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ELEMENTARY PARTICLE PHYSICS AND FIELD THEORY

ON THE EQUIVALENCE OF TWO APPROACHES TO THE CONSTRUCTION OF INTERACTIONS IN HIGHER-DERIVATIVE THEORIES

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*It is shown that two previously developed approaches to the construction of nonlinear interactions in higher-derivative theories (Eur. Phys. J., **C74** (2014); J. Phys., **A49** (2016)) lead to equivalent nonlinear models. A substitution of variables that is invertible on the mass shell is presented, which maps the classical trajectories of one of the models onto the trajectories of the other model.*

Keywords: higher-derivative theories, stability, theories of derivative type, proper deformation.

INTRODUCTION

Higher-derivative theories possess a number of remarkable properties and have attracted interest for many years in different branches of field theory [1–3]. In particular, in many cases, inclusion of higher-derivative terms improves the convergence at the classical and the quantum level or allows one to realize a wider symmetry. In this case, higher derivative theories face a notorious problem of classical and quantum stability which has been studied for decades. It is well known that except the cases in which higher derivative terms, contributing to the initial data, are strongly bounded by the constraints, as, for example, in theories of $f(R)$-gravity [2, 4], the canonical energy is unbounded; therefore, its conservation does not guarantee stability of the theory. In [3] it was noted that higher-derivative theories can admit other conserved quantities besides the canonical energy-momentum, which stabilize dynamics in many cases. The existence of such conservation laws is a specific feature of a definite class of models and is associated with the presence of a wider symmetry group.

In the present work we consider higher-derivative field theories where the wave operator of the free part of the theory is a polynomial of the other self-adjoint operator. For such models, the conservation laws can be systematically found using procedures described in [3, 5–7], and have been used to construct interactions in agreement with a stability condition. There exist two ways to construct stable interaction vertices: by using the procedure of reducing the order [3, 5–9] and by making use of the proper deformation transformation [10]. In both cases, the nonlinear theory is in agreement with the condition of existence of a bounded from below conservation law, which ensures stability of the model. The expressions for the vertices in the different approaches in these cases are different. Therefore, the question of the relationship between the two methods is of interest.

The main result of the present work is a proof of the equivalence of the two approaches to the construction of interactions. In Section 3, formula (17), we introduce a substitution of variables that maps trajectories of one of the models onto trajectories of the other. The substitution of variables is local and invertible with account of the equations of motion for the corresponding variables (on the mass shell), but is not local outside the mass shell. This latter

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condition is weaker than the existence of an invertible change of variables in the configuration space of the theory; however, it still ensures a one-to-one correspondence between the main invariants of the two models, such as symmetries, conservation laws, and Lagrange structures. In particular, it is shown that the substitution takes the conservation laws constructed in [3, 5–9] into the conservation law from [10], and *vice versa*. The general procedure for constructing an equivalence relation between the formulations is illustrated by the example of the nonlinear theory of the Pais–Uhlenbeck oscillator and by the example of Podolsky's electrodynamics with spinor matter.

1. THEORIES OF DERIVATIVE TYPE

Let us consider a set of fields $\varphi^J(x)$ on a d-dimensional Minkowski space with local coordinates x^μ, $\mu = 0, 1, \ldots, d-1$. Here the multi-index J contains all discrete indices of the tensor field, including tensor indices and spinor indices. We assume that the theory admits a constant metric that allows us to raise and to lower the multi-indices. In this notation, each linear local system of equations can be represented in the following form:

$$M(\partial_\mu)_{IJ}\varphi^J = 0\,, \tag{1}$$

where M is a square matrix, whose elements are formal polynomials in the variable ∂_μ. If ∂_μ denotes the partial derivatives with respect to the coordinate x^μ, then Eq. (1) will be a partial differential equation. The operator $M(\partial_\mu)$ is known as the *wave operator*. The formally adjoint operator to M is defined by the relation

$$M^+(\partial_\mu)_{IJ} = M(-\partial_\mu)_{JI}\,. \tag{2}$$

Field equations (1) are Lagrangian if the wave operator is self-adjoint $M^+ = M$. The action functional in this case has the form

$$S = \frac{1}{2}\int\limits_X d^d x \varphi^I M(\partial_\mu)_{IJ}\varphi^J\,. \tag{3}$$

We will call linear theory (1) *a theory of derived type (or a derived theory)* if the wave operator is a polynomial of finite order in some other self-adjoint operator

$$M(\partial_\mu) \equiv M(W(\partial_\mu)) = \sum_{p=0}^{n} a_p W^p(\partial_\mu), \quad a_n \neq 0\,. \tag{4}$$

Here the real numbers a_p are the coefficients of the polynomial $M(W)$. The degree n of the polynomial $M(W)$ is called *the order of the theory*. The order of the theory is found from relation (4). The Pais–Uhlenbeck oscillator [11], including its odd-order extension [9], Podolsky's generalized electrodynamics [12], and linearized critical gravitation [13] are the examples of theories of the derived type.

