DOI 10.1007/s11182-017-0997-7

Russian Physics Journal, Vol. 59, No. 11, March, 2017 (Russian Original No. 11, November, 2016)

EXTENSION OF THE CHERN–SIMONS THEORY: CONSERVATION LAWS, LAGRANGE STRUCTURES, AND STABILITY

D. S. Kaparulin, I. Yu. Karataeva, and S. L. Lyakhovich UDC 530.2; 530.13

We consider the class of higher derivative 3d vector field models with the wave operator being a polynomial of the Chern–Simons operator. For the nth order theory of this type, we provide a covariant procedure for constructing n-parameter family of conservation laws associated with spatiotemporal symmetries. This family includes the canonical energy that is unbounded from below, whereas others conservation laws from the family can be bounded from below for certain combinations of the Lagrangian parameters, even though higher derivatives are present in the Lagrangian. We prove that any conserved quantity bounded from below is related with invariance of the theory with respect to the time translations and ensures the stability of the model.

Keywords: higher-order derivative theories, stability, Lagrange anchor.

INTRODUCTION

We consider the class of models of the vector field $A = A_\mu dx^\mu$ on the $3d$ Minkowski space with the action functional

$$S = \frac{1}{2\gamma^2}\int *A\wedge(a_0 A + a_1(*\gamma d)A + a_2(*\gamma d)^2 A + \ldots + a_n(*\gamma d)^n A), \qquad a_n \neq 0\,, \tag{1}$$

where γ is the constant whose dimension is inverse to that of the mass, $a_0, a_1, a_2, \ldots, a_n$ are some real numerical parameters, and $*$ denotes the Hodge dual. The signature $(+,-,-)$ of the metric is used. Due to the special structure of the action that includes powers of the Chern–Simons (CS) operator $w = *\gamma d$, model (1) is an extension of the CS theory. We define the order of extension as the maximal order of the derivative in the action. Hereinafter, we consider only finite-order extensions.

The Euler–Lagrange equations have the form

$$\frac{\delta S}{\delta A} \equiv M(w)A = 0\,, \tag{2}$$

with the wave operator M given by the expression

National Research Tomsk State University, Tomsk, Russia, e-mail: dsc@phys.tsu.ru; karin@phys.tsu.ru; sll@phys.tsu.ru. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 11, pp. 172–177, November, 2016. Original article submitted July 23, 2016.

$$M(w) = \frac{1}{\gamma^2}\sum_{k=0}^{n} a_k w^k = \frac{a_n}{\gamma^2}\prod_{i=1}^{r}(w-\lambda_i)^{p_i}\prod_{j=1}^{s}(w^2 - 2\mathrm{Re}\,\omega_j w + |\omega_j|^2)^{q_j} \ . \tag{3}$$

The real numbers λ_i and the complex conjugate numbers ω_j and $\bar{\omega}_j$ are roots of the polynomial $M(w)$ with multiplicities p_i and q_j, respectively. Depending on values of the coefficients a_k, the extension of the CS theory describes the well-known models of field theory. The extension of order 1 includes the original CS theory and its massive analog, the CS–Proca model [1, 2]. The extension of order 2 describes the Maxwell–CS–Proca theory and its modifications [3, 4]. The extension of order 3 has been studied in [5]. The generalized Podolsky electrodynamics [6] in three dimensions is an example of extension of order 4. Due to simplicity and generality, the extension of the CS theory can be considered as a useful test model for studying various aspects of field theory with higher derivatives.

In the present work, we study the stability of extension of the CS model from the viewpoint of existence of conserved quantities bounded from below. In particular, using the Lagrange anchor concept [7], we study the relationship between the conserved values bounded from below and the invariance of the theory with respect to the translations of time. The existence of a bounded conservation law related to the invariance of the theory with respect to the translations of time is a less restrictive condition than the requirement for the canonical energy to be bounded from below, but this condition still ensures the presence of the Hamiltonian formulation with the Hamiltonian bounded from below. For the theories with higher derivatives, the use of the alternative Hamiltonian formulations provides methods of stability control at the classical and quantum levels [8–12]. Therefore, the construction of bounded conservation laws and the study of their relationship with time translations is of interest.

The stability of the extension of the CS theory has been studied in [13], where the method for construction of the second-rank conserved tensors has been proposed and, in the extension of the third order, the bounded conservation laws have been classified. The present work further develops these results in two ways. First, we propose a covariant procedure for construction of bounded conservation laws in extensions of any order. Second, we prove that any positive (non-negative and equal to zero only on purely gauge solutions of the equation of motion) conserved quantity can be related to the invariance of the theory with respect to the translations of time. In total, our results demonstrate the stability of the extension of the CS theory even though the canonical energy of this model is unbounded from below.

