Russian Physics Journal, Vol. 51, No. 1, 2008

ELEMENTARY PARTICLE PHYSICS AND FIELD THEORY

A DYNAMIC MODEL OF SPHERICAL PERTURBATIONS IN THE FRIEDMANN UNIVERSE. I

Yu. G. Ignat'ev and N. Elmakhi UDC 530.12:531.51

A self-consistent set of equations describing the evolution of linear spherically symmetrical perturbations in the Friedmann world is derived for an arbitrary equation of state. A singular part of perturbations corresponding to a massive particle-like source is separated, an evolution equation for calculating the source mass is obtained and solved exactly. An exact solution to evolution equations for perturbations at an arbitrary equation of state is constructed.

INTRODUCTION

In [1–3], a theory of spherical perturbations in the Friedmann world was constructed in connection with the necessity to develop a relativistic kinetic theory with allowance for gravitational interactions. A procedure of averaging local fluctuations of the gravitational field applied to derivation of the kinetic equations revealed an interesting fact: mean square fluctuations of the gravitational field act as energy-momentum tensor of the ideal liquid with an extremely rigid equation of state [3]. In [4], exact solutions were obtained for equations of spherically symmetrical perturbations of the ultrarelativistic Friedmann world with an arbitrary radius of curvature. However, the solutions obtained were not analyzed and the mean square corrections for the Friedmann metric were not calculated. In relation to the problem of dark energy and dark matter in cosmology, the problem on a possible change in the macroscopic equation of state of the Friedmann universe by local gravitational interactions becomes important, since the potential energy of gravitation interaction is negative and this can correspond to a negative macroscopic pressure. In this work, we examine this possibility, constructing self-consistent dynamic theory of the Friedmann Universe with allowance for local gravitational perturbations.

1. SPHERICALLY SYMMETRICAL SPACE-TIME

1.1. Symmetry of space and an energy-momentum tensor

Let us consider space-time with spherical symmetry, whose metric in the isotropic coordinate system[1] $(r, \theta, \varphi, \eta)$, where η is the time and r is the radial variables, can be written as

$$ds^2 = e^{\nu} d\eta^2 - e^{\lambda}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{1}$$

where

$$\lambda = \lambda(r, \eta), \quad \nu = \nu(r, \eta)$$

[1] That is in the coordinate system where 3D space metric takes conformal-flat form.

Tatar State Humanitarian-Pedagogical University, e-mail: ignatev_yu@rambler.ru. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 1, pp. 66–76, January, 2008. Original article submitted April 17, 2007.

are the arbitrary scalar functions of their arguments. Isotropic coordinates are convenient as the 3D space metric takes an explicit conformal-flat form in these coordinates. It is known that metric (1) assumes the rotation group G_3 with the following Killing vectors (see, e.g.[7]):

$$\frac{dx^i}{ds} = \frac{\partial H}{\partial P_i}, \quad \frac{dP_i}{ds} = -\frac{\partial H}{\partial x^i}. \tag{2}$$

therefore, the symmetry of space-time is retained in the energy-momentum tensor due to the Einstein equations, and we have

$$\underset{\substack{\xi \\ \alpha}}{\mathrm{L}} T^{ik} = 0, \quad (\alpha = \overline{1,3}), \tag{3}$$

where $\underset{\xi}{\mathrm{L}}$ is the Lie variable with respect to the direction ξ (see, e.g. [7]) written as follows:

$$\underset{\xi}{\mathrm{L}} A^i = A^i_{,k}\xi^k + A^k \xi^i_{,k}, \tag{4}$$

and other tensor indices are treated in a way similar to the Lie variable.

Thus, in conditions of spherical symmetry of space-time, the energy-momentum tensor is known to take the form of the energy-momentum tensor of ideal isotropic liquid

$$T^{ik} = (\varepsilon + p)u^i u^k - pg^{ik}, \tag{5}$$

where the scalars $\varepsilon(r,\eta)$ and $p(r,\eta)$ are the energy density and pressure in the liquid, respectively, and u^i is the unit time-like vector of the dynamic velocity of the liquid

$$g_{ik}u^i u^k = 1, \tag{6}$$

with

$$u^i = (u^r(r,\eta), 0, 0, u^\eta(r,\eta)). \tag{7}$$

Assuming

$$u^r = vu^\eta e^{\frac{\nu-\lambda}{2}}, \quad v^2 < 1,$$

where $v(r,\eta)$ is the radial 3D velocity of the liquid, one can find from Eqs. (5) and (6)

$$u^\eta = e^{-\frac{\nu}{2}} \frac{1}{\sqrt{1-v^2}}, \tag{8}$$

$$T^1_4 = (\varepsilon + p)e^{(\nu-\lambda)/2} \frac{v}{1-v^2}, \quad T^4_4 = \frac{\varepsilon + v^2 p}{1-v^2}; \; T^1_1 = -\frac{\varepsilon v^2 + p}{1-v^2}; \; T^2_2 = T^3_3 = -p. \tag{9}$$

