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ELECTRIC FIELD STRENGTH AT AND NEAR THE CATHODE EDGE IN A MAGNETICALLY INSULATED COAXIAL DIODE

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For an annular cathode in a coaxial diode it has been shown that the averaged electric field strength at the end face of the cathode, E_n, *depends on the edge thickness* h *as* $E_n \propto 1/\sqrt{h}$. *It has been found that the field strength varies with distance from the edge approximately as* $\propto 1/\sqrt{r}$. *The problem of the electric field strength at the edge of the cathode in a magnetically insulated coaxial diode has been solved for the case where the cathode emissivity is limited with the use of a model assuming a given internal resistance of the voltage source.*

INTRODUCTION

High-current annular electron beams are known to be used in relativistic microwave electronics. To produce an electron beam of this type, the thickness of the cathode tube should be made small enough to provide a necessary electric field strength for explosive emission to develop at the cathode edge. It is generally believed that once explosive emission has developed, the cathode has an unlimited emissivity, and this results in that the electric field strength at the cathode edge drops to zero [1]. However, it is of interest to know the field strength at the cathode edge and the field distribution near the edge before the development of explosive emission when the field is still not substantially distorted by the space charge, i.e. to solve an electrostatic problem. This problem can be solved in good approximation analytically. In the present work, for this purpose two methods have been used: one supposing the conservation of the longitudinal component of the electromagnetic field momentum and the other seeking an approximate solution of the Laplace equation. Based on the solution of the electrostatic problem, the problem of incomplete screening of the electric field at the cathode edge in the case of limited emissivity of the cathode has been considered. For this case, an averaged electric field strength at the cathode edge has been derived.

1. THE LAW OF CONSERVATION OF THE *Z*-COMPONENT OF THE FIELD MOMENTUM IN A COAXIAL DIODE

Let us consider a coaxial diode (Fig. 1) with cathode and anode radii R_c and R_a, respectively, and with cathode tube thickness h_c to which a potential difference U is applied.

The potential inside the coaxial diode obeys the Laplace equation (1), which, in cylindrical coordinates, has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}+\frac{\partial^2 \varphi}{\partial z^2}=0\,, \tag{1}$$

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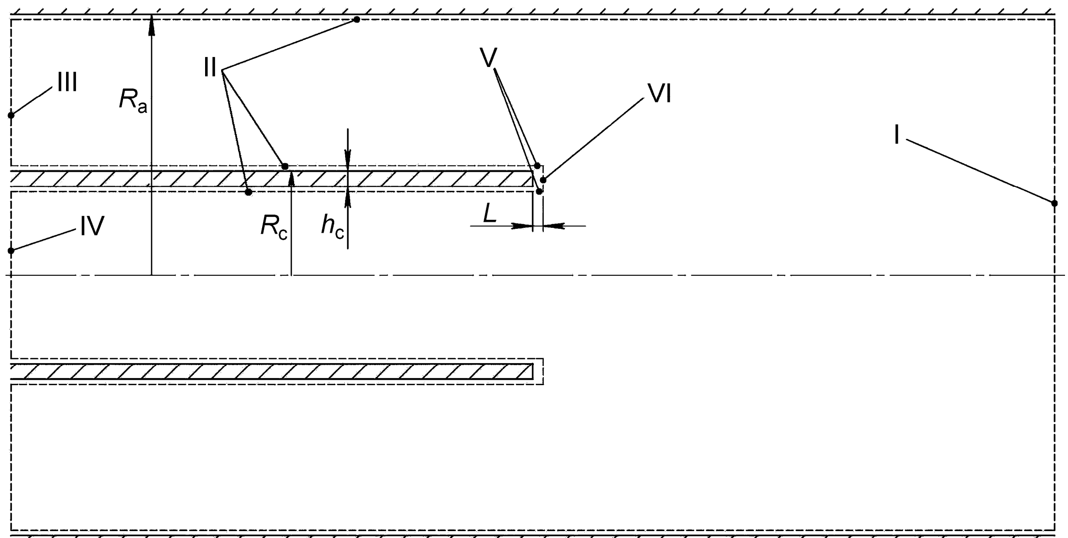

Fig. 1. System of conductors of a coaxial diode.

