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THE HAMILTON EQUATIONS OF FERROHYDRODYNAMICS WITH THE LANDAU–LIFSHITZ EQUATION FOR MAGNETIZATION

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A system of Hamilton equations of motions for the ideal nonconducting magnetic fluid with the Landau–Lifshitz equation for the magnetization vector is constructed based on the functional of total energy derived in this work and the Poisson bracket method.

Keywords: the Poisson bracket, ferrofluid, effective field, magnetization relaxation.

INTRODUCTION

A new section of hydrodynamics – ferrohydrodynamics has appeared quite recently [1, 2]. The subject of its investigation is a magnetic fluid or ferrofluid – an artificially synthesized medium which is a colloid solution of nanoparticles of a solid magnetic material with a size of 10 nm in the carrier fluid. In contrast to magnetic hydrodynamics [3] which studies the interaction between magnetic fields and conducting fluids, the majority of magnetic fluids are synthesized based on liquid dielectrics and do not conduct electric current.

The purpose of this work is to derive Hamilton equations of ideal ferrohydrodynamics with the Landau–Lifshitz equation for the magnetization vector.

In Section 1, for construction of the Hamilton system a functional of the total energy of the system is developed and the choice of physical fields in whose phase space the Hamilton equations of motion of ferrohydrodynamics are found is substantiated.

In Section 2, the equations of motion of a non-conducting magnetic fluid in which specific magnetization evolution occurs according to the Landau–Lifshitz equation are derived. Based on linearized equations, the spectrum of eigenmodes of the system is investigated. The results of our work are discussed in Section 3.

1. THE ENERGY FUNCTIONAL AND A CHOICE OF PHYSICAL FIELDS

Using the first law of thermodynamics we concreticize a potential part of the Lagrangian of the system. Each physically infinitesimally small element of the volume of magnetic fluid is in the magnetic field determined by the distribution of magnetization in the medium and by the external field [4]. On the other hand, a complete system of equations of an ideal material in ferrohydrodynamics must contain a continuity equation expressing the law of conservation of mass and the adiabatic condition of motion of the fluid. Therefore, we should choose the density $\rho = \rho(\boldsymbol{x},t)$, specific entropy $s = s(\boldsymbol{x},t)$, vector of specific magnetization $\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{x},t)$, and velocity of the liquid $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x},t)$ as physical variables describing the motion of the fluid.

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For the medium considered here the initial equation for changing of volumetric energy density without hysteresis can be written in the form [5]

$$dw = d\varepsilon + d\left(-\boldsymbol{MH} - \frac{\boldsymbol{H}^2}{8\pi}\right), \tag{1}$$

where $\varepsilon = \varepsilon(\rho, s, m)$ is the internal energy density, $\boldsymbol{M} = \rho\boldsymbol{m}$ is the magnetization, and $\boldsymbol{H}$ is the magnetic field strength.

Thus, the functional of the total energy of a magnetized non-conducting fluid is as follows

$$W(\rho, s, \boldsymbol{m}, \boldsymbol{v}) = \int d\boldsymbol{x}\left[\rho\frac{v^2}{2} + \rho\psi(\rho, s, \boldsymbol{m}) - \rho(\boldsymbol{mH}) - \frac{\boldsymbol{H}^2}{8\pi}\right]. \tag{2}$$

To formulate the Hamilton complete system of equations in ferrohydrodynamics, in addition to Eq. (2), a continuity equation and an adiabatic condition, an equation of magnetization evolution is necessary. To substantiate this equation for the ideal non-conducting magnetic fluid we assume a condition of the system conservatism, that is, dissipative processes in the first approximation are neglected [6, 7].

Thus, for all models of the ideal non-conducting ferrohydrodynamics the energy functional (2), a continuity equation, an adiabatic condition, and the Maxwell equations where bias current is neglected are general. In this case, for a medium with an arbitrary relation between induction $\boldsymbol{B}$ and intensity $\boldsymbol{H}$ of the magnetic field the Maxwell magnetostatic equations can be written as

$$\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M}, \quad \boldsymbol{H} = -\mathrm{grad}\varphi, \tag{3}$$

$$\Delta\varphi = 4\pi\mathrm{div}\boldsymbol{M}, \tag{4}$$

where the scalar potential φ of the magnetic field is introduced. The difference between possible models is in the law of magnetization evolution and, hence, in the Euler equation.

