with respect to t, $\dot{b}(t)=db(t)/dt$. Equation (17) is linear with respect to the symmetries σ; therefore, the set of symmetries forms a linear space: if σ_1 and σ_2 are two solutions of Eq. (17), then any linear combination of them, $c_1\sigma_1+c_2\sigma_2$, with arbitrary constants c_1 and c_2 will also be a symmetry – a solution of Eq. (17).

Let us consider linearly independent Lie symmetries for different functions $b(t)$ following from Eqs. (18) and (19):

1. Let $b(t)$ be an arbitrary function of t, $(\dot{b}(t)\neq 0)$. Then it follows from Eq. (19) that $\alpha=\gamma=0$ and consequently from Eq. (18) we have $\sigma=u_1$. The given symmetry, according to [9–12], corresponds to the generator

$X = \partial_x$ of the Lie group of transformations of the variables (t,x,u) – the independent variables x and t and one dependent variable u – leaving Eq. (11) invariant. One group of transformations of the variables (t,x,u) to the variables $(\overline{t},\overline{x},\overline{u})$ is the group of translations in x: $\overline{t} = t$, $\overline{x} = x + a$, $\overline{u} = u$, and a is the group parameter. One example of a group invariant solution is $u(t,x) = u(t)$ – the homogeneous solution.

2. Let $\dot{b}(t) \neq 0$ and $\alpha \neq 0$. Then, from Eq. (19) we find $b(t) = b_0 \exp(\gamma t/\alpha)$, where b_0 is a constant. In this case, the constants b_0, α, and γ enter into Eq. (12) and are prescribed parameters of the problem. Expression (18) is a linear combination of $\sigma_1 = \alpha H(u) + \gamma u$ and $\sigma_2 = u_1$ with arbitrary constant β. Substituting into σ_1 in place of $H(u)$ the variable $u_{(1)}$ from Eq. (16), we obtain the first-order symmetry $\sigma_1 = \alpha u_{(1)} + \gamma u$. The symmetries σ_1 and σ_2 form the basis of a linear space of Lie symmetries.

3. Let $\dot{b}(t) = 0$ in Eq. (19), i.e., b is an arbitrary ($\neq 0$) constant. Then $\gamma = 0$ and $\sigma = \alpha H(u) + \beta u_1$, where α and β are arbitrary constants. Substituting the variable $u_{(1)}$ from Eq. (16) in place of $H(u)$, we obtain the first-order symmetry $\sigma = \alpha u_{(1)} + \beta u_1$, which is a linear combination of $\sigma_1 = u_{(1)}$ and $\sigma_2 = u_1$, which form the basis of a linear space of Lie symmetries in the given case. The symmetry σ_2, as in the preceding case, corresponds to the transformation of a translation in x. Similarly, the symmetry $\sigma_1 = u_{(1)}$ corresponds to the transformation of a translation in time t, and, correspondingly, the solution $u(t,x) = u(x)$ – a stationary solution – will be a group invariant solution. If we take the linear combination $\sigma = u_{(1)} + cu_1$, where c is a constant, then the given symmetry corresponds to the generator $X = \partial_t + c\partial_x$, and the corresponding group invariant solution will have the form of a traveling wave:

$$u(x,t) = u(\eta), \quad \eta = x - ct. \tag{20}$$

As can be seen from Eqs. (18) and (19), the Lie symmetries of Eq. (11) particularized by Eq. (12) constitute a very narrow class of symmetries. Of the corresponding group invariant solutions, the solutions in the form of a traveling wave can be of definite interest. Substitution of Eqs. (20) into Eq. (11) together with Eq. (12) reduces it to an ordinary differential equation:

$$cu' + Du'' + b\left(6(u')^2 + u'u'''\right) = 0, \tag{21}$$

where the prime denotes differentiation with respect to η, $u' = du(\eta)/d\eta$. Equation (21) admits a double lowering of the order effected by the substitution $s(\eta) = u'(\eta)$ and introduction of the new unknown function $g(s)$ in place of $u(\eta)$ proceeding from the condition $s'(\eta) = g(s)$. As a result, Eq. (21) reduces to a first-order equation in the function $g(s)$:

$$s\,g(s)\frac{dg(s)}{ds} + \mu g(s) + 6s^2 + c_1 s = 0, \tag{22}$$

where the parameters μ and c_1 are determined by the conditions $D = \mu b$ and $c = c_1 b$. Equation (22) reduces by a substitution of variables to the Abel equation [13], which in the general case cannot be integrated in closed form. A detailed investigation of the solutions of Eq. (22) requires a special treatment and goes beyond the scope of the given work.