The work is organized as follows. In Section 1 we associate with each Killing vector of the Minkowski space the *n*-parameter family of symmetry transformations that keep the action of the theory invariant. In Section 2 we find the family of conserved currents and identify those corresponding to bounded conserved quantities. In Section 3 we study a relationship between the bounded conserved quantities and the invariance of the theory with respect to the translation of time. In conclusion, we briefly summarize the results.

1. HIGHER SYMMETRIES

The extension of the CS theory of order n admits the following *n*-parameter family of symmetry transformations associated with the Killing vector of the Minkowski space X:

$$\delta_X A = \sum_{l=0}^{n-1} æ^l w^l \mathcal{L}_X A \ , \tag{4}$$

where $\mathcal{L}_X = i_X d + d i_X$ is the Lie derivative along X, and the real numbers $æ^l$, $l = 0,\ldots,n-1$ are the parameters of the transformation. The symmetries that include the higher orders of the CS operator are equivalent to the lower-order transformations and can be excluded from consideration. The variation of action (1) under transformation (4) has the form

$$\delta_X S = \sum_{l=0}^{n-1} æ^l \int d(i_X(w^l A \wedge MA) + \sum_{s=1}^{s=l} w^{s-1} A \wedge w^{l-s} \mathcal{L}_X MA) .$$

The symmetries in the form defiend by Eq. (4) is inconvenient for us, because it is difficult to keep track positivity of the corresponding Noether conserved quantities. To construct bounded conserved quantities, we chose another parameterization of the family of symmetries

$$\delta_X A = \sum_{i=1}^{r} \sum_{p=0}^{p_i - 1} æ_i^p w^p \mathcal{L}_X \xi_i + \sum_{j=1}^{s} \sum_{q=0}^{2q_i - 1} \hat{æ}_j^q w^q \mathcal{L}_X \zeta_j . \tag{5}$$

Here the parameters are real numbers $æ_i^p$ and $\hat{æ}_j^q$, and the notations

$$\xi_i = \Lambda_i A, \qquad \Lambda_i = \prod_{k=0 k\neq i}^{r} (w - \lambda_k)^{p_k} \prod_{j=1}^{s} (w^2 - 2\mathrm{Re}\omega_k w + |\omega_j|^2)^{q_j} , \tag{6}$$

$$\zeta_j = \Omega_j A, \qquad \Omega_j = \prod_{i=1}^{r} (w - \lambda_i)^{p_i} \prod_{k=0 k\neq j}^{s} (w^2 - 2\mathrm{Re}\omega_k w + |\omega_k|^2)^{q_k} .$$

have been used. The relationship between the parameters $æ_i^p, \hat{æ}_j^q$, and $æ^l$ is defined from the requirement of equality of the numerical coefficients at the corresponding powers of the operator w in the relation

$$\sum_{i=1}^{r} \sum_{p=0}^{p_i - 1} æ_i^p w^p \Lambda_i + \sum_{j=1}^{s} \sum_{q=0}^{2q_i - 1} \hat{æ}_j^q w^q \Omega_j = \sum_{l=0}^{n-1} æ^l w^l . \tag{7}$$

The one-to-one correspondence between the numbers $æ_i^p, \hat{æ}_j^q$, and $æ^l$ is a mere consequence of the Bezout lemma for the univariate polynomials $\Lambda_i(w)$ and $\Omega_j(w)$.

2. CONSERVED CURRENTS

According to the Noether theorem, the extension of the CS theory of order n admits the following n-parameter family of the conserved currents:

$$j_X = \sum_{k=0}^{n-1} æ^l (j_X)^l = \sum_{i=1}^{r} \sum_{p=0}^{p_i - 1} æ_i^p (j_X)_i^p + \sum_{j=1}^{s} \sum_{q=0}^{2q_j - 1} \hat{æ}_i^p (\hat{j}_X)_j^q . \tag{8}$$