which, by virtue of the azimuthal symmetry, does not contain the term $\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$. If we multiply this equation by $-\frac{1}{4\pi}\frac{\partial\varphi}{\partial z}$, it is reduced to the following one:

$$\operatorname{div}\mathbf{\Sigma} = 0\,, \tag{2}$$

where the components of the vector $\mathbf{\Sigma}$ are given by

$$\Sigma_z = \frac{1}{8\pi}\left[\left(\frac{\partial\varphi}{\partial r}\right)^2 - \left(\frac{\partial\varphi}{\partial z}\right)^2\right] = \frac{E_r^2 - E_z^2}{8\pi},$$

$$\Sigma_r = -\frac{1}{4\pi}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial z} = -\frac{E_r E_z}{4\pi}\,, \tag{3}$$

$$\Sigma_\theta = 0\,.$$

Integrating equation (2) throughout the volume bounded by the surface S, depicted by the dashed line in Fig. 1, and going, following the Ostrogradskii–Gauss theorem, to a surface integral, we write

$$\oint_S \mathbf{\Sigma}\, d\mathbf{s} = 0\,. \tag{4}$$

Let us elucidate the physical meaning of the vector $\mathbf{\Sigma}$. Consider the z-components of a Maxwellian stress tensor in cylindrical coordinates for a region possessing azimuthal symmetry [2]:

$$\sigma_{zz} = \frac{E_r^2 - E_z^2}{8\pi},$$

$$\sigma_{zr} = -\frac{E_r E_z}{4\pi}, \quad (5)$$

$$\sigma_{z\theta} = \frac{E_r E_\theta}{4\pi} = 0.$$

Thus, the vector $\mathbf{\Sigma}$ is a vector formed of the z-components of the tensor of an electromagnetic field momentum flux density for an azimuthally symmetric system in cylindrical coordinates [3]. Equation (4) can be rewritten as [3]

$$-\frac{dG_z}{dt} = \oint_S \mathbf{\Sigma}\, d\mathbf{s} = 0\,, \quad (6)$$

where G_z is the z-component of the field momentum in the volume bounded by the surface S. Obviously, this equation expresses the law of conservation of the z-component of the field momentum in the volume under consideration.

Let us now consider the derivation of the expression for the field strength at the end face of an edged cathode based on the conservation law (4) for two shapes of the end face: plane and semicircular.

2. THE AVERAGED FIELD STRENGTH AT THE END FACE OF A CATHODE

Let us evaluate the integral (4) over the surface S for the case where the end face of the cathode is a plane surface, as shown in Fig. 1. Surfaces I and IV are chosen so far from the cathode end face that the field at them is equal to zero, i.e. $\Sigma = 0$. At surfaces II, which are the surfaces of the conductors, we have $E_z = 0$, and, hence, $\Sigma\, d\mathbf{s} = \Sigma_r ds = 0$. The nonzero contribution to the integral (4) is made by surfaces III, V, and VI. We are interested in the case where the length L of surface V (see Fig. 1) tends to zero. The flux of the vector $\mathbf{\Sigma}$ through this surface is proportional to the integral $\int_0^L E_r E_z dl$. The relation between the field strength and the distance l from the right angle of the conductor, as follows from the solution of Problem 3, Sec. 3 in Ref. 4, for small l has the form

$$E(l) \propto l^{-1/3}.$$

Thus, the integral $\int_0^L E_r E_z dl$ is proportional to $L^{1/3}$, so that the flux of the vector $\mathbf{\Sigma}$ through surface V also tends to zero with L as $L^{1/3}$. The flux of the vector $\mathbf{\Sigma}$ through surface III can be written as

$$\int_{\mathrm{III}} \Sigma\, d\mathbf{s} = -\frac{U^2}{4 \ln R_\mathrm{a}/R_\mathrm{c}}, \quad (7)$$

and the flux of the vector $\mathbf{\Sigma}$ through surface IV (the cathode end face) with L tending to zero in the form

$$\int_{\mathrm{VI}} \Sigma\, d\mathbf{s} = \frac{1}{8\pi} \int_{R_\text{к} - h_\text{к}}^{R_\text{к}} \left[E_z(r)\big|_{z=+0} \right]^2 2\pi r dr\,. \quad (8)$$

From here, in view of equation (4), we obtain for the field strength at the cathode end face averaged over the surface

$$\left\langle E_z(r)\big|_{z=+0}\right\rangle = -\left(\frac{U^2}{\left(R_c h_c - h_c^2/2\right)\ln R_a/R_c}\right)^{1/2} \underset{h_c \ll R_c}{\approx} E_c\sqrt{\ln(R_a/R_c)\frac{R_c}{h_c}}\,, \tag{9}$$

where $E_c = -\dfrac{U}{R_c \ln R_a/R_c}$ is the field at the exterior cylindrical surface of the cathode far from the end face.