If the operator W commutes with the Lie derivative L_ξ along some time-like Killing vector ξ, then the derived theory is invariant under the action of an n-parametric symmetry transformation of the form

$$\delta_\xi\varphi = \left(\sum_{p=0}^{n}\alpha_p W^p\right) L_\xi\varphi\,. \tag{5}$$

In contrast to the canonical energy, which is always bounded from below, the n-parameter family of conservation laws corresponding to symmetry (5) can for some values of the parameters include a conservation law bounded from below,

and hence stabilize the dynamics. In what follows, we consider only such models, where a suitable time-like Killing vector ξ exists. We will not explicitly trace out the boundedness-from-below for the conservation law here. The corresponding problem was considered in [3, 5–10].

2. NONLINEAR EXTENSIONS IN METHODS OF FACTORIZATION AND PROPER DEFORMATION

In [3, 5–10] two different approaches to the construction of stable interactions in theories of derivative type were formulated: using the procedure of lowering the order and the proper deformation method. The following nonlinear equations were obtained using the procedure of reduction of order:

$$E(\varphi) \equiv M(W)\varphi + U'(V\varphi) = 0 . \tag{6}$$

Here we have used the notation

$$V = \sum_{p=0}^{n-1} \alpha_p W^p , \tag{7}$$

and the vector function U' denotes the Euler–Lagrange derivative of the local function $U(\varphi, \partial\varphi, \ldots, \partial^m\varphi)$ with respect to its argument:

$$U'(\varphi) \equiv \frac{\delta U}{\delta \varphi} = \sum_{p=0}^{m} (-1)^p \partial_{\mu_1} \ldots \partial_{\mu_p} \left(\frac{\partial U}{\partial(\partial_{\mu_1} \ldots \partial_{\mu_p} \varphi)} \right). \tag{8}$$

The numerical parameters α_p are the interaction parameters.

The conserved current J associated with the Killing vector ξ is defined by the relation

$$\partial_\mu J^\mu = -\left\{ L_\xi V\varphi \right\} E(\varphi) . \tag{9}$$

Depending on the values of the interaction parameters α_p, the conserved quantity defined by the conserved current J can be bounded from below, even if the canonical energy of the theory is not bounded from below. If the nonlinear theory given by Eq. (6) admits bounded from below conserved quantity, then such a model should be considered to be stable.

The nonlinear equations obtained using the proper deformation method have the form

$$E'(\psi) \equiv M(W)\psi + VU'(\psi) = 0 , \tag{10}$$

where the operator V is assigned by formula (7) and the quantity U' is defined by Eq. (8). The quantities ψ^I are the dynamical fields of the theory given by Eq. (11). The conserved current J' is defined by the relation

$$\partial_\mu J'^\mu = -\left\{ L_\xi (V^{-1}\psi + KU'(\psi)) \right\} E'(\psi) , \tag{11}$$

where the operator V^{-1} is defined by the relation

$$VV^{-1} = 1 + KM , \tag{12}$$

which is understood as an identity for the univariate polynomials $V(W)$, $V^{-1}(W)$, $K(W)$, and $M(W)$. We assume that $V(W)$ and $M(W)$ are coprime. In this case, the solution of Eq. (11) with respect to the unknowns $V^{-1}(W)$ and $K(W)$ exists and is

unique. The existence and uniqueness of the solution follow from Bézout's Lemma. In explicit form, the conserved current J' can be found using formulas from [10].

3. EQUIVALENCE OF NONLINEAR EXTENSIONS

We call two theories equivalent if there exists an invertible on the mass shell transformation that takes classical trajectories of one theory into the trajectories of the other. Equivalent theories are indistinguishable from a physical point of view and should be considered as two different forms of one and the same theory. If the dynamics of the theories is described by systems of partial differential equations [1]

$$E(\varphi) = 0, \quad E'(\psi) = 0\,, \tag{13}$$

then the models are equivalent under the condition that there exists a transformation

$$\varphi = \Phi(\psi, \partial\psi, ..., \partial^m\psi), \quad \psi = \Psi(\varphi, \partial\varphi, ..., \partial^m\varphi)\,, \tag{14}$$

such that

$$E(\Phi(\psi, \partial\psi, ..., \partial^m\psi)) \approx 0, \quad E'(\Psi(\varphi, \partial\varphi, ..., \partial^m\varphi)) \approx 0\,, \tag{15}$$

$$\varphi \approx \Phi(\Psi, \partial\Psi, ..., \partial^m\Psi)), \quad \psi \approx \Psi(\Phi, \partial\Phi, ..., \partial^m\Phi)\,. \tag{16}$$

Here the sign ≈ denotes equality on the mass shell for the corresponding variable. By virtue of relations (15), there exists a one-to-one correspondence between the conserved quantities in the equivalent theories. In particular, if one of the models admits bounded from below conserved quantity, then the other theory should also admit a conservation law bounded from below. Therefore, two equivalent theories are always stable or unstable simultaneously.