Hence, the following algebraic relations are valid:

$$a \sim t \quad \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\delta\nu = 8\pi a^2 m(\eta)\delta(r). \tag{10}$$

1.2. The Einstein equations

The nontrivial Einstein equations with respect to metric (1) have the form (see, e.g. [6])[2]

$$\frac{1}{2}e^{-\lambda}\left(\frac{\lambda'^2}{2}+\lambda'\nu'+\frac{2}{r}(\lambda'+\nu')\right)-e^{-\nu}\left(\ddot{\lambda}-\frac{1}{2}\dot{\lambda}\dot{\nu}+\frac{3}{4}\dot{\lambda}^2\right)=8\pi\frac{\varepsilon v^2+p}{1-v^2} \quad (=-8\pi T_1^1), \tag{11}$$

$$\frac{1}{4}e^{-\lambda}\left[2(\lambda''+\nu'')+\nu'^2+\frac{2}{r}(\lambda'+\nu')\right]-e^{-\nu}\left(\ddot{\lambda}-\frac{1}{2}\dot{\lambda}\dot{\nu}+\frac{3}{4}\dot{\lambda}^2\right)= 8\pi p \quad (=-8\pi T_2^2), \tag{12}$$

$$-e^{-\lambda}\left(\lambda''+\frac{1}{4}\lambda'^2+\frac{2}{r}\lambda'\right)+\frac{3}{4}e^{-\nu}\dot{\lambda}^2=8\pi\frac{\varepsilon+v^2p}{1-v^2} \quad (=8\pi T_4^4), \tag{13}$$

$$\frac{1}{2}e^{-\lambda}(2\dot{\lambda}'-\nu'\dot{\lambda})=8\pi(\varepsilon+p)e^{(\nu-\lambda)/2}\frac{v}{1-v^2} \quad (=8\pi T_4^1), \tag{14}$$

where f' denotes a derivative of the function f with respect to the radial variable r, $\dot{f}$ is a derivative with respect to time variable η, and a universal system of units where $G=c=\hbar=1$ is adopted in all cases. Subtracting the respective parts of Eq. (12) from both parts of Eq. (11), we get the following corollary with allowance for Eq. (9):

$$\frac{1}{2}e^{-\lambda}\left[\frac{1}{2}\lambda'^2+\lambda'\nu'-\frac{1}{2}\nu'^2+\frac{1}{r}(\lambda'+\nu')-(\lambda''+\nu'')\right]=8\pi(\varepsilon+p)\frac{v^2}{1-v^2}. \tag{15}$$

2. BACKGROUND SPACE-TIME

In isotropic spherical coordinates, an unperturbed gravitational field is described by the metric of the isotropic homogeneous Universe:

$$ds^2=a^2(\eta)\left(d\eta^2-\frac{1}{\rho^2(r)}[dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)]\right), \tag{16}$$

where

$$\rho(r)=1+\frac{1}{4}kr^2, \tag{17}$$

and the *index of curvature* $k=0$ is for the spatially flat Universe and $k=\pm1$ is for the Universe with positive and negative curvature of 3D space, respectively. In this case, r and η are the dimensionless variables, and the *scaling factor* $a(\eta)$ has the dimension of length.

Hence, in the unperturbed state, the introduced scalar metric functions λ and ν are written as

[2] In order to derive these equations, one should assume $\mu=\lambda+2\ln r$ in the formulas adopted in [6].

$$\nu_0 = \ln a^2(\eta), \quad \lambda_0 = \ln\left(\frac{a(\eta)}{\rho(r)}\right)^2, \tag{18}$$

and due to homogeneity of space-time

$$p = p_0(\eta), \quad \varepsilon = \varepsilon_0(\eta), \quad v_0 = 0. \tag{19}$$

Substituting Eqs. (18), (19) into the Einstein equations (11)–(15), we derive a set of equations describing the dynamics of the Friedmann Universe

$$\frac{1}{a^2}\left(\frac{\dot{a}^2}{a^2} + k\right) = \frac{8\pi}{3}\varepsilon; \tag{20}$$

$$\frac{1}{a^2}\left(2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + k\right) = -8\pi p. \tag{21}$$

It is known that Eq. (21) can be replaced by the algebraically differential corollary (20) and (21) (see, e.g. [6]), and we have

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + p) = 0. \tag{22}$$

In this case it is convenient to go from the time variable η to the physical time t using the following formula:

$$a(\eta)d\eta = dt, \quad \Rightarrow t = \int a(\eta)d\eta, \tag{23}$$

in so doing

$$\frac{\partial}{\partial\eta} = a\frac{\partial}{\partial t} \to \dot{f} = a\dot{f}_t. \tag{24}$$

The independent Einstein equations can be written in a more compact form (see, e.g. [6]) as follows:

$$\frac{1}{a^2}(\dot{a}_t^2 + k) = \frac{8\pi}{3}\varepsilon; \tag{25}$$

$$\dot{\varepsilon}_t + 3\frac{\dot{a}_t}{a}(\varepsilon + p) = 0. \tag{26}$$

Provided we know the equation of state, that is, the dependence of the form

$$p = p(\varepsilon), \tag{27}$$

Eq. (22) is integrated in the quadratures as

$$-3\ln a = \int\frac{d\varepsilon}{\varepsilon + p(\varepsilon)} + \text{const.} \tag{28}$$

Substituting solution (28) into Eq. (20), we get a first-order closed differential equation with respect to ε(η). For the case of *barotropic* equation of state[3]

$$p = \kappa\varepsilon \tag{29}$$

Eq. (24) is easily integrated

$$\varepsilon = c_1 a^{-3(\kappa+1)}, \tag{30}$$

and Eq. (20) is integrated in the quadratures

$$\int \frac{da}{a\sqrt{\frac{8\pi c_1}{3} a^{2-3(\kappa+1)} - k}} = c_2\eta, \tag{31}$$

where c_1, c_2 are arbitrary constants. The foregoing equations are integrated in the elementary functions for the early Universe ($t \to 0$). It is well-known that in this case the behavior of solutions does not depend on the index of curvature k (see, e.g. [6]) and does not differ from that for the spatially flat Universe ($k = 0$)

$$a = a_1\eta^{2/(3\kappa+1)}, \ \varepsilon = c_1 a_1^{-3(1+\kappa)}\eta^{-6(1+\kappa)/(3\kappa+1)}, \quad \kappa+1 \neq 0, \tag{32}$$

where the constants a_1 and c_1 are related by the equation

$$a_1 = \left((3\kappa+1)\sqrt{\frac{2\pi c_1}{3}}\right)^{2/(3\kappa+1)}, \quad 1+3\kappa \neq 0. \tag{33}$$

The variables t and η are found to be related as (Eq. 23)

$$t = a_1 \frac{3\kappa+1}{3(1+\kappa)}\eta^{3(1+\kappa)/(3\kappa+1)}. \tag{34}$$

Taking into account Eqs. (33) and (34), we find from Eq. (32)

$$\varepsilon = \left(2\pi(\kappa+1)^2 t^2\right)^{-1}. \tag{35}$$

Note that the solution to the Einstein equations for $\kappa = -1/3$ is singular

$$a = c_2 \exp\left(\sqrt{\frac{8\pi}{3}} c_1\eta\right), \quad \kappa = -\frac{1}{3}, \ \eta \in (-\infty, +\infty).$$

However, this singularity is merely coordinate. Indeed, in going from the time variable η to the physical time t

[3] The authors call reader's attention to the difference in notation: k is the index of curvature and κ (kappa) is the coefficient of barotrophics.

$$t = \frac{c_2}{\sqrt{\frac{8\pi}{3}c_1}} \exp\left(\sqrt{\frac{8\pi}{3}c_1}\,\eta\right)$$

we get $a \sim t$ and a formula for the energy density (35), in which it is necessary to substitute only $\kappa = -1/3$. For $\kappa = -1$ we obtain the so called inflationary solution from Eq. (32)

$$a = -\frac{1}{\Lambda\eta}, \quad t = -\ln\eta,$$

$$a = a_1 e^{\Lambda t}; \quad \varepsilon = \frac{3\Lambda^2}{8\pi} = \text{const.} \tag{36}$$

Solutions (33) corresponding to $\kappa < -1$ describe so called *dark matter*.

3. LINEAR SPHERICALLY SYMMETRICAL PERTURBATIONS OF THE FRIEDMANN SPACE-TIME

3.1. Equations for spherically symmetrical perturbations

Let us consider small spherically symmetrical perturbations of the homogeneous isotropic cosmological solution (18), assuming

$$\lambda = \ln a^2(\eta) + \delta\lambda; \quad \nu = \ln a^2(\eta) + \delta\nu;$$

$$p = p_0(\eta) + \left.\frac{dp}{d\varepsilon}\right|_{\varepsilon_0} \delta\varepsilon; \quad \varepsilon = \varepsilon_0(\eta) + \delta\varepsilon, \tag{37}$$

where the scalar functions $\delta\lambda(r,\eta)$, $\delta\nu(r,\eta)$, $\delta\varepsilon(r,\eta)$, $v(r,\eta)$ will be considered small of one order of smallness. Substituting Eq. (37) into Eq. (15), we derive one closed equation with respect to the function $\delta\lambda + \delta\nu$ in the first approximation over smallness of the perturbations $\delta\lambda, \delta\nu, \delta\varepsilon$, v

$$\frac{\partial}{\partial r}\left(\frac{\rho(r)}{r}(\lambda+\nu)'\right) = 0. \tag{38}$$

Integrating Eq. (38), we find (see also [4])

$$\lambda + \nu = \begin{cases} C_1(\eta) + C_2(\eta) r^2, & k = 0, \\ C_1(\eta) + \dfrac{C_2(\eta)}{\rho(r)}; & k = \pm 1, \end{cases} \tag{39}$$

where $C_1(\eta)$, $C_2(\eta)$ are arbitrary functions.

Further we will seek only such solutions to the Einstein equation of class C^1 which coincide with the homogeneous isotropic unperturbed solution beyond a certain sphere

$$\lambda(r,\eta)\big|_{rr_0(\eta)} = \lambda_0(r,\eta); \quad \nu(r,\eta)\big|_{rr_0(\eta)} = \nu_0(\eta),$$
$$\lambda'(r,\eta)\big|_{rr_0(\eta)} = \lambda'_0(r,\eta); \quad \nu'(r,\eta)\big|_{rr_0(\eta)} = \nu'_0(\eta). \tag{40}$$

These solutions correspond to retarded solutions to equations of hyperbolic type. Physical meaning of the solutions we will discuss later. According to Eq. (39), it follows that (see [4])

$$\delta\lambda + \delta\nu = 0, \Rightarrow \delta\lambda = -\delta\nu. \tag{41}$$

In [4], spherically symmetrical perturbations are examined in the ultrarelativistic Universe ($\kappa = 1/3$) alone, however, solutions to all linearized Einstein equations are derived for the Friedmann Universe of all types. In our work, we restrict ourselves to the spatially flat Universe ($k = 0$), but the coefficient of barotrophics κ is considered to be arbitrary. Thus, with allowance made for the background Einstein equations (20) – (21) which, for $k = 0$, have the corollary

$$2\frac{\ddot{a}}{a} = \frac{\dot{a}^2}{a^2}(1-3\kappa), \tag{42}$$

we derive a closed set of three differential Einstein equations linearized around the background solution (16) with respect to three unknowns $\delta\nu(r,\eta)$, $\delta\varepsilon(r,\eta)$, and $v(r,\eta)$

$$\delta\ddot{\nu} + 3\delta\dot{\nu}\frac{\dot{a}}{a} - 3\kappa\delta\nu\frac{\dot{a}^2}{a^2} = 8\pi a^2\delta p\ , \tag{43}$$

$$3\delta\dot{\nu}\frac{\dot{a}}{a} + 3\delta\nu\frac{\dot{a}^2}{a^2} - \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\delta\nu = -8\pi a^2\delta\varepsilon\ , \tag{44}$$

$$\frac{1}{a^3}\frac{\partial}{\partial\eta}a\delta\nu' = -8\pi\varepsilon_0(1+\kappa)v. \tag{45}$$

Equation (45) in this set is determination of the radial velocity $v(r,\eta)$. One of the Eqs. (43), (44) defines perturbation of the energy density $\delta\varepsilon(r,\eta)$.

3.2. Separating out particle-like solutions

Let us consider canonical equations of motion for a classical gravitating point particle in a gravitational field, bearing in mind that particle-like solutions to equations for perturbations are possible later on. A δ -like distribution of the energy density corresponds to the point. Due to two competing processes – accretion of the surrounding medium and the inverse process – evaporation of matter, the mass of the classical point particle in a medium cannot be constant. Therefore, the invariant Hamilton function of the classical massive particle can be as follows[4]:

$$H(x,P) = \sqrt{g^{ik}P_iP_k} - m, \quad (=0),$$

where $m = m(s)$ is the scalar function. We derive the following normalization relation from Eq. (45):

[4] For details, see [8].

$$(P,P)=m^2(s). \tag{46}$$

The relativistic *canonical* equations of particle motion are written as

$$\frac{dx^i}{ds}=\frac{\partial H}{\partial P_i};\quad \frac{dP_i}{ds}=-\frac{\partial H}{\partial x^i}. \tag{47}$$

It follows from the first pair of canonical equations, taking into account the normalization relation, that

$$\frac{dx^i}{ds}=\frac{P^i}{m}\Rightarrow g_{ik}\frac{du^i}{ds}\frac{du^k}{ds}=1. \tag{48}$$

The second pair of canonical equations of motion leads to the Lagrange equations for the classical massive particle of variable mass, and we have:

$$\frac{d^2x^i}{ds^2}+\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=(\ln m)_{,k}\left(g^{ik}-\frac{du^i}{ds}\frac{du^k}{ds}\right). \tag{49}$$

