Russian Physics Journal, Vol. 50, No. 3, 2007

METHOD OF ORBITS OF COASSOCIATED REPRESENTATION IN THERMODYNAMICS OF THE LIE NONCOMPACT GROUPS

V. V. Mikheev and I. V. Shirokov UDC 530.145.531.19

In the present paper, an efficient method of solving the main problem of thermodynamics of homogeneous spaces is presented. The method is based on the formalism of the noncommutative harmonic analysis relying on the method of orbits of coassociated representation. In the present work, the formula is derived that allows one to construct effectively the density matrix and the statistical sum in space of any arbitrary generally noncompact non-unimodular Lie group. To illustrate the method, an example is given of exact calculation of the statistical sum and density matrix in the Riemannian space of the noncompact Lie group with left-invariant metric.

INTRODUCTION

The present work is aimed at the construction of a method for solving the main problem of thermodynamics of homogeneous spaces for an arbitrary Lie group unrestricted by a unimodular group. The main problem of thermodynamics of homogeneous spaces is the calculation of the statistical sum (the distribution function)

$$Z_\beta = \sum_n d_n \exp(-\beta E_n)\,, \tag{1}$$

where summation is carried out over the states of the system, d_n is the degree of degeneracy of the corresponding energy E_n, and β is the reciprocal temperature. The statistical sum can be found as the trace of the density matrix (the thermal kernel):

$$Z_\beta = \int \rho_\beta(x,x)d\mu(x), \qquad d\mu(x) = \sqrt{|g|}dx\,, \tag{2}$$

where integration is carried out over the entire volume of the manifold.

This problem is of interest not only from the physical viewpoint, because the statistical sum and density matrix are important quantities for a given manifold and determine the thermodynamic properties of particles in this space [1]. In addition, a solution of the main problem of thermodynamics of homogeneous spaces for an arbitrary manifold brings us one step closer to a solution of the problem formulated by Katz: can we hear the drum shape? Or in other words, it brings us closer to the understanding of the influence of the global geometrical and topological space characteristics on the spectral properties of the Laplacian operator acting in it [2–4].

A great volume of the results was obtained in this direction for compact and noncompact spaces with limited volumes. Generally, there is no algorithm for constructing the statistical sum for any arbitrary noncompact manifold, since in this case series (1) and integral (2) diverge by virtue of the infinite volume of the manifold. We have already considered in [5, 6] the problem of finding the density matrix in spaces of the noncompact unimodular Lie groups with the left-invariant Riemannian metric. In the present work, the results are generalized to an arbitrary Lie group with the left-invariant Riemannian metric without assumption about the unimodularity.

The density matrix (the thermal kernel) is determined from the Bloch equation (the equation for the thermal kernel) in the manifold with the special initial condition:

Omsk State University, e-mail: vvm125@mail.ru. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 3, pp. 84–88, March, 2007. Original article submitted September 6, 2006.

$$\frac{\partial \rho_\beta(x,x')}{\partial \beta} + H(x)\rho_\beta(x,x') = 0, \quad \rho_\beta(x,x')|_{\beta=0} = \delta(x,x'). \tag{3}$$

A solution of Eq. (3) faces two serious problems that can hardly be solved by the existing methods of integration of linear differential equations, in particular, by the widely used method of separation of variables. First, a global solution of the equation should be constructed in the entire manifold, but the method of separation of variables depends significantly on the system of coordinates in the manifold and hence can give only local solutions. Second, a solution satisfying the initial condition in the form of the δ-function should be constructed from the basic solutions of Bloch equation (3), which is also a challenge.

1. NONCOMMUTATIVE INTEGRATION OF THE BLOCH EQUATION ON THE LIE GROUPS

Let us consider Bloch equation (3) on the n-dimensional real Lie group G with operator H being a quadratic function of left-invariant vector fields ξ on the group. Thus, the operator H is the Laplacian operator in the group manifold supplied with the Riemannian metric:

$$H(-i\hbar\xi) = -\hbar^2 G^{ab}\xi_a\xi_b = -\hbar^2\Delta\,. \tag{4}$$

In the present work, a solution of Eq. (3) is derived within the formalism of the noncommutative harmonic analysis on the Lie groups based on the method of orbits.

To this end, we now introduce a special irreducible representation of the Lie algebra G (the so-called λ-representation) in the Lagrangian submanifold to the orbit of the coassociated representation $O_\lambda \in G^*$, where G^* is the space dual to the Lie algebra G:

$$\left[l_i(q,\partial_q,\lambda),\, l_j(q,\partial_q,\lambda)\right] = C_{ij}^k l_k(q,\partial_q,\lambda)\,, \tag{5}$$

C_{ij}^k are the structural constants of the Lie algebra G, and $l_k(q,\partial_q,\lambda)$ are the first-order differential operators.