3. ADOMIAN'S ITERATIVE METHOD

To solve the Cauchy problem for Eqs. (11) and (12), we apply one of the asymptotic methods that have been in active use in recent decades to find approximate solutions of nonlinear equations (see, for example, the review in [14]), namely, the iterative decomposition method developed by George Adomian (the Adomian decomposition method) [15–17]. Let us consider the Cauchy problem for Eqs. (11) and (12):

$$\partial_t u(x,t) = Du_{xx}(x,t) + b(t)\hat{N}(u)(x,t)\,,\ u(x,0) = g(x)\,. \tag{23}$$

Here $g(x)$ is the initial function and $\hat{N}$ is the nonlinear operator entering into Eq. (12):

$$\hat{N}(u) = 6u_x^{\ 2} + u_x u_{xxx}\,. \tag{24}$$

We introduce the inverse operator to ∂_t: $\partial_t^{-1}(.) = \int_0^t (.)\,dt$, and integrate Eq. (23) with respect to t:

$$u(x,t) = g(x) + \partial_t^{-1}\left[Du_{xx}(x,t) + b(t)\hat{N}(u)(x,t)\right]. \tag{25}$$

Following [15–17], we seek the solution of Eq. (25) in the form of a series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)\,, \tag{26}$$

and represent the action of the nonlinear operator (Eq. (24)) on the series (Eq. (26)) as a sum:

$$\hat{N}\left(\sum_{n=0}^{\infty} u_n(x,t)\right) = \sum_{n=0}^{\infty} A_n\left(u(x,t)\right), \tag{27}$$

where $A_n\left(u\right)$, $n = 0,1,\ ...$ are the Adomian polynomials, defined by the expression

$$A_n(u) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\hat{N}\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]\Bigg|_{\lambda-0}\,. \tag{28}$$

According to the Adomian method, the iterative procedure for constructing series (26) with Eqs. (25)–(28) taken into account is represented as follows:

$$u_0(x,t) = g(x)\,, \tag{29}$$

$$u_{n+1}(x,t) = \partial_t^{-1} Du_{n\,xx}(x,t) + \partial_t^{-1} b(t) A_n\left(u(x,t)\right). \tag{30}$$

Let us construct the first terms of asymptotic series (26). We write out the first three Adomian polynomials for nonlinear operator (24) (for brevity, the arguments of the functions are not indicated):

$$A_0(u) = 6u_{0\ x}^{\ \ 2} + u_{0\ x}u_{0\ xxx} = A_0(g) = 6g_x^{\ 2} + g_x g_{xxx}\,,$$

$$A_1(u) = 12u_{0\ x}u_{1\ x} + u_{0\ xxx}u_{1\ x} + u_{0\ x}u_{1\ xxx} = hu_{1\ x} + g_x u_{1\ xxx}\,,$$

$$A_2(u) = 6u_{1x}^2 + 12u_{0\,x}u_{2\,x} + u_{0\,xxx}u_{2\,x} + u_{0\,x}u_{2\,xxx} + u_{1\,x}u_{1\,xxx}$$
$$= 6u_{1x}^2 + u_{1\,x}u_{1\,xxx} + hu_{2\,x} + g_{\,x}u_{2\,xxx}\,. \tag{31}$$

Here and below, $h = h(x) = \partial A_0 / \partial g_x = 12g_x + g_{xxx}$. Substituting polynomials (31) into Eq. (30), we find the following expressions for the first terms of the asymptotic series in terms of $g(x)$ and $A_0(g(x))$:

$$u_1 = Dg_{xx}\,t + A_0\,B(t) = Dg_{xx}\,t + (6g_x^{\ 2} + g_x g_{xxx})B(t)\,, \tag{32}$$

$$u_2 = \frac{1}{2}D^2\frac{d^4 g}{dx^4}t^2 + DA_{0\,xx}B_0(t) + \left(h\,A_{0\,x} + g_x A_{0\,xxx}\right)B_1(t) + D\left(g_{xxx}h + g_x\frac{d^5 g}{dx^5}\right)b_1(t)\ , \tag{33}$$

where we have employed the notation $B(t) = \int_0^t b(t)dt$, $B_0(t) = \int_0^t B(t)dt$, $B_1(t) = \int_0^t b(t)B(t)dt$, and $b_1(t) = \int_0^t tb(t)dt$. In order to write down an expression for u_3, we calculate the auxiliary function

$$w(x,t) = \int_0^t b(t)u_2\,dt = \frac{1}{2}D^2\frac{d^4 g}{dx^4}b_2(t) + DA_{0\,xx}B_{01}(t) + \left(hA_{0\,x} + g_x A_{0\,xxx}\right)B_2(t)$$
$$+ D\left(g_{xxx}h + g_x\frac{d^5 g}{dx^5}\right)b_{01}(t), \tag{34}$$

where $B_{01}(t) = \int_0^t b(t)B_0(t)dt$, $B_2(t) = \int_0^t b(t)B_1(t)dt$, $b_2(t) = \int_0^t t^2 b(t)dt$, and $b_{01}(t) = \int_0^t b(t)b_1(t)dt$. Then

$$u_3 = \frac{1}{3!}D^3\frac{d^6 g}{dx^6}t^3 + D^2\frac{d^4 A_0}{dx^4}\int_0^t B_0(t)dt + D\frac{d^2}{dx^2}\left(hA_{0\,x} + g_x A_{0\,xxx}\right)\int_0^t B_1(t)dt$$
$$+ D^2\frac{d^2}{dx^2}\left(g_{xxx}h + g_x\frac{d^5 g}{dx^5}\right)\int_0^t b_1(t)dt + D^2 g_{xxx}\left(6g_{xxx} + \frac{d^5 g}{dx^5}\right)b_2(t)$$
$$+ D\left(g_{xxx}A_{0\,xxx} + h_{xx}A_{0\,x}\right)\int_0^t tb(t)B(t)dt + A_{0\,x}\left(6A_{0\,x} + A_{0\,xxx}\right)\int_0^t b(t)B^2(t)dt$$
$$+ hw_x + g_x w_{xxx}\,. \tag{35}$$

