Russian Physics Journal, Vol. 49, No. 12, 2006

THEORY OF QUATERNION ANGULAR MOMENTUM FOR UNIFIED FIELDS OF DYONS

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UDC 530.12

Unified quaternionic angular momentum for the fields of dyons and gravito-dyons has been developed and the commutation relations for dynamical variables are obtained in compact and consistent manner. Demonstrating the quaternion forms for unified fields of dyons (electromagnetic fields) and gravito-dyons (gravito-Heavisidian fields of linear gravity), corresponding quantum equations are reformulated in compact, simpler and manifestly covariant forms.

The question of existence of monopole [1–3] has become a challenging new frontier and the object of more interest in connection with quark confinement problem of quantum choromodynamics. The eight decades of this century witnessed a rapid development of the group theory and gauge field theory to establish the theoretical existence of monopoles and to explain their group properties and symmetries. Keeping in mind t'Hooft's solutions [4–5] and the fact that despite the potential importance of monopoles, the formalism necessary to describe them has been clumsy and not manifestly covariant, Rajput *et al*. [6–7] developed a self-consistent quantum field theory of generalized electromagnetic fields associated with dyons (particles carrying electric and magnetic charges). The analogy between linear gravitational and electromagnetic fields leads to the asymmetry in Einstein's linear equation of gravity and suggests the existence of gravitational analogue of magnetic monopole [8–9]. Like magnetic field, Cantani [10] introduced a new field (i.e. Heavisidian field) depending upon the velocities of gravitational charges (masses) and derived the covariant equations (Maxwell's equations) of linear gravity. Avoiding the use of arbitrary string variable [1], we have formulated manifestly covariant theory of gravito-dyons [11–12] in terms of two four-potentials leading to the structural symmetry between generalized electromagnetic fields of dyons [13–14] and generalized gravito-Heavisidian fields of gravito-dyons.

In this paper we have used quaternion analysis to combine the complex description of dyons and gravito-dyons. Generalized fields of dyons and gravito-dyons are combined together and the corresponding covariant field equations and equation of motion have been derived. It has been shown that the theory leads four different chirality parameters associated to electric, magnetic, gravitational, and Heavisidian charges. The study of gauge invariant and rotationally symmetric angular momentum operator for unified quaternionic fields of dyons and gravito-dyons has been undertaken and the commutation relations associated with the components of angular momentum and other dynamical parameters have been derived. The symmetry of these commutation relations requires an additional potential term, of appropriate strength depending upon the magnetic, gravitational, and Heavisidian parameter, besides the usual Coulombian potential term, in the Hamiltonian. It has been shown that this unified theory reproduces the dynamics of individual charges (masses) in the absence of others.

In order to formulate unified theory of generalized electromagnetic fields (associated with dyons) and generalized gravito-Heavisidian fields (associated with gravito-dyons) of linear gravity, we describe the properties of quaternion algebra with the use of natural units ($c = \hbar = 1$), and gravitational constant is taken unity. Unified quaternionic charge is described as [15]

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$$Q = (e, g, m, h) = e - ig - jm - kh\,, \tag{1}$$

where e, g, m, and h are respectively electric, magnetic, gravitational, and Heavisidian charges (masses) and i, j, and k are the quaternion units. The properties of quaternion described

$$ij = -ji = k \text{ (say)}, \tag{2a}$$

and

$$\begin{aligned} i(ij) &= (ii)j = -j, \\ (ij)j &= i(jj) = -i, \\ ik &= -ki = -j, \\ kj &= -jk = -i, \\ i^2 &= j^2 = k^2 = -1\,. \end{aligned} \tag{2b}$$