We can demonstrate that any irreducible representation of the Lie algebra can be expressed as an λ-representation constructed for a definitely chosen linear functional $\lambda \in G^*$. This linear functional depends on the parameters $j:\ \lambda \equiv \lambda(j)$, whose number is equal to the number of the Casimir functions, that is, to the index of the Lie algebra G. Therefore, the measure $d\mu(\lambda)$ is a spectral measure of the Casimir operators on the corresponding Lie group.

Let us consider the representation of group G in the functional space $C^\infty(Q)$ that acts on functions from this space as follows:

$$T_g^\lambda \phi(q) = \int D^{\lambda}_{q\bar{q}'}(g)\phi(q')d\mu(q') \tag{6}$$

and boosts the λ-representation of the Lie algebra to the group

$$l_i(q,\partial_q,\lambda) = \frac{\partial}{\partial g^i} T_g^\lambda|_{g=e}\,. \tag{7}$$

In this case, the linear functional λ should satisfy the integrality condition:

$$\int_{\gamma\in H^2(O_\lambda)} \omega_\lambda = 2\pi i n, \quad n \in \mathbb{Z}\,, \tag{8}$$

where ω_λ is the well-known Kirillov form in the orbit [8].

The generalized functions $D^{\lambda}_{q\overline{q'}}(g)$ are matrix elements of representation (6) and are a solution of the system of linear differential equations

$$\left[\xi_i(g)+\overline{l_i(q',\partial_{q'},\lambda)}\right]D^{\lambda}_{q\overline{q'}}(g)=0, \quad D^{\lambda}_{q\overline{q'}}(e)=\delta(q,\overline{q'}) . \tag{9}$$

Here e is unity of the group (the identical element).

The generalized functions $D^{\lambda}_{q\overline{q'}}(g)$ carry out the generalized Fourier transform on the Lie group, thereby solving the main problem of harmonic analysis [9]:

$$\varphi(g)=\int\limits_{Q\times Q\times J}\hat{\varphi}_j(q,q')D^{\lambda}_{q\overline{q'}}(g)d\mu(q)d\mu(q')d\mu(\lambda) . \tag{10}$$

Thus, the action of right- and left-invariant vector fields on the group is transformed into the action of operators of λ-representation on the Lagrangian submanifold of the orbit of coassociated representation [7]:

$$\xi_i\varphi(g)\Leftrightarrow l_i(q',\partial_{q'},\lambda)\hat{\varphi}_j(q,q'), \quad \eta_i\varphi(g)\Leftrightarrow \overline{l_i(q,\partial_q,\lambda)}\hat{\varphi}_j(q,q') . \tag{11}$$

After transition from the group space to the Lagrangian submanifold of the orbit O_λ, the Bloch equation for the density matrix is reduced to the equation in the orbit with a smaller number of independent variables [10]:

$$\frac{\partial \mathrm{R}_\beta(q,\bar{\bar{q}},j)}{\partial\beta}+H(-i\hbar l)\mathrm{R}_\beta(q,\bar{\bar{q}},j)=0, \quad \mathrm{R}_\beta(q,\bar{\bar{q}},j)|_{\beta=0}=\delta(q,\bar{\bar{q}}) , \tag{12}$$

which is an ordinary differential equation integrable in quadratures when the condition

$$(\dim G-\mathrm{ind}G)/2=1 \tag{13}$$

is satisfied.

The density matrix in the orbit $\mathrm{R}_\beta(q,\bar{\bar{q}},j)$ in Eq. (12) is connected to the density matrix in the initial space $\rho_\beta(g,g')$ by the expression

$$\rho_\beta(g,g')=\int \mathrm{R}_\beta(q,\bar{\bar{q}},j)D^{\lambda}_{\bar{\bar{q}}\tilde{q}}(g'^{-1}g)d\mu(q)d\mu(\tilde{q})d\mu(\lambda) . \tag{14}$$

Determining the density matrix from Eq. (12), we can obtain the statistical sum on any arbitrary noncompact Lie group using the properties of the generalized functions $D^{\lambda}_{q\overline{q'}}(g)$:

$$Z_\beta=\int\limits_G d\mu(x)\int\limits_{Q\times J}\mathrm{R}_\beta(q,q,j)d\mu(q)d\mu(\lambda)=\mathrm{Vol}_G\int\limits_{Q\times J}\mathrm{R}_\beta(q,q,j)d\mu(q)d\mu(\lambda) . \tag{15}$$

It can be seen that integration over the volume of the group manifold in Eq. (15) is independent of integration over the measure $d\mu(q)$ in the orbit of coassociated representation and the spectral measure $d\mu(\lambda)$. Thus, we can separate as a multiplier of the statistical sum the diverging quantity associated with the infinite volume of the noncompact manifold and obtain the following expression for the specific (per unit volume) statistical sum:

$$z_\beta = Z_\beta / \mathrm{Vol}_G = \int\limits_{Q\times J} \mathrm{R}_\beta(q,q,j)d\mu(q)d\mu(\lambda)\,, \tag{16}$$

which is essentially finite.

As a result, instead of solving Bloch equation (2) on the group with n independent variables, Eq. (12) must be solved in the orbit with a much smaller number of independent variables.

2. AN EXAMPLE

As an example of application of the developed method, we now consider the widely known 3-D Lie group $E(2)$ frequently used for similar purposes. It is the group of motions of the 2-D Euclidean plane. In spite of the fact that the structure of group $E(2)$ is simple enough, it can illustrate the influence of the nontrivial topology of a noncompact manifold on the thermodynamic properties of particles comprised in it. The Lie algebra of group $E(2)$ is specified by the following commutation relations:

$$[e_1,e_2]=0, [e_2,e_3]=\varepsilon e_1, [e_1,e_3]=-\varepsilon e_2\,.$$

In this case, the group manifold possesses the topology of the cylinder with the parameter $\varepsilon \sim 1/R$ that means the curvature. The contraction of the Lie algebra for $\varepsilon \to 0$ carries out a transition to the 3-D Abel group with the topology of the 3-D Euclidean space R^3.

The matrix is $G^{ab} = \mathrm{diag}(A,B,C)$, and in the case under consideration, we assume $A,B,C>0$.

The left- and right-invariant vector fields on the group $E(2)$ have the following form:

$$\xi_1 = \cos(\varepsilon x_3)\partial_1 + \sin(\varepsilon x_3)\partial_2, \quad \xi_2 = \cos(\varepsilon x_3)\partial_2 - \sin(\varepsilon x_3)\partial_1, \quad \xi_3 = \partial_3,$$
$$\eta_1 = -\partial_1, \quad \eta_2 = -\partial_2, \quad \eta_3 = \varepsilon x_2\partial_1 - \varepsilon x_1\partial_2 - \partial_3.$$

The operator $H(-i\hbar\xi)$ of Bloch equation (3) on the group assumes the form

$$H = -\hbar^2(A\xi_1^2 + B\xi_2^2 + C\xi_3^2)\,.$$

The operators of λ-representation of the Lie algebra with the chosen functional $\lambda = (j,0,0)$ assume the form

$$l_1 = -(\frac{ij}{\hbar})\cos(\varepsilon q), \quad l_2 = (\frac{ij}{\hbar})\sin(\varepsilon q), \quad l_3 = \frac{\partial}{\partial q}.$$

Thereby, functions on the Lagrangian submanifold of a simplectic folium to the orbit – a function of the variable q – are functions on the sphere S^1, that is, $2\pi/\varepsilon$ periodic functions.

The matrix elements $D^j_{qq'}(x)$ obtained by solving system (9) have the form

$$D^j_{qq'}(x) = \exp\left[\frac{ij}{\hbar}(-x_1\cos(\varepsilon q) + x_2\sin(\varepsilon q))\right]\delta(x_3 + q - q')\,.$$

The spectral measure $d\mu(j)$ was chosen in the form $d\mu(j) = \varepsilon j dj/(2\pi\hbar)^2$.

The example considered below is the Boltzmann gas (the gas of noninteracting particles with mass m). This imposes the following restrictions on the elements of matrix G^{ab}: $A=B=C=1/2m$, and hence the operator $H(-i\hbar l)$ is the Hamiltonian of a free particle.

The density matrix $\mathrm{R}_\beta(q,\tilde{q},j)$ in the Lagrangian submanifold Q of the symplectic folium to the orbit O_λ should be $2\pi/\varepsilon$ periodic.

As a result, we obtain the density matrix $\mathrm{R}_\beta(q,\tilde{q},j)$ on the Lagrangian submanifold Q of the symplectic folium to the orbit O_λ as a solution of Eq. (12):

$$\mathrm{R}_\beta(q,\tilde{q},j)=\frac{\varepsilon}{2\pi}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{1}{2m}\left((n\hbar\varepsilon)^2+j^2\right)-in\varepsilon(q-\tilde{q})\right)$$
$$=\frac{\varepsilon}{2\pi}\exp\left(-\frac{j^2\beta}{2m}\right)\theta_3\left(\frac{\varepsilon(q-\tilde{q})}{2},\exp\left(-\frac{\hbar^2\beta\varepsilon^2}{2m}\right)\right).$$

From this expression, according to formula (16), the specific statistical sum z_β over the volume can be obtained:

$$z_\beta=\left(\frac{m}{2\pi\beta\hbar^2}\right)^{3/2}\theta_3\left(0,\exp\left(-\frac{2m\pi^2}{\hbar^2\beta\varepsilon^2}\right)\right). \tag{17}$$

Let us rewrite formula (17) in the form more convenient for investigations of thermodynamic properties:

$$z_\beta=\left(\frac{mk_\mathrm{B}T}{2\pi\hbar^2}\right)^{3/2}\theta_3\left(0,\exp\left(-\frac{T}{T_0(\varepsilon)}\right)\right). \tag{18}$$

Here k_B is the Bolzmann constant and $T_0(\varepsilon)=\hbar^2\varepsilon^2/2m\pi^2k_\mathrm{B}$. We have used the standard designation for θ_3 – the well-known Yacobi θ-function. It can be seen that after contraction at $\varepsilon=0$, we obtain the statistical sum for the gas of free noninteracting particles. Here θ_3 is the correction factor caused by the nontrivial topology of the examined space.