The obtained expressions for the terms of asymptotic series (Eq. (26)) allow us to construct an approximate solution of problem (29) and (30) for the initial function $g(x)$ and function $b(t)$ of general form.

In some cases, the method can lead to exact solutions. This is possible, for example, if the iterative series (Eq. (26)) truncates or it is possible to make good use of some regularity in the formation of the terms of the series. Let us consider two simple examples illustrating such a situation:

Example 1. We assign the initial function in the form $u(x,0) = g(x) = x$. From Eqs. (31) and (32) we find $A_0 = 6$ and $u_1 = 6B(t)$. From Eq. (33) we have $u_2 = 0$. Next, by direct verification it is possible to show that

$u_n = 0,\ n > 1$, i.e., series (26) truncates and the exact solution of the Cauchy problem has the following form: $u(x,t) = x + 6B(t)$.

Example 2. Let us consider the initial function

$$u(x,0) = g(x) = \frac{1}{2}x^2. \tag{36}$$

From Eqs. (31) and (32) we find the polynomial $A_0 = 6x^2$ and the first iteration $u_1 = Dt + 6x^2 B(t)$, respectively. Next, from Eqs. (33) and (34) we find the following iterations: $u_2 = 12DB_0(t) + 12^2 x^2 B_1(t)$ and $u_3 = 2\cdot 12^2 D\int_0^t B_1(t)dt + 12^2 x^2 \left(6\int_0^t b(t)B^2(t)dt + 24B_2(t) \right)$. From the form of the found iterative terms of series (26) it can be conjectured that solution (26) has the following structure:

$$u(x,t) = \alpha(t) + \beta(t)x^2, \tag{37}$$

where $\alpha(t)$ and $\beta(t)$ are functions of t satisfying the initial conditions

$$\alpha(0) = 0,\ \beta(0) = \frac{1}{2}. \tag{38}$$

Substituting expression (37) into Eqs. (23) and (24) leads to the equations

$$\dot{\alpha}(t) = 2D\beta(t),\ \dot{\beta}(t) = 24b(t)\beta^2(t). \tag{39}$$

Integration of system (39) with initial conditions (38) gives

$$\beta(t) = \frac{1}{2\left(1 - 12B(t)\right)},\ \alpha(t) = 2D\int_0^t \frac{dt}{2\left(1 - 12B(t)\right)}. \tag{40}$$

Thus, the exact solution of the Cauchy problem given by Eqs. (23) and (24) with the initial function (36) has the form (37), where $\alpha(t)$ and $\beta(t)$ are determined by conditions (40).

CONCLUSIONS

In this paper we have investigated the nonlinear Fokker–Planck equation in one-dimensional spacetime with nonlocal drift, being a nonlinear integrodifferential equation of diffusion type. In the semiclassical approximation in a small parameter characterizing the size of the localization region, the initial equation reduces to a partial differential equation. The main goal of this work consisted in the application of the method of symmetry analysis and the Adomian decomposition method to the reduced equation (Eq. (9)).

Calculation of the Lie symmetries showed that the set of these symmetries is extremely limited. From amongst the set of group invariant solutions corresponding to them, we considered a solution of traveling wave type. Finding the given solution led to the Abel ordinary differential equation. Other approaches can also be applied to an investigation of the symmetries of the nonlocal FP equation (Eqs. (2) and (3)). Thus, in [18] the FP equation (Eqs. (2) and (3)) was reduced in the semiclassical approximation to a unified system consisting of an equation of diffusion type and a dynamical system of the moments entering into the coefficients of the equation. The symmetries were calculated for the unified system. Thus, the approach based on the quasilocal approximation proposed in this paper and the approach of [18] allow one to investigate the symmetries of the nonlocal FP equation from different points of view.

The iterative Adomian method was used in this work to calculate the first terms of the asymptotic series representing the solution of the Cauchy problem for the reduced equation (Eqs. (11) and (12)). Two examples of an exact solution of the Cauchy problem for a linear and a quadratic initial function were found. Note that it is possible to investigate Eq. (6) without introducing the potential $w(\xi,\tau)$. In this case, it is possible to eliminate the parameter ε from the equation with the help of the transformation $t'=\varepsilon^{-1}\tau$, $x'=\varepsilon^{-1}\xi$ and the condition of smallness of the diffusion coefficient D, $D=\varepsilon C$, where C is a constant. This and other schemes of quasilocal reduction go beyond the scope of this work.

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