Two different parameterizations of this family correspond to two different parameterizations of family of symmetries (4) and (5). The conservation laws under consideration have the following form:

$$(*j_X)^l = \frac{1}{2} i_X(w^l A \wedge *MA) + \frac{1}{2\gamma}\sum_{k=0}^{n} a_k t^{k,l}(A),$$

$$(*j_X)_i^p = \frac{1}{2} i_X(w^p \xi_i \wedge *MA) + \frac{a_n}{2\gamma}\sum_{l=0}^{p_i} \frac{p_i!(-\lambda_i)^{p_i-l}}{l!(p_i-l)!} t^{p,l}(\xi_i), \tag{9}$$

$$(*\hat{j}_X)_j^q = \frac{1}{2} i_X(w^q \zeta_j \wedge *MA) + \frac{a_n}{2\gamma}\sum_{l=0}^{q_j}\sum_{k=0}^{q_j-l} \frac{q_j!(-2\mathrm{Re}\,\omega_j)^k |\omega_j|^{2(q_j-l-k)}}{l!k!(q_j-l-k)!} t^{q,2l+k}(\zeta_j),$$

where we have used the notation $t^{k,l}(A) = \sum_{s=1}^{l-k} w^{k+s-1} A \wedge w^{l-s}\mathcal{L}_X A - \sum_{s=1}^{k-l} w^{k-s} A \wedge w^{l+s-1}\mathcal{L}_X A$.

As far as the stability is concerned, we are interested in the conserved quantities that are both related to the time translations and bounded from below. Such conserved quantities are generated by the family of conserved currents j_0 that are associated with the timelike Killing vector $X = \partial/\partial t$. The quantity bounded from below is controlled by the 0 component $j_{00} \equiv (j_0)_0$ of the conserved current j_0. The canonical energy is always reproduced by the family j_0, but is not bounded from below whenever $n > 2$.

Consider the problem of existence of the conserved quantities bounded from below that are defined by the family of conserved currents j_0. Since the sign of the conserved quantity coincides with that of j_{00}, it is necessary to identify the values of the parameters æ^l (or, equivalently, æ$_i^p$ and $\hat{æ}_j^q$) such that $j_{00} \geq 0$. There are two obvious cases when the extension of the CS theory admits conservation laws bounded from below:

i) The roots of the polynomial $M(w)$ are real and simple:

$$*j_X \approx \frac{a_n}{2\gamma}\sum_{i=1}^{n} æ_i^0 \xi_i \wedge \mathcal{L}_X \xi_i, \qquad j_{00} \approx \frac{a_n}{2\gamma^2}\sum_{i=1}^{n} æ_i^0 \lambda_i (\xi_i)_\mu (\xi_i)_\mu, \qquad æ_i^0 \geq 0. \tag{10}$$

ii) The polynomial $M(w)$ has the zero root $\lambda_1 = 0$ with the multiplicity $p_1 = 2$; all other roots are real and simple:

$$*j_0 \approx \frac{a_n}{2\gamma}\left\{\sum_{i=2}^{n-1} æ_i^0 \xi_i \wedge \mathcal{L}_X \xi_i + æ_1^0(\xi_1 \wedge w\mathcal{L}_X \xi_1 + w\xi_1 \wedge \mathcal{L}_X \xi_1)\right\},$$

$$j_{00} \approx \frac{a_n}{2\gamma^2}\left\{\sum_{i=2}^{n-1} æ_i^0 \lambda_i (\xi_i)_\mu (\xi_i)_\mu + æ_1^0 (w\xi_1)_\mu (w\xi_1)_\mu\right\}, \tag{11}$$

In formulas (10) and (11) the weak equality is understood modulo natural equivalence in the definition of the conserved current, and the summation over the repeated at the same level index $\mu = 0,1,2$ is assumed. If æ$_i^0 \geq 0$, the constructed conserved values are positive (that is, nonnegative and equal to zero only for pure gauge solutions to equations of motion (2)) for all possible values of the subscript i.

Let us prove that the extension of the CS theory admits no other positive conserved currents. Due to a smooth dependence of the conserved current on the parameters æ^l and roots λ_i and ω_i of the polynomial $M(w)$, the condition of existence of bounded conservation law defines an open subset in the space of parameters æ$^l, \lambda_i$, and ω_i. Thus, any point of this subset is contained in it with its neighborhood. The last requirement, however, is impossible in all cases except i) and ii). The presence of complex roots is excluded from the condition that each multiple complex root by a small change can be replaced by the appropriate number of simple complex roots, and the conserved currents associated with simple complex roots are unbounded from below (this was shown in [13]). The presence of multiple real roots is excluded due to the fact that one real root of multiplicity 2 can be replaced by a pair of complex conjugate roots. Finally, the case of zero root is special due to the presence of the gauge symmetry. Without breaking of the gauge symmetry, we can replace the zero root of multiplicity 3 by a pair of complex conjugate roots and a simple zero root. Thus, the multiplicity of zero root cannot exceed 2.