Complex structure (e, g) represents the generalized charge of dyons in electromagnetic fields while (m, h) is the generalized charge of gravito-dyons. The norm of unified quaternion charge is

$$N(Q) = Q\bar{Q} = (e^2 + g^2 + m^2 + h^2)\,, \tag{3}$$

where

$$\bar{Q} = (e, -g, -m, -h) = e + ig + jm + kh\,. \tag{4}$$

The interaction of a^{th} quaternionic charge Q_a in the field of b^{th} quaternionic charge Q_b depends on the quantity

$$\bar{Q}_a Q_b = (W_{ab}, X_{ab}, Y_{ab}, Z_{ab})\,, \tag{5}$$

where

$$\begin{aligned} W_{ab} &= e_a e_b + g_a g_b + m_a m_b + h_a h_b, \\ X_{ab} &= e_a g_b - g_a e_b + m_a h_b - h_a m_b, \\ Y_{ab} &= e_a m_b - m_a e_b + h_a g_b - g_a h_b, \\ Z_{ab} &= e_a h_b - h_a e_b + g_a m_b - m_a g_b. \end{aligned} \tag{6}$$

These four coupling parameters W_{ab}, X_{ab}, Y_{ab}, and Z_{ab} may be identified as electric, magnetic, gravitational, and Heavisidian parameters associated with the basis elements $(1, i, j, k)$ of a quaternion, respectively. The coupling parameters W_{ab} may also be obtained by taking the scalar product of quaternions Q_a and Q_b thus equals to norm (3) when $a = b$. It shows that if same quaternions were interacting with each other, their behavior would be Coulomb-like. W_{ab} may then be called Coulomb-like parameter while the second one X_{ab} corresponds to the combined chirality parameter associated with the interaction of two dyons (electromagnetic and gravito-dyons). Parameter Y_{ab} corresponds to the interaction between electric-gravitational and magnetic-Heavisidian charges while the fourth one Z_{ab} is the coupling parameter that correspond to the interaction of electric-Heavisidian and magnetic-gravitational charges.

Unified quaternion valued potential of dyons may then be defined as

$$V_\mu = (A_\mu, B_\mu, C_\mu, D_\mu) = A_\mu - iB_\mu - jC_\mu - kD_\mu , \tag{7a}$$

where A_μ is the four-potential associated with the dynamics of electric charge, B_μ is for magnetic charge, C_μ is the gravitational charge (mass) while D_μ is associated with the gravi-magnetic (Heavisidian) charge (mass). Then above equation (7a) is written in the following general form:

$$V = (A, B, C, D) = A - iB - jC - kD ,$$

where

$$\begin{aligned} A &= (A_0, A_1, A_2, A_3) = A_0 - iA_1 - jA_2 - kA_3, \\ B &= (B_0, B_1, B_2, B_3) = B_0 - iB_1 - jB_2 - kB_3, \\ C &= (C_0, C_1, C_2, C_3) = C_0 - iC_1 - jC_2 - kC_3, \\ D &= (D_0, D_1, D_2, D_3) = D_0 - iD_1 - jD_2 - kD_3. \end{aligned} \tag{7b}$$

Similarly one may define the quaternion valued unified fields as

$$\Im_{\mu\nu} = F_{\mu\nu} - i F^d_{\mu\nu} - j f_{\mu\nu} - k f^d_{\mu\nu} , \tag{8a}$$

where d stands for dual part of field tensors with usual definition and

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} - i\varepsilon_{\mu\nu\rho\sigma} B^{\rho\sigma} , \tag{8b}$$

$$F^d_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} - i\varepsilon_{\mu\nu\rho\sigma} A^{\rho\sigma} , \tag{8c}$$

$$f_{\mu\nu} = C_{\mu,\nu} - C_{\nu,\mu} - i\varepsilon_{\mu\nu\rho\sigma} D^{\rho\sigma} , \tag{8d}$$

$$f^d_{\mu\nu} = D_{\mu,\nu} - D_{\nu,\mu} - i\varepsilon_{\mu\nu\rho\sigma} C^{\rho\sigma} , \tag{8e}$$

and

$$\begin{aligned} F_{\mu\nu,\nu} &= j^{(e)}_\mu, \\ F^d_{\mu\nu,\nu} &= j^{(m)}_\mu, \\ f_{\mu\nu,\nu} &= j^{(G)}_\mu, \\ f^d_{\mu\nu,\nu} &= j^{(H)}_\mu. \end{aligned} \tag{8f}$$

In equations (8b) to (8e), the field tensors $F_{\mu\nu}$ and $F^d_{\mu\nu}$ are associated with dyons, while $f_{\mu\nu}$ and $f^d_{\mu\nu}$ are associated with gravito-dyons [16].