In spherically symmetrical metric, a time line corresponding to the particle rest at the origin of coordinates is a solution to the equations of motion (48) violating no spherical symmetry

$$r=0,\, x^4=\eta, \tag{50}$$

in this case, the rest mass can be an arbitrary function of the coordinate time:

$$m=m(\eta). \tag{51}$$

Let us write the energy density corresponding to the singular part in the invariant form as follows:

$$\delta\varepsilon_m=m(\eta)\delta(r), \tag{52}$$

where $\delta(r)$ is the invariant Dirac δ-function in the spherical coordinates and is treated as the integral relation

$$\int d^3V\delta^3(x)=a^3\int d\Omega\int\limits_0^{r_0} r^2dr\delta(r)=4\pi a^3\int\limits_0^{r_0}\delta(r)r^2dr=1, \tag{53}$$

hence

$$\int d^3V\delta\varepsilon_m=m(\eta). \tag{54}$$

Dropping for some time the derivatives with respect to the variable η in the left-hand side of Eq. (43), we obtain the following equation for the singular part corresponding to the singular part of density:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\delta\nu=8\pi a^2 m(\eta)\delta(r). \tag{55}$$

Multiplying both parts of Eq. (55) by $ar^2 dr$ and integrating, performing integration by parts in the left-hand side of the equation, and assuming

$$\lim_{r\to 0} r^2 \frac{\partial \delta\nu}{\partial r} = 0\,, \tag{56}$$

we derive

$$ar^2 \frac{\partial \delta\nu}{\partial r} = 2m(\eta). \tag{57}$$

Integrating this equation once more, we find

$$\delta\nu = -\frac{2m(\eta)}{ar}.$$

Thus, there is a relation similar to the known one with allowance for the redefined invariant δ-function (53)

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left(-\frac{m}{r}\right) = 4\pi a^3 m \delta(r). \tag{58}$$

Therefore, for separating out a particle-like singular part of the solution, it is convenient to introduce a new field function $\psi(r,\eta)$, such that [9]

$$\delta\nu = -\delta\lambda = 2\frac{\psi(r,\eta) - m(\eta)}{ar} \equiv 2\frac{\Phi(r,\eta)}{ar}, \tag{59}$$

in so doing, according to Eq. (56), the following relation should be fulfilled :

$$\left|\lim_{r\to 0}\frac{\psi}{r}\right| < \infty. \tag{60}$$

Separating out a singular part of energy density in the right-hand side of Eq. (43), substituting the function $\delta\nu$ in the form of Eq. (59) into Eqs. (42)–(44), and leaving out the singular part with allowance for Eqs. (58) and (60), we obtain a set of linear equations with respect to the function Φ and perturbations of energy density and velocity

$$\ddot{\Phi} + \frac{\dot{a}}{a}\dot{\Phi} - \frac{3}{2}(1+\kappa)\frac{\dot{a}^2}{a^2}\Phi = 4\pi r a^3 \kappa \delta\varepsilon. \tag{61}$$

$$3\frac{\dot{a}}{a}\dot{\Phi} - \psi'' = -4\pi r a^3 \delta\varepsilon. \tag{62}$$

$$\frac{\partial}{\partial r}\frac{\dot{\Phi}}{r} = -4\pi r a^3 (1+\kappa)\varepsilon_0 v. \tag{63}$$

Multiplying Eq. (62) by κ and summing its both parts with the corresponding parts of Eq. (61), we derive the following closed equation:

$$\ddot{\Phi}+\frac{\dot{a}}{a}(1+3\kappa)\dot{\Phi}-\frac{3}{2}(1+\kappa)\frac{\dot{a}^2}{a^2}\Phi-\kappa\psi''=0. \tag{64}$$

Assuming, according to Eq. (59),

$$\Phi(r,\eta)=\psi(r,\eta)-m(\eta) \tag{65}$$

and separating the variables in Eq. (64), we get two equations for the functions $m(\eta)$ and $\psi(r,\eta)$

$$\ddot{m}+\frac{\dot{a}}{a}(1+3\kappa)\dot{m}-\frac{3}{2}(1+\kappa)\frac{\dot{a}^2}{a^2}m=\Theta(\eta); \tag{66}$$

$$\ddot{\psi}+\frac{\dot{a}}{a}(1+3\kappa)\dot{\psi}-\frac{3}{2}(1+\kappa)\frac{\dot{a}^2}{a^2}\psi-\kappa\psi''=-\Theta(\eta), \tag{67}$$

where $\Theta(\eta)$ is the arbitrary function of its argument.