If the cross-section of the cathode end face in Fig. 1 ends not with a vertical line but with a semicircle of radius $r_c = h_c/2$, surface V–VI should also be chosen as a semicircle of radius $r = r_c$, and the flux of the vector $\mathbf{\Sigma}$ through this surface is approximately given by

$$\int\limits_{\text{V–VI}} \Sigma\, d\boldsymbol{s} \underset{r_c \ll R_c}{\approx} \frac{1}{4} R_c r_c \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} E_n^2(\theta)\, d\theta\,, \tag{10}$$

where $E_n = \sqrt{E_z^2 + E_r^2}$ is the normal component of the field strength vector at the surface of the semicircle. The polar angle θ is measured counterclockwise from the middle of the semicircle. Thus formula (9) will be rewritten as

$$\left\langle E_n(\theta)\right\rangle \underset{r_c \ll R_c}{\approx} E_c\sqrt{\ln(R_a/R_c)\frac{2R_c}{\pi h_c}}\,. \tag{11}$$

The dependence of the averaged field strength at the edge on h_c as $\propto 1/\sqrt{h_c}$ holds not only for a coaxial diode. Based on the conservation law for the *z*-component of the field momentum, it can be shown that the averaged field strength at the flat end face of a semi-infinite plane capacitor (strip line) can be given by

$$\left\langle E_z\big|_{z=+0}\right\rangle \underset{h_c \ll d}{\approx} E_0\sqrt{\frac{d}{2h_c}}\,, \tag{12}$$

where $E_0 = \dfrac{U}{d}$ is the field strength inside the capacitor far from its edges. Thus, the relation $\left\langle E_z\big|_{z=+0}\right\rangle \propto 1/\sqrt{h_c}$ seams to be considered general for the whole class of thin-edged conductors when the problem on the potential distribution near the edge can be reduced to a local two-dimensional problem. Such a problem is solved below.

3. THE FIELD STRENGTH AT A DISTANCE FROM THE EDGE OF A CATHODE

For the electric field strength in a coaxial diode at some distance from the edge $\gg h_c$, we can use an approximate solution of the Laplace equation for a potential. It corresponds to the solution of the problem on the potential distribution near the edge of a plane blade [4]:

$$\varphi \approx \text{const}\sqrt{r}\sin\frac{\theta}{2}\,. \tag{13}$$

This solution is the first term of the series being a solution of the Laplace equation. Formula (13) is written in polar coordinates r, θ for a plane perpendicular to the border of the edge. The angle θ is measured from the surface of the blade and r from the edge of the blade.

Expression (13) approximately describes the behavior of the potential in some range of r values outside which it has no physical meaning. The bottom limit is related to the thickness of the edge: $r \gg h_c$, and it corresponds to the

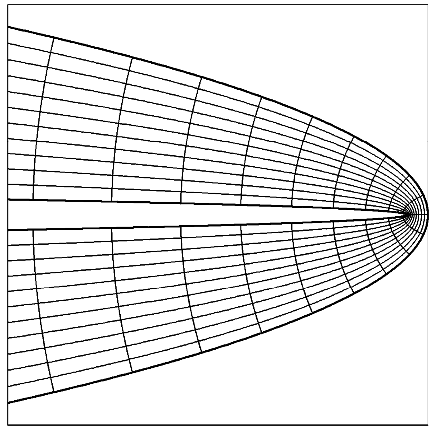

Fig. 2. The family of force and equipotential lines of the solution (13) between two charged confocal parabolas.

approximation of an infinitely thin edge. The limit superior for r can be written as $r \ll D$, where D is the distance from a point at the border of the edge to the nearest charged body. If there is no charged body in the neighborhood of the edge, i.e. the field of a solitary charged conductor, such as, a thin plate, is considered, the characteristic size of the conductor, such as the plate width or length, is taken for the parameter D. The limit superior for r can also be imposed by the radius of curvature R_c of the edge: $r \ll R_c$.

The electric field strength, according to (13), depends on r as $E \propto 1/\sqrt{r}$. It should be noted that the apparent problem of the field strength singularity at the point $r = 0$ disappears by itself since this point is outside the domain of applicability of the solution (13).

Expression (13) for a potential is valid for all thin-edged conductors [4] if a range of values $h_c \ll r \ll \{D, R_c\}$ does exist. The value of the constant in equation (13) can be found by solving the problem for the field as a whole and expanding it in a power series of r.