The following transformation establishes equivalence between theories (6) and (11):

$$\varphi = V^{-1}\psi + KU'(\psi), \quad \psi = V\varphi\,. \tag{17}$$

Conditions (14) and (15) are verified by straightforward calculation:

$$E(V^{-1}\psi + KU'(\psi)) = V^{-1}E'(\psi) \approx 0, \quad E'(V\varphi) = VE(\varphi) \approx 0\,. \tag{18}$$

The following relation holds between the conserved currents J and J':

$$J = J' + J''\,, \tag{19}$$

where the trivial conserved current $J'' \approx 0$ is defined by the relation

$$\partial_\mu J''^\mu = \left\{L_\xi V\varphi\right\}E(\varphi) - \left\{L_\xi\varphi\right\}VE(\varphi)\,. \tag{20}$$

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[1] In this Section we assume that the spaces of the fields of the considered theories coincide, although in general this is not required.

The conserved current J'' is obtained by a finite number of integrations by parts since the operator V is a polynomial of finite degree in the formally self-adjoint operator W. Thus, theories (6) and (10) are equivalent and the equivalence between these models is established by transformation (17).

4. EXAMPLES

4.1. The Pais–Uhlenbeck fourth-order oscillator

The dynamical variable of the theory $\varphi(t)$ enjoys an equation of derived type

$$M(W)\varphi \equiv (W^2 + a_1 W + a_0)\varphi = 0, \tag{21}$$

where the self-adjoint operator W and the coefficients a_1 and a_0 have the form

$$W = W^+ = \frac{1}{\omega_2^2 - \omega_1^2}\frac{d^2}{dt^2}, \quad a_0 = \frac{\omega_2^2\omega_1^2}{(\omega_2^2 - \omega_1^2)^2}, \quad a_1 = \frac{\omega_2^2 + \omega_1^2}{\omega_2^2 - \omega_1^2}, \quad a_2 = 1, \tag{22}$$

and the frequencies of the oscillator are assumed to be different $\omega_1 \neq \omega_2$. The order of the theory is equal to 2, $n = 2$; therefore, the operator defined by Eq. (7) has the following representation:

$$V = \alpha_1 W + \alpha_0 = \tilde{\alpha}_1 \frac{d^2}{dt^2} + \alpha_0, \tag{23}$$

where α_1 and α_0 are numerical parameters.

Nonlinear extensions (6) and (10) in the case of the Pais–Uhlenbeck theory have the form

$$M(W)\varphi + U'\left(\tilde{\alpha}_1 \frac{d^2\varphi}{dt^2} + \alpha_0\varphi\right) = 0, \tag{24}$$

$$M(W)\psi + \left(\tilde{\alpha}_1 \frac{d^2}{dt^2} + \alpha_0\right)U'(\psi) = 0, \tag{25}$$

respectively. The conserved quantities associated with nonlinear extensions (24) and (26) have the form

$$J = \frac{1}{2}\frac{\tilde{\alpha}_1\omega_1^2 - \alpha_0}{\omega_2^2 - \omega_1^2}\left(\dot{\xi}_1^2 + \omega_1^2\xi_1^2\right) - \frac{1}{2}\frac{\tilde{\alpha}_1\omega_2^2 - \alpha_0}{\omega_2^2 - \omega_1^2}\left(\dot{\xi}_2^2 + \omega_2^2\xi_2^2\right) + \left(\dot{\psi}\frac{\partial U}{\partial\dot{\psi}} - U\right)\Big|_{\psi = V\varphi}, \tag{26}$$

$$J' = \frac{1}{2}\frac{\tilde{\alpha}_1\omega_1^2 - \alpha_0}{\omega_2^2 - \omega_1^2}\left(\dot{\zeta}_1^2 + \omega_1^2\zeta_1^2\right) - \frac{1}{2}\frac{\tilde{\alpha}_1\omega_2^2 - \alpha_0}{\omega_2^2 - \omega_1^2}\left(\dot{\zeta}_2^2 + \omega_2^2\zeta_2^2\right) + \dot{\psi}\frac{\partial U}{\partial\dot{\psi}} - U, \tag{27}$$

where a dot above a symbol indicates the time derivative and we have used the notation

$$\xi_1 = \frac{\ddot{\varphi} + \omega_2^2\varphi}{\omega_2^2 - \omega_1^2}, \quad \xi_2 = \frac{\ddot{\varphi} + \omega_1^2\varphi}{\omega_1^2 - \omega_2^2}, \quad \zeta_i = \xi_i\big|_{\varphi=\psi} - \alpha_1 U'(\psi), \quad i = 1,2.$$

Upon redefinition of interaction parameters, formulas (24) and (26) reproduce the nonlinear equation and the conserved quantity obtained in [3, 6, 8]. Formulas (25) and (27) reproduce the nonlinear equation and the conserved quantity obtained in [10].