3.3. Fundamental theorem

Let hereafter $m=M(\Theta,\eta)$ be a particular solution to Eq. (66) corresponding to the specified function $\Theta(\eta)$. Due to linearity of Eq. (67), the function $\psi_1=-M(\Theta,\eta)$ is its partial solution, and the general solution can be written as

$$\psi(r,\eta)=\psi_0(r,\eta)-M(\Theta,\eta), \tag{68}$$

where $\Psi(r,\eta)$ is the general solution to the corresponding homogeneous equation

$$\ddot{\Psi}+\frac{\dot{a}}{a}(1+3\kappa)\dot{\Psi}-\frac{3}{2}(1+\kappa)\frac{\dot{a}^2}{a^2}\Psi-\kappa\Psi''=0. \tag{69}$$

Further, due to linearity of Eq. (66), its general solution is a sum of a general solution of the corresponding homogeneous equation $\mu(\eta)$ and a partial solution of the inhomogeneous one

$$m(\eta)=\mu(\eta)+M(\Theta,\eta). \tag{70}$$

However, in this case

$$\Phi(r,\eta)=\psi(r,\eta)-m(\eta)=\Psi(r,\eta)-\mu(\eta), \tag{71}$$

where $\mu(\eta)$ is the general solution to the homogeneous equation

$$\ddot{\mu}+\frac{\dot{a}}{a}(1+3\kappa)\dot{\mu}-\frac{3}{2}(1+\kappa)\frac{\dot{a}^2}{a^2}\mu=0\,. \tag{72}$$

The rest equations of the set (61)–(63) describe evolution of a nonsingular part of the energy density and perturbation velocity as

$$\delta\varepsilon = -\frac{1}{4\pi r a^3}\left(3\frac{\dot{a}}{a}(\dot{\Psi}-\dot{\mu}) - \Psi''\right), \tag{73}$$

$$\frac{\partial}{\partial r}\frac{\dot{\Psi}-\dot{\mu}}{r} = -4\pi r a^3(1+\kappa)\varepsilon_0 v. \tag{74}$$

Thus the theorem is proved.

Theorem. *Linear spherically symmetric perturbations of the Friedmann metric are described by a set of two independent linear homogeneous equations (69), (72) with respect to two functions* $\mu(\eta)$ *and* $\psi(r,\eta)$ *nonsingular at the origin of coordinates. Spherically symmetric perturbations of energy density and velocity are found through perturbations of metric by Eqs. (73)–(74) .*

For $\kappa > 0$, the homogeneous equation (69) is hyperbolic, for $\kappa < 0$ – elliptical, for $\kappa = 0$ this equation coincides with Eq. (72).

4. EVOLUTION EQUATIONS FOR PERTURBATIONS AT A CONSTANT COEFFICIENT OF BAROTROPHICS

4.1. Evolution of mass of a particle-like source

Let us examine cosmological evolution of mass of a particle-like source. Going from the variable η to the variable $a(\eta)$ in the equation of evolution of mass (72) and taking into account Eq. (42), we find

$$\frac{d^2\mu}{da^2} + \frac{3}{2}\frac{1+\kappa}{a}\frac{d\mu}{da} - \frac{3}{2}(1+\kappa)\frac{\mu}{a^2} = 0. \tag{75}$$

The general solution to this equation is easily derived as

$$\mu = C_+ a + C_- a^{-\frac{3}{2}(1+\kappa)}, \tag{76}$$

where C_+ and C_- are arbitrary constants. The term with the coefficient C_+ in this solution corresponds to accretion processes while that with the coefficient C_- corresponds to evaporation processes. Substituting the dependence of scaling factor on the time variable η from Eq. (32) into Eq. (76) and using relation (34) between the time variable η and physical time *t*, we get the law of evolution of mass of a particle-like source in the explicit form

$$\mu = \tilde{C}_+ t^{\frac{2}{3(1+\kappa)}} + \tilde{C}_- t^{-1}, \quad (1+\kappa) \neq 0, \tag{77}$$

where $\tilde{C}_+$ and $\tilde{C}_-$ are new arbitrary constants. It is seen from solutions (76), (77) that $C_- = \tilde{C}\_0$ corresponds to the finite particle mass at $t = 0$. Solutions (76)–(77) generalize the solutions obtained earlier in [3, 4] for two particular values of the adiabatic coefficient $\kappa = 0$ and 1/3 corresponding to nonrelativistic and ultrarelativistic equations of state, respectively. In the foregoing cases, we have from Eq. (77)

$$\mu = \tilde{C}_+ t^{\frac{2}{3}} + \tilde{C}_- t^{-1}, \quad \kappa = 0; \tag{78}$$

$$\mu = \tilde{C}_+ t^{\frac{1}{2}} + \tilde{C}_- t^{-1}, \quad \kappa = \frac{1}{3}. \tag{79}$$