The family of equipotential lines of the general solution of (13) represents confocal parabolas (Fig. 2) and describes a plane field. Therefore, the field between two confocal parabolas (heavy lines in Fig. 2) to which the potential difference is applied will exactly obey the solution of (13).

If for a coaxial diode there exists a range of distances from the cathode edge $h_c \ll r \ll \{R_c, R_a - R_c\}$, the potential in this range is approximately described by expression (13). The value of the constant in this expression can be determined from the conservation law (4). To evaluate the integral in (4), it is necessary to choose surface V–VI having the shape of a torus [5] whose greater radius is the radius of the cathode, R_c, and the smaller radius $r_0 \ll \{R_c, R_a - R_c\}$. The flux of the vector **Σ**, defined by relation (13), through the surface of the torus is given by

$$\int_{\mathrm{V-VI}} \Sigma \, d\boldsymbol{s} = \frac{\pi R_c \, \mathrm{const}^2}{8} \tag{14}$$

and it does not depend on r_0. Proceeding from the equality of the sum of expressions (7) and (14) to zero, we obtain an expression for the potential:

$$\varphi \approx -E_c \sqrt{\ln(R_a/R_c)\frac{2R_c r}{\pi}} \sin\frac{\theta}{2}, \tag{15}$$

where $E_c = -\dfrac{U}{R_c \ln R_a / R_c}$ is the field at the cathode far from the end face.

The electric field strength corresponding to the potential (13) is given by

$$\begin{aligned} E_r &\approx E_c \sqrt{\ln(R_a/R_c)\frac{R_c}{2\pi r}} \sin\frac{\theta}{2}, \\ E_\theta &\approx E_c \sqrt{\ln(R_a/R_c)\frac{R_c}{2\pi r}} \cos\frac{\theta}{2}. \end{aligned} \tag{16}$$

Since a strong guide magnetic field is used in a coaxial diode, we write the expression for the z-component of the electric field strength at $r \approx R_c$. Sewing together solutions (16), (9) and (16), (11), we set, in local polar coordinates, $r = z$ and $\theta = \pi$. Then, in cylindrical coordinates, we can write the following expressions:

$$E_z(r \approx R_c, z) \approx E_c \sqrt{\ln(R_a/R_c)\frac{R_c}{h_c}} \frac{1}{\sqrt{1 + 2\pi z/h_c}} \tag{17}$$

for the case that the edge end face is flat and

$$E_z(r \approx R_c, z) \approx E_c \sqrt{\ln(R_a/R_c)\frac{2R_c}{\pi h_c}} \frac{1}{\sqrt{1 + 4z/h_c}} \tag{18}$$

for the case that the edge end face is rounded for $0 \le z << \{R_c, R_a - R_c\}$ and for some r lying in the range $R_c - h_c < r < R_c$. Since this r is not known, the writing $r \approx R_c$ is used in formulas (17), (18). The solution of the electrostatic problem in the form of (17) rather well agrees with the prediction of a numerical simulation.

4. THE AVERAGED FIELD STRENGTH AT THE END FACE OF A CATHODE UNDER THE PASSAGE OF CURRENT WITH MAGNETIC INSULATION

In the case of the current passage in a magnetically insulated coaxial diode, the electric field strength at the end face of the cathode depends on the emissivity of the cathode. As shown by Fedosov and co-authors [1], if the cathode emissivity is unlimited, the field at the cathode edge vanishes; the corresponding current has received the name Fedosov's current. The authors of Ref. 5 contend that in the case of an arbitrary emissivity, the field at the cathode does not vanish, but increases unrestrictedly as h_c tends to zero. In our opinion, this approach is wrongful since the procedure of mathematical trending of h_c to zero is made. It would be more justified to choose the edge thickness h_c to be greater than the "physically infinitesimal" linear dimensions of the volumes over which microscopic quantities are averaged [4].

With this limitation for h_c the averaged field strength at the cathode edge always has a finite value. An expression for it can be derived by using the laws of conservation for the z-component of the field momentum and for the beam electrons [3]:

$$\oint_S \Sigma\, d\mathbf{s} = \int_V (E_z \rho)\, dV, \tag{19}$$

where ρ is the volumetric charge density in the beam and S is the surface that confines the volume V (it is depicted by the dashed line in Fig. 1).