The transformation establishing equivalence has the form of Eqs. (18), where the operators V are defined in Eqs. (21) and (23), and the operators $V^{-1}(W)$ and K have the form

$$V^{-1} = \frac{1}{(\alpha_0 - \tilde{\alpha}_1\omega_1^2)(\alpha_0 - \tilde{\alpha}_1\omega_2^2)}\left(-\tilde{\alpha}_1\left(\frac{d^2}{dt^2} + \omega_2^2 + \omega_1^2\right) + \alpha_0\right), \quad K = -\frac{\alpha_1^2}{(\alpha_0 - \tilde{\alpha}_1\omega_1^2)(\alpha_0 - \tilde{\alpha}_1\omega_2^2)}.$$

It is easy to see that this transformation takes the interaction represented by Eqs. (24) and (26) into the interaction represented by Eqs. (25) and (27), and *vice versa*. Thus, the two nonlinear extensions of the Pais–Uhlenbeck theory obtained earlier are equivalent.

4.2. Podolsky's electrodynamics with spinor matter

In [3], using the order lowering method, we obtained the following nonlinear equations describing the couplings of the Podolsky vector field $\varphi_\mu(x)$ with a massive field O with spin 1/2 and charge e on the Minkowski space:

$$(\partial^\rho\partial_\rho + m^2)\partial^\nu F_{\nu\mu} - j_\rho = 0, \quad \left(\gamma^\mu(i\partial_\mu - e(V\varphi)_\mu) + m\right)\Xi = 0. \tag{28}$$

Here we have used the notation

$$F_{\nu\mu} = \partial_\nu\varphi_\mu - \partial_\mu\varphi_\nu, \quad j_\rho = e\bar{\Xi}\gamma_\rho\Xi, \quad V\varphi_\mu = \alpha_1\partial^\nu F_{\nu\mu} + \alpha_0\varphi_\mu, \tag{29}$$

where the quantity j_ρ is the current of the spinor field, m is a constant with units of mass, ∂_μ is the partial derivative with respect to the coordinate x^μ, and the metric signature is (+, –, …, –). The real numbers α_1 and α_0 are the interaction parameters. In the general case, Eq. (28) describes non-minimal interactions of the vector field with spinor matter, where a non-minimal interaction arises in the spinor field sector, while the equations of motion of the Podolsky field φ_μ include interaction term.

In the vector field sector, theory (28) is a theory of derived type since the corresponding equation of motion can be reduced to the form given by Eq. (6)

$$M\varphi + U' = 0, \quad M = W^2 + m^2W, \tag{30}$$

where $W_\mu{}^\nu = \partial^\rho\partial_\rho\delta_\mu{}^\nu + \partial_\mu\partial^\nu$ is the Maxwell operator and $U = -e\Xi\gamma^\mu\varphi_\mu\Xi$. The conserved current J^μ is defined by the 0-component of the energy-momentum tensor $\Theta^\mu{}_0$ and can be found in formula (74) of [4]. The equivalent form of Eq. (11) for Eqs. (30) has the form

$$(\partial^\mu\partial_\mu + m^2)\partial^\nu G_{\nu\rho} = Vj_\rho, \quad \left(\gamma^\mu(i\partial_\mu - e\psi_\mu) + m\right)\Xi = 0, \quad G_{\nu\mu} = \partial_\nu\psi_\mu - \partial_\mu\psi_\nu. \tag{31}$$

Here, in contrast to Eqs. (28), the interaction in the electromagnetic field sector is modified in a non-minimal way while the interaction in the spinor field sector remains minimal. Thus, there are two types of non-minimal interactions of Podolsky's electrodynamics with spinor matter: in one of them the spinor field sector is non-minimal, and in the other, the electromagnetic field sector is non-minimal. These two classes of nonlinear models are connected by the transformation

$$\varphi = V\psi, \quad \psi = \frac{1}{\alpha_0}\left(\frac{\alpha_1 W}{(m^2\alpha_1 - \alpha_0)} + 1\right)\psi + \frac{1}{\alpha_0}\frac{\alpha_1^2}{(m^2\alpha_1 - \alpha_0)}U', \tag{32}$$

which is invertible on the mass shell.

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