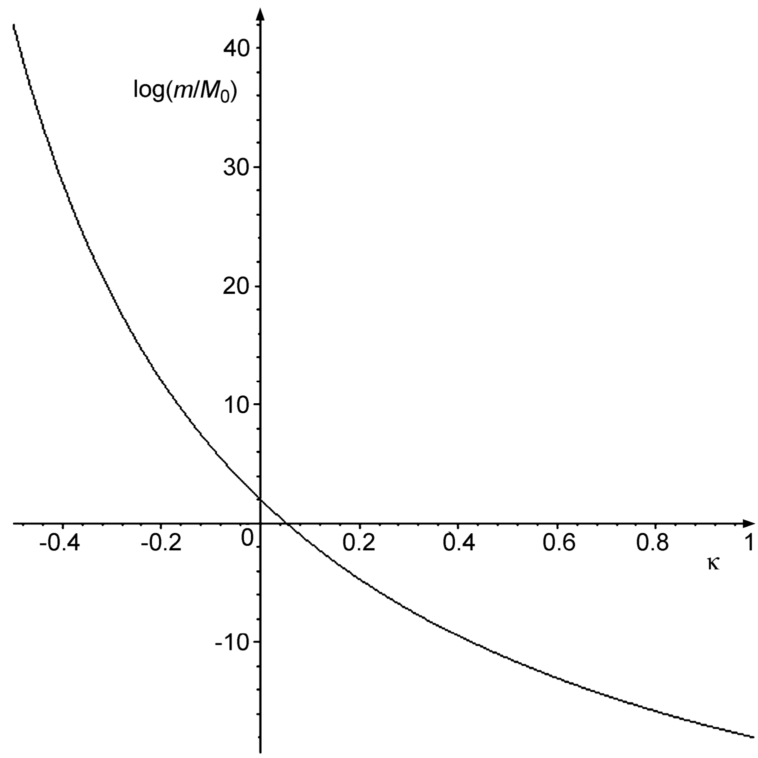

Fig. 1. Current mass of a particle-like source in the Friedmann world whose mass was Planckian at the Planckian instant of time versus barotrophics κ. Mass logarithm values in the units of mass of the Sun $M_\odot \approx 2\cdot10^{33}$ $M_\odot \approx 2\cdot10^{33}$ g are plotted as ordinates.

Let us consider a numerical example. Let the mass of particle-like source at the Planckian instant of time $t = t_{\rm Pl}$ be equal to the Planckian mass, hence, according to Eq. (77), at the current instant of time $t \sim 10^{60} t_{\rm Pl}$, the current mass of the "particle" is varied from $10^{-18} M_\odot$ for $\kappa = 1$ to $10^2 M_\odot$ for $\kappa = 0$. For negative values of the adiabatic coefficient, the perturbation mass is rapidly increased and becomes comparable with the mass of the apparent Universe in the order of magnitude at $\kappa \approx -0.5$ (Fig. 1).

In the case of inflationary solution (36), the evolution equation (72) becomes

$$\ddot{\mu} = -\frac{2}{\eta}\dot{\mu} = 0. \tag{80}$$

Solving Eq. (80), we find

$$m = C_+ + \frac{C_-}{\eta}. \tag{81}$$

Using the relation of the time variable η with the physical time t for Eq. (36), we eventually derive for inflationary solution

$$\mu = C_+ + C_- e^{-\Lambda t}. \tag{82}$$

In this case, $C_- = 0$ corresponds to finite mass at $t \to -\infty$.

4.2. Evolution equation for a nonsingular perturbation mode

We get the following relations from the general unperturbed solution (32):

$$\frac{\dot{a}}{a} = \frac{2}{3\kappa+1}\frac{1}{\eta}; \quad \frac{\ddot{a}}{a} = \frac{2(1-3\kappa)}{(3\kappa+1)^2}\frac{1}{\eta^2} \quad (1+\kappa \neq 0), \tag{83}$$

using these relations we reduce Eq. (69) for the nonsingular perturbation mode to the form

$$\ddot{\Psi} + \frac{2}{\eta}\dot{\Psi} - \frac{6(1+\kappa)}{(1+3\kappa)^2}\frac{\Psi}{\eta^2} - \kappa\Psi'' = 0. \tag{84}$$