Using the similar considerations, we can show that for any bounded (but not necessarily positive) conserved current

$$\ae_i^0 = \hat{\ae}_j^0 = \hat{\ae}_j^1 = 0 \tag{12}$$

whenever $p_i > 1$ and $q_j > 0$; $p_i > 2$ if $\lambda_i = 0$. To prove that, one should consider the positivity of conserved current (8) on the particular solution such that only nonzero component $\xi_i(\zeta_j)$ among ξ and ζ obeys the equation

$$(w-\lambda_i)\xi_i = 0 \qquad \left((w^2 - 2\mathrm{Re}\,\omega_j w + |\omega_j|^2)\zeta_j = 0\right).$$

Families (10) and (11) generalize the families of the second-rank conserved tensors that have been found in the third-order extensions in [13]. Our result demonstrates that depending on the structure of the roots of polynomial (3), the positive conserved currents may exist in extensions of any order, even though the canonical energy for these models is unbounded. The existence of the positive conservation law is a sufficient condition for the stability only at the classical level. The quantum stability requires that the spectrum of the Hamiltonian has to be bounded from below. It is natural to expect that the classical Hamiltonian is also bounded from below. To identify the positive conservation law with the Hamiltonian of the theory, one should find the Lagrange anchor that relates the positive conservation law with the invariance of the theory with respect to the translations of time.

3. THE LAGRANGE ANCHORS

In case of linear variational theories, the Lagrange anchor is given by the formally self-adjoint operator acting on the space of fields (see [10], Appendix C). The extension of the CS theory of order n admits n-parameter family of the Lagrange anchors

$$V = \sum_{k=0}^{n-1} V^k (*\ae d)^k = \sum_{i=1}^{r}\sum_{p=0}^{p_i-1} V_i^p w^p \Lambda_i + \sum_{j=1}^{s}\sum_{q=0}^{2q_i-1} \hat{V}_j^q w^q \Omega_j,$$

where two sets of numbers, V^k and V_i^p and $\hat{V}_j^q$, correspond to two different parameterizations of this family. The terms that include the higher orders of the CS operator are equivalent to the lower-order Lagrange anchors and can be excluded from our consideration.

The Lagrange anchor defines the map from the space of conservation laws to the space of symmetries (for more details, see [14]). For conserved current (8), the corresponding symmetry reads

$$\delta_X^V A = VQ\mathcal{L}_X A,$$

with $Q_X \equiv \delta_X A = Q\mathcal{L}_X A$ being the characteristic of the conserved current j_X. Here we have used the notation

$$Q=\sum_{k=0}^{n-1}æ^l w^l=\sum_{i=1}^{r}\sum_{p=0}^{p_i-1}æ_i^p w^p\Lambda_i+\sum_{j=1}^{s}\sum_{q=0}^{2q_j-1}\hat{æ}_i^p w^q\Omega_j .$$

In particular, for the family j_0, which is associated with the Killing vector $X=\partial/\partial t$, the symmetry transformation reads

$$\delta_0^V A=\frac{\partial}{\partial t}\{VQA\}.$$

To relate the conserved current j_0 with the time translation, it is sufficient to impose the requirement that

$$VQ\approx 1, \tag{13}$$

where the weak equality denotes the equality modulo wave operator (3). Whenever the univariate polynomials $Q(w)$ and $M(w)$ are relatively prime, the existence of the Lagrange anchor V satisfying Eq. (13) is a consequence of the Bezout lemma. Conversely, if the polynomials $Q(w)$ and $M(w)$ have nonzero common divisor, the solution to Eq. (13) does not exist. When applied to the CS extension, this result suggests that the conservation law of j_0 from family (8) can be related with the time translation symmetry if and only if

$$æ_i^0\neq 0,\quad \hat{æ}_j^0\neq 0,\quad i=1,\dots,r,\quad j=1,\dots,s . \tag{14}$$

Conversely, if at least one coefficient among $æ_i^0$ and $\hat{æ}_j^0$ equals to zero, the conserved current cannot be related with the time translations. Requirement (14) is not compatible with condition (12) whenever the wave operator has the complex roots and multiple real roots with exception of the multiplicity 2 zero root. Therefore, any bounded conservation law that is related with the time translation has to be positive.