2. A NONCONDUCTING MAGNETIC FLUID WITH THE LANDAU-LIFSHITZ EQUATION FOR MAGNETIZATION EVOLUTION

Here we obtain the Hamilton system of equations of motion for a non-conducting magnetic fluid with the Landau–Lifshitz equation for magnetization evolution.

The calculations presented below are based on formula (2) for the Hamiltonian of ferrohydrodynamics and on the following properties of the Poisson bracket following from its definition [6–8]:

$$\{F_1F_2, F_3\} = F_1\{F_2, F_3\} + F_2\{F_1, F_3\}, \tag{5}$$

$$\{G(F_1, ..., F_k, ...F_n), F_k\} = \sum_{i=1}^{n}\int d\boldsymbol{x}'\frac{\delta G}{\delta F_i(\boldsymbol{x}')}\{F_i(\boldsymbol{x}'), F_k\}. \tag{6}$$

Thus, to formulate the Hamilton equations of motion, it is necessary to calculate the mutual Poisson brackets for the values, whose functional is the system energy.

The equations of motion of liquid ferromagnetic with infinite conductivity and spectra of low-amplitude oscillations of such microscopically ferromagnetic fluid were investigated in [9]. The Hamilton form of the system of equations of motion for ferromagnetic fluid presented in [9] was found in [10]. The problem is discussed in detail for example, in [7].

Generally, we assume that $s = s_0 = \text{const}$ that is, entropy is not considered as a dynamical variable. In this case, the density of the fluid is a solution to the continuity equation, the specific magnetization vector is changed according to the Landau–Lifshitz equation, and the total energy of the system is set by formula (2). The explicit form of the mutual Poisson brackets of interest for physical variables has the following form [8,10]:

$$\left\{\pi_i, \pi_k'\right\} = \partial'\left(\pi_k'\delta\right) - \partial_k\left(\pi_i\delta\right),\ \left\{\rho, \pi_k'\right\} = -\partial_k\left(\rho\delta\right), \tag{7}$$

$$\{M_i, \rho'\} = \{\rho, \rho'\} = 0, \tag{8}$$

$$\left\{M_i, M_k'\right\} = \gamma\varepsilon^{ike} M_e \delta(x' - x),\ \left\{M_i, \pi_k'\right\} = -\partial\left(M_i\delta\right). \tag{9}$$

The Poisson brackets of the arbitrary functional F with the Hamiltonian of the system W can be written as

$$\begin{gathered}\{F, W\} = \int d\boldsymbol{x}\left\{\frac{\delta F}{\delta\pi_i}\left[\{\pi_i, \pi_k\}\frac{\delta W}{\delta\pi_k} + \{\pi_i, \rho\}\frac{\delta W}{\delta\rho} + \left\{\pi_i, M_\beta\right\}\frac{\delta W}{\delta M_\beta}\right]\right. \\ \left. + \frac{\delta F}{\delta\rho}\{\rho, \pi_k\}\frac{\delta W}{\delta\pi_k} + \frac{\delta F}{\delta M_\alpha}\left[\{M_\alpha, \pi_k\}\frac{\delta W}{\delta\pi_k} + \left\{M_\alpha, M_\beta\right\}\frac{\delta W}{\delta M_\beta}\right]\right\},\end{gathered} \tag{10}$$

and for the density of hydrodynamical forces, we obtain the formula

$$F_i = \int d\boldsymbol{x}'\left[\frac{\delta U}{\delta\rho'}\{\pi_i, \rho'\} + \frac{\delta U}{\delta M_n'}\left\{\pi_i, M_n'\right\}\right], \tag{11}$$

where U is the potential part of the total energy of the system. Formulas (8)–(11) result in the Hamilton system of equations of motion for a non-conducting magnetic fluid where the evolution of specific magnetization occurs according to the Landau–Lifshitz equation

$$\frac{\partial\rho}{\partial t} = \{W, \rho\} = -\frac{\partial}{\partial x_k}(\rho v_k), \tag{12}$$

$$\frac{\partial M_i}{\partial t} = \{W, M_i\} = -\frac{\partial(M_i v_k)}{\partial x_k} + \gamma\varepsilon^{ike} H_k^{\text{eff}} M_e, \tag{13}$$

$$\rho\left[\frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial x_k}\right] = -\frac{\partial p}{\partial x_i} + (\boldsymbol{M}\nabla)H_i + \frac{\partial}{\partial x_i}\left(\boldsymbol{M}(\boldsymbol{H}^{\text{eff}} - H)\right), \tag{14}$$

where ε^{ike} is a completely antisymmetric unit tensor and $\boldsymbol{H}^{\text{eff}}$ is the strength of the effective magnetic field [4].