Now, for the known statistical sum, we can easily find the average kinetic energy of the particle and the specific (per particle) heat in this space:

$$u=k_\mathrm{B}T^2\frac{\partial\ln z}{\partial T},\quad c_v=\frac{\partial u}{\partial T}.$$

In this case, the asymptotic transition (contraction) for $\varepsilon\to 0$ corresponds to the specific heat of the Boltzmann gas of free particles in the 2-D space R^3.

Figure 1 shows the plot of the specific (per particle) heat of the Bolzmann gas as a function of the parameter $t=T/T_0(\varepsilon)$.

It can be seen that at low temperatures, one degree of freedom is frozen, and at high temperatures, we have the specific heat corresponding to the gas of free particles in R^3. At intermediate temperatures comparable with $T_0(\varepsilon)$, anomalous behavior of the specific heat is observed for particles in $E(2)$, which is undoubtedly caused by the nontrivial topology of this space.

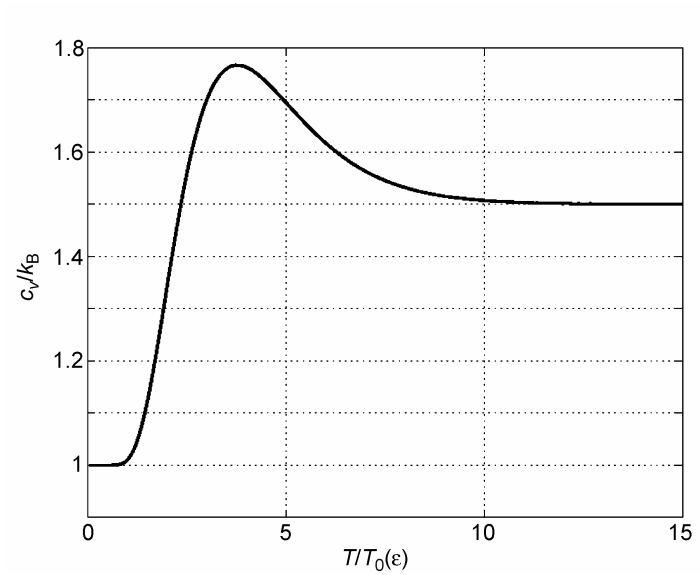

Fig. 1

3. RESULTS AND DISCUSSION

In the present work, the method of solving the main problem of thermodynamics of homogeneous spaces for any arbitrary Lie group has been constructed, including noncompact cases, without assumption about the unimodularity. The application of the method of noncommutative integration to the Bloch equation allowed us to avoid the complexities of the search for a global solution and to bypass the problem of divergence of the statistical sum in noncompact spaces for the Lie groups. This method gives no final universal formula for calculation of the statistical sum on the Lie groups supplied with the left-invariant Lorentz metric, being as a matter of fact the algorithm that allows one to find and to investigate the density matrix and the statistical sum in these spaces.

REFERENCES

1. N. Heart, Geometrical Quantization in Action [Russian translation], Nauka, Moscow (1984).
2. S. Minakshisundaram and A. Pleijel, Can. J. Math., **1**, 242–256 (1949).
3. S. R. S. Varadhan, Comm. Pure Appl. Math., **20**, 431–455 (1967).
4. V. F. Molchanov, Usp. Matem. Nauk, **30**, No. 6, 1–64 (1975).
5. V. V. Mikheev and I. V. Shirokov, Russ. Phys. J., No. 1, 6–14 (2003).
6. V. V. Mikheev and I. V. Shirokov, Electr. J. Theor. Phys., **2**, No. 7, 1–10 (2005).
7. I. V. Shirokov, Theor. Matem. Fiz., **123**, No. 3, 407–423 (2000).
8. A. A. Kirillov, Elements of Representation Theory [in Russian], Nauka, Moscow (1984).
9. A. Barut and R. Ronchka, Theory of Group Representations and Its Applications [in Russian], Ainshtain, Bishkek (1997).
10. S. P. Baranovskii, V. V. Mikheev, and I. V. Shirokov, Teor. Matem. Fiz., **129**, No. 1, 3–14 (2001).