The following values of the parameters V_i^p and $\hat{V}_j^q$ define the Lagrange anchors that relate positive conservation laws (10) and (11) with the invariance of theory with respect to the time translations:

i) The roots of polynomial $M(w)$ are real and simple

$$V_i^0=\frac{1}{\Lambda_i^2(\lambda_i)æ_i^0},\qquad i=1,\dots,n . \tag{15}$$

ii) The polynomial $M(w)$ has the zero root $\lambda_1=0$ with the multiplicity $p_1=2$, given that all other roots are real and simple

$$V_1^1=-\sum_{k=2}^{n-1}V_k^0,\qquad V_i^0=\frac{1}{\Lambda_i^2(\lambda_i)æ_i^0},\qquad i=1,\dots,n-1 . \tag{16}$$

Relation (13) is verified explicitly considering that $\Lambda_i\Lambda_j\approx 0$ for all $i\neq j$. For this reason, any positive conservation law can be related with the time translation, and the extensions of the CS theory that are compatible with the existence of the positive conservation law should be considered as stable.

CONCLUSIONS

In the present work, we have demonstrated that the extension of the CS theory of order n admits n-parameter family of conservation laws. The procedure of construction of these conservation laws is explicitly covariant and admits generalization for the case of curved space-time. Depending on the structure of roots in polynomial (3), the positive tensors exist among the conserved quantities, while in the other cases, none of the conserved quantities is positive. The positive conservation laws are encountered in the extensions of any order, even though the canonical energy is unbounded in all the extensions with higher derivatives. By means of the Lagrange anchor concept, we proved the relationship between the positive conservation laws and interpret this result as the stability of the theory. As shown in [15], any Lagrange anchor, which is admissible by the equations of motion, ensures existence of the Hamiltonian form of the equations of motion. The Hamiltonian is then the conserved value that is related to the invariance of the theory with respect to the time translations by the Lagrange anchor. The nonequivalent Lagrange anchors define the canonically nonequivalent Poisson brackets. If the equations of motion admit several nonequivalent Lagrange anchors, the theory is multi-Hamiltonian, and this takes place in the considered model. We should also mention that the existence of different Lagrange anchors implies the existence of different Peierls brackets [16], even though the model has one action. The other important statement is that the Lagrange anchor allows to systematically include the interactions that are compatible with the stability condition [17]. This observation makes it possible to construct the stable interactions in the extension of the CS theory with higher derivatives.

The authors thank A. A. Sharapov for useful discussions. The work was partially supported by the Tomsk State University Competitiveness Improvement Program and by the Russian Foundation for Basic Research (Grant No. 16-02-00284). S. L. Lyakhovich was partially supported by the Russian Foundation for Basic Research (Grant No. 14-01-00489).

REFERENCES

1. P. K. Townsend, K. Pilch, and P. van Nieuwenhuizen, Phys. Lett., **B136**, 38–42 (1984).
2. S. Deser and R. Jackiw, Phys. Lett., **B139**, 371–373 (1984).
3. S. Deser, R. Jackiw, and S. Templeton, Ann. Phys., **140**, 372–411 (1982).
4. S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett., **48**, 975–978 (1982).
5. S. Deser and R. Jackiw, Phys. Lett., **B451**, 73–76 (1999).
6. B. Podolsky, Phys. Rev., **62**, 68–71 (1942).
7. P. O. Kazinski, S. L. Lyakhovich, and A. A. Sharapov, J. High Energy Phys., **0507**, 076 (2005).
8. K. Bolonek and P. Kosinski, Acta Phys. Polon., **B36**, 2115–2131 (2005).
9. E. V. Damaskinsky and M. A. Sokolov, J. Phys., **A39**, 10499 (2006).
10. D. S. Kaparulin, S. L. Lyakhovich, and A. A. Sharapov, Eur. Phys. J., **C74**, 3072 (2014).
11. D. S. Kaparulin and S. L. Lyakhovich, Russ. Phys. J., **57**, No. 11, 1261–1265 (2015).
12. I. Masterov, Nucl. Phys., **B902**, 95–114 (2015).
13. D. S. Kaparulin, I. Yu. Karataeva, and S. L. Lyakhovich, Eur. Phys. J., **C75**, 552 (2015).
14. D. S. Kaparulin, S. L. Lyakhovich, and A. A. Sharapov, J. Math. Phys., **51**, 082902 (2010).
15. D. S. Kaparulin, S. L. Lyakhovich, and A. A. Sharapov, J. Geom. Phys., **74**, 164–184 (2013).
16. A. A. Sharapov, Int. J. Mod. Phys., **A29**, 145057 (2014).
17. D. S. Kaparulin, S. L. Lyakhovich, and A. A. Sharapov, J. Phys., **A49**, 155204 (2016).