In terms of the system of equations (12)–(14), we consider the propagation of small perturbations in the model of a non-conducting magnetic fluid with the Landau–Lifshitz equation. In the stationary ferrofluid,

$$\boldsymbol{B}_0 = \boldsymbol{H}_0 + 4\pi\boldsymbol{M}_0,\ \boldsymbol{M}_0 \uparrow\uparrow \boldsymbol{H}_0, \tag{15}$$

where $\boldsymbol{H}_0$ is the constant magnetic field.

We now consider the perturbations of the density, magnetic field strength, and specific magnetization relative to their equilibrium values, assuming that

$$\rho = \rho_0 + \rho', \; \boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{h}, \; \boldsymbol{M} = M_0 + \boldsymbol{m}^э \tag{16}$$

with allowance for the exchange interaction, the effective magnetic field has the form [9]

$$\boldsymbol{H}_{\text{eff}} = \alpha \Delta M + \boldsymbol{H}, \tag{17}$$

where α is the exchange interaction constant. In Eqs. (12)–(14), we neglect small values whose order is higher than the first order, considering the velocity $\boldsymbol{v}$ to be a value of the same order as ρ', $\boldsymbol{h}$ and $\boldsymbol{m}'$. Since the absolute value of the specific magnetization is conserved, the condition $\boldsymbol{m}' \perp M_0$ must be also fulfilled.

As a result, a linerized system of equations, whose solution can be found in the form of plane waves,

$$\rho' = \rho_0' e^{-i(\omega t - \boldsymbol{kr})}, \; \boldsymbol{v} = \boldsymbol{v}_0 e^{-i(\omega t - \boldsymbol{kr})},$$

$$\boldsymbol{m}' = \boldsymbol{m}_0 e^{-i(\omega t - \boldsymbol{kr})}, \; \boldsymbol{h} = \boldsymbol{h}_0 e^{-i(\omega t - \boldsymbol{kr})},$$

is reduced to system of algebraic equations

$$\boldsymbol{v}_0 = s^2 \frac{(\boldsymbol{v}_0 \boldsymbol{k})}{\omega^2} \boldsymbol{k} - (\boldsymbol{M}_0 \boldsymbol{h}_0) \frac{\boldsymbol{k}}{\omega \rho_0}, \tag{18}$$

$$i\omega \boldsymbol{m}_0 + i\boldsymbol{M}_0 (\boldsymbol{kv}) = \gamma \left[\boldsymbol{m}_0, \boldsymbol{H}_0 + \alpha k^2 \boldsymbol{M}_0 \right] + \gamma [\boldsymbol{M}_0, \boldsymbol{h}_0], \tag{19}$$

where s is the velocity of sound in a nonmagnetized fluid. Equation (18) shows that $\boldsymbol{v}_0 \parallel \boldsymbol{k}$. Further, we direct the Ox axis parallel to the vectors $\boldsymbol{k}$, $\boldsymbol{v}_0$, and $\boldsymbol{h}_0$, and choose the vector $\boldsymbol{M}_0$ on the plane xOy expressing the angle between the Ox axis and $\boldsymbol{M}_0$ as θ. From the compatibility condition of the system of homogeneous linear equations derived, we obtain the dispersion equation

$$\omega^4 - \omega^2 \left[\omega_S^2 + \Omega^2 \right] + \omega_S^2 \Omega^2 - \omega_0^4 = 0, \tag{20}$$

where we introduce the designations

$$\omega_S^2 = \gamma^2 \cos^2 \theta \left(H_0 + \alpha k^2 M_0 \right) + \gamma^2 \sin^2 \theta \left(H_0 + \alpha k^2 M_0 \right) \left(B_0 + \alpha k^2 M_0 \right), \tag{21}$$

$$\Omega^2 = S^2 k^2 + (M_0 \cos \theta)^2 \frac{4\pi k^2}{\rho_0}, \tag{22}$$

$$\omega_0^4 = \gamma^2 \cos^2 \theta \left(\alpha k^2 + \frac{H_0}{M_0} \right) \left(4\pi M_0^2 \right) (M_0 \cos \theta)^2 \frac{4\pi k^2}{\rho_0}. \tag{23}$$