At $(1+\kappa) = 0$ Eq. (74) ceases to be the equation for determination of a radial perturbation velocity, and becomes a differential eqution with respect to the function ψ

$$\frac{\partial}{\partial r}\frac{\dot{\Psi} - \dot{\mu}}{r} = 0, \tag{85}$$

which must be solved along with Eq. (84), the latter becoming in this case

$$\ddot{\Psi} + \frac{2}{\eta}\dot{\Psi} + \Psi'' = 0. \tag{86}$$

5. A GENERAL SOLUTION TO THE EVOLUTION EQUATION FOR A NONSINGULAR PERTURBATION MODE

Assuming

$$\Psi(r,\eta) = R(r)\Theta(\eta)$$

and separating the variables in Eq. (84), we get the following ordinary differential equations:

$$\kappa R'' + e\alpha^2 R = 0, \tag{87}$$

$$\eta^2\ddot{\Theta} + 2\eta\dot{\Theta} + \left[e\alpha^2\eta^2 - 6\frac{1+\kappa}{(1+3\kappa)^2}\right]\Theta = 0. \tag{88}$$

To provide convergence of the solutions, the sign of separation constant is chosen opposite to the sign of the barotrophics

$$e = -\mathrm{sgn}(\kappa). \tag{89}$$

Solving Eq. (87), we find

$$R = C_1 \sin\frac{\alpha}{\sqrt{\kappa}} r + C_2 \cos\frac{\alpha}{\sqrt{\kappa}} r. \tag{90}$$

For the function $\Psi(r,\eta)$ to contain no singularity at the origin of coordinates, it is necessary and sufficient that $C_2 = 0$ in Eq. (90), hence

$$R(r) = C(\alpha)\sin\frac{\alpha}{\sqrt{|\kappa|}} r, \tag{91}$$

where $C(\alpha)$ is an arbitrary constant.

For $\kappa > 0$, Eq. (88) has the following solution:

$$\Theta(\eta) = \frac{\tilde{C}_1}{\sqrt{\eta}} J_s(\alpha\eta) + \frac{\tilde{C}_2}{\sqrt{\eta}} Y_s(\alpha\eta), \tag{92}$$

where $J_s(z)$ and $Y_s(z)$ are the Bessel functions of the first and second kind, respectively,

$$s = \frac{1}{2}\left|\frac{5+3\kappa}{1+3\kappa}\right|\frac{1}{2}, \quad \kappa \in [-1, +\infty). \tag{93}$$

At $\kappa < 0$ the solution to Eq. (88) is expressed through the Bessel functions of the imaginary argument $I_s(z)$ and $K_s(z)$, (see, e.g. [10]) as

$$\Theta(\eta) = \frac{\tilde{C}_1}{\sqrt{\eta}} I_s(\alpha\eta) + \frac{\tilde{C}_2}{\sqrt{\eta}} K_s(\alpha\eta). \tag{94}$$

Since the functions $J_s(z)/\sqrt{z}$ and $K_s(z)/\sqrt{z}$ tend to infinity at $z \to 0$, in order to obtain restricted solutions at $\eta \to 0$, it is necessary to assume in Eqs. (92) and (94) that

$$\tilde{C}_2 = 0. \tag{95}$$

Thus, the nonsingular solution to the evolution equation for perturbations can be written as

$$\Phi(r,\eta) = \frac{1}{\sqrt{\eta}} \int_0^\infty C(\alpha)\sin\frac{\alpha}{\sqrt{|\kappa|}} r J_s(\alpha\eta) d\alpha, \quad \kappa > 0. \tag{96}$$

For $\kappa < 0$, we obtain in a similar way

$$\Phi(r,\eta) = \frac{1}{\sqrt{\eta}} \int_0^\infty C(\alpha)\sin\frac{\alpha}{\sqrt{|\kappa|}} r I_s(\alpha\eta) d\alpha, \quad \kappa < 0. \tag{97}$$

REFERENCES

1. Yu. G. Ignat’ev and A. A. Popov, The Problems of Gravitation Theory, Relativistic Kinetics, and Evolution of the Universe [in Russian], Izd. KSPI, Kazan, 1988.
2. Yu. G. Ignat’ev and A. A. Popov, Sov. Phys. J., No. 5, 391 (1989).

3. Yu. G. Ignat’ev and A. A. Popov, Astrophys. Spac. Sci., **163**, 153 (1990).
4. Yu. G. Ignat’ev and A. A. Popov, Phys. Lett. A, **220**, 22 (1996).
5. J. L. Synge, Relativity: The General Theory, North-Holland Publishing Company, Amsterdam, 1960.
6. L. D. Landau and E. M. Lifshitz, Field Theory [in Russian], Nauka, Moscow, 1973.
7. A. Z. Petrov, New Methods in the General Relativity Theory [in Russian], Nauka, Moscow, 1966.
8. Yu. G. Ignat’ev, Sov. Phys. J., No. 8, 686 (1983).
9. Yu. G. Ignat’ev, Diss. Doct. Fiz. Mat. Nauk, FIAN BSSR, Minsk, 1988.
10. N. N. Lebedev, Special Functions and Their Applications [in Russian], Gos. Izd. Fiz. Mat. Literatury, Moscow, 1963.