Formulas (9) and (11) for the averaged field strength at the cathode edge correspond to no current passage in the coaxial diode, i.e. to the zero emissivity of the cathode. The zero field strength at the edge corresponds to an unlimited emissivity of the cathode [1]. Let us derive an expression for the averaged field strength at the cathode edge for the cathode emissivity varying from zero to an unlimited value and relate it to the current flowing in the diode. Assume that the voltage across the diode is a constant equal to U . This statement of the problem, i.e. the assumption of a variable current at a fixed voltage across the diode physically corresponds to a controlled variation of the internal resistance of the voltage source.

Neglecting the dependence of the energy of electrons in the generated beam on radius $\left(\frac{\partial\gamma}{\partial r}=0\right)$, we can obtain from equation (19) the relation

$$(\Gamma-1)^2\left(1-\xi^2\right)-(\Gamma-\gamma_0)^2-2\frac{\gamma_0^2-1}{\gamma_0}(\Gamma-\gamma_0)=0\,, \tag{20}$$

where $\xi=\dfrac{\langle E_n\rangle}{E_c\sqrt{\ln(R_a/R_c)\dfrac{R_c}{h_c}}}$ is the averaged field strength at the end face of the cathode during the passage of current, referred to its electrostatic value (9); $(\Gamma-1)=\dfrac{eU}{mc^2}$ and $(\gamma_0-1)=\dfrac{e\varphi_0}{mc^2}$ are the dimensionless applied voltage and the dimensionless potential difference between the generated beam and the anode. The value of ξ lies in the interval $[0,1]$. The first summand in equation (20) represents the sum of the field momentum fluxes through surfaces III and VI in Fig. 1. The second summand is the field momentum flux through surface I. The third summand corresponds to the beam electron momentum flux through surface I (the beam is not shown in Fig. 1). The solution of equation (20) has the form

$$\gamma_0=2\sqrt{\frac{2\Gamma+1+(\Gamma-1)^2\,\xi^2}{3}}\cos\frac{\phi}{3}\,, \tag{21}$$

where $\phi=\arccos\left[(-\Gamma)\left(\dfrac{3}{2\Gamma+1+(\Gamma-1)^2\,\xi^2}\right)^{3/2}\right]$.

For $\xi=0$ the solution (21) is reduced to the well-known solution [1]

$$\gamma_F=\sqrt{\frac{1}{4}+2\Gamma}-\frac{1}{2} \tag{22}$$

and corresponds to Fedosov's current [1]

$$I(\gamma_F)=\frac{mc^3}{2e\ln(R_a/R_c)}\left(\frac{(\Gamma-\gamma_F)\sqrt{\gamma_F^2-1}}{\gamma_F}\right). \tag{23}$$

Normalizing the beam current for $I(\gamma_F)$ at an arbitrary ξ, we obtain an expression for the dimensionless current ς as a function of Γ and ξ:

$$\varsigma(\Gamma,\xi)=\frac{I(\gamma_0)}{I(\gamma_F)}=\left(\frac{(\Gamma-\gamma_0)\sqrt{\gamma_0^2-1}}{\gamma_0}\right)\Bigg/\left(\frac{(\Gamma-\gamma_F)\sqrt{\gamma_F^2-1}}{\gamma_F}\right). \tag{24}$$

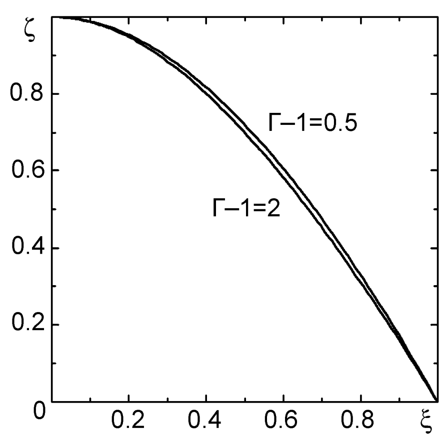

Fig. 3. The dimensionless current in a magnetically insulated coaxial diode as a function of the dimensionless averaged field strength at the edge of the cathode.

Figure 3 presents the plot of the dimensionless current (24) versus the dimensionless averaged field strength for two values of the applied voltage. As can be seen from the figure, the field strength at the end face of the cathode increases rather abruptly even at an inappreciable decrease in current from $I(\gamma_F)$ and then, more smoothly, approaches its electrostatic value. This is in direct opposition to the conclusion of the authors of Ref. 5 that the field strength at the end face of a cathode has a singularity throughout the range of currents. It should also be noted that the form of relation (24) is practically the same for different values of the applied voltage.

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