Equation (22) has two solutions

$$\omega = \frac{1}{\sqrt{2}} \left[\omega_S^2 + \Omega^2 + \sqrt{\left(\omega_S^2 - \Omega^2 \right)^2 + 4\omega_0^4} \right]^{1/2}, \tag{24}$$

$$\omega = \frac{1}{\sqrt{2}}\left[\omega_S^2 + \Omega^2 - \sqrt{\left(\omega_S^2 - \Omega^2\right)^2 + 4\omega_0^4}\right]^{1/2}. \quad (25)$$

Formula (24) determines the law of dispersion of magnons in a compressible non-conducting magnetic fluid. In the extreme case of incompressible fluid, the law of dispersion of spin waves in cubic crystals follows from Eq. (24), if the magnetic anisotropy energy is neglected (formula (70.11) in [4]).

Formula (25) determines the law of dispersion of a longitudinal sound wave with allowance for the influence of the external magnetic field $\boldsymbol{H}_0$ and exchange interaction. If the influence of the magnetic field $\boldsymbol{h}$ on the system dynamics due to fluctuations of the magnetization vector is disregarded, the velocity of the sound wave is found from Eq. (25) as

$$\upsilon \cong S\sqrt{1 + (M_0 \cos\theta)^2 \frac{4\pi}{s^2 \rho_0}}, \quad (26)$$

where θ is the angle between the direction of the external field strength and that of the wave propagation.

A similar result was obtained earlier within the framework of quasi-stationary ferrohydrodynamics in [11].

CONCLUSION

Currently the correct use of equations of magnetization relaxation in ferrohydrodynamics has been fruitfully discussed [12, 13]. Recent experiments with rotations of a magnetic fluid in the constant external field show significant disagreement of model calculations not only with each other but also with experimental results [13]. The models with viscous mechanism of relaxation fail to account for one more experimental result – the effect of magnetic anisotropy of the velocity of sound propagation in the ferrofluids [14]. It is possible that the real behavior of ferrofluids requires new ideas, for example, consideration of the fact that the magnetization vector is not always related closely to the particle itself.

In this case, of interest can be a system of equations of motion of the ideal non-conducting magnetic fluid with the Landau–Lifshitz equation for the magnetization vector allowing for the dissipative term in the form suggested by Landau and Lifshitz obtained in this work [15]. Applicability of the latter for ferrofluids with the Néel mechanism of relaxation is physically justified.

If one assumes existence of a microscopically ferromagnetic non-conducting fluid which corresponds to a dielectric analog of the model discussed in [9], the flow of this fluid must also be described within the derived system (12)–(14).

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REFERRENCES

1. R. E. Rosensweig, Ferrohydrodynamics, Cambridge Univ. Press, New York, 1985.
2. V. V. Sokolov and V. V. Tolmachev, Magnetic Hydrodynamics, **32**, No. 3, 318 (1996).
3. L. D. Landau and E. M. Lifshitz, Electrodynamics of Continua [in Russian], Nauka, Moscow, 1992.
4. L. D. Landau and E. M. Lifshitz, Statistical Physics, P. 2 Condensed State Theory [in Russian], Nauka, Moscow, 1978.
5. I. E. Tamm, Fundamentals of the Theory of Electricity [in Russian], Nauka, Moscow, 1988.
6. V. E. Zakharov and E. A. Kuznetsov, Usp. Fiz. Nauk., **167**, No. 11, 1137 (1997).
7. V. P. Goncharov and V. I. Pavlov, The Hamilton Vortex and Wave Dynamics [in Russian], GEOS, Moscow, 2008.
8. I. E. Dzyaloshinskii and G. E. Volovic, Ann. Phys., **125**, 67 (1980).

9. I. A. Akhiezer and I. T. Akhiezer, Fiz. Tv. Tela, **26**, No. 2, 453 (1984).
10. D. D. Holm and B. A. Kupershmidt, Phys. Lett., **91A**, No. 9, 425 (1982); **129A**, No. 2, 93 (1988).
11. F. V. Bunkin, A. I. Lipkin, and G. A. Lyakhov, Pis'ma Zh. Teor. Fiz, Vol. 9, No. 12, 714 (1983).
12. B. U. Felderhof, Phys. Rev., **E62**, 3848 (2000); **64**, 063502 (2001).
13. J. P. Embs, S. May, and C. Wagner, Phys. Rev., **E73**, 063502 (2006).
14. T. Sawada, H. Nishiyama, and T. Tabata, JMMM, **252**, 186 (2002).
15. L. D. Landau, in: Collection of Works, Vol. 1, Nauka, Moscow, 1969.