As such the quaternion valued current may then be defined as

$$J_\mu = j^{(e)}_\mu - i\, j^{(m)}_\mu - j\, j^{(G)}_\mu - k\, j^{(H)}_\mu \tag{9a}$$

or in general

$$J = \left(j^{(e)}, j^{(m)}, j^{(G)}, j^{(H)} \right),$$

which is the quaternion valued expression for field equation of dyons and gravito-dyons, i.e.,

$$\Im_{\mu\nu,\nu} = J_{\mu}. \tag{9b}$$

Lagrangian density may then be written in terms of compact, simpler, and consistent form as follows:

$$L = -M - \tfrac{1}{8}\bar{\Im}_{\mu\nu}\Im_{\rho\sigma}\eta^{\mu\nu}\eta^{\rho\sigma} + \tfrac{1}{2}\bar{V}_{\mu}J_{\nu}\eta^{\mu\nu} + \text{c.c.}, \tag{10}$$

where M is the effective mass of the particle and $\eta^{\mu\nu}$ is the flat space-time metric with signature –2. Combined linear momentum may then be written as

$$P_{\mu} \to p_{\mu} - \tfrac{1}{2}(\bar{Q}V_{\mu} + Q\overline{V_{\mu}}) \to p_{\mu} - eA_{\mu} - gB_{\mu} - mC_{\mu} - hD_{\mu}, \tag{11}$$

where V_{μ} is the quaternion valued potential defined by equation (7a), which leads to equation of motion in the following compact and simpler form [17]:

$$M\ddot{x}_{\mu} = \bar{Q}\Im_{\mu\nu}u^{\nu}, \tag{12}$$

where $\ddot{x}_{\mu}$ is the four-acceleration of the particle and u^{ν} is the four-velocity of the particle. The expression for linear momentum given by equation (11) is not consistent with the gauge invariant linear momentum described by us earlier [12–13]. To make it consistent in terms of quantization parameter, we define the gauge invariant linear as

$$\pi_j = p_j - \mathrm{Im}(\bar{Q}_a Q_b)V_j^T \qquad (j = 1, 2, 3), \tag{13}$$

where $\mathrm{Im}(\bar{Q}_a Q_b) = iX_{ab} + jY_{ab} + kZ_{ab}$, V^T is the transverse part of the four-potential. Thus the gauge invariant and rotationally symmetric angular momentum in the unified quaternionic field of dyons and gravito-dyons may be written as

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{\pi} + (X_{ab} + Y_{ab} + Z_{ab})\frac{\boldsymbol{r}}{r}, \tag{14}$$

where $(X_{ab} + Y_{ab} + Z_{ab})\frac{\boldsymbol{r}}{r}$ is the residual angular momentum [12]. The angular momentum given by Eq. (14) leads to the following commutation relations:

$$[J_k, J_l] = i\varepsilon_{klm}J_m, \tag{15a}$$

$$[J^2, J_k] = [J^2, J_l] = [J^2, J_m] = 0, \tag{15b}$$

$$[J_k, r_l] = i\varepsilon_{klm}r_m, \tag{15c}$$

$$[\pi_k, r_l] = -i\delta_{kl}, \tag{15d}$$

$$[\pi_k, \pi_l] = i(X_{ab} + Y_{ab} + Z_{ab})\varepsilon_{klm}\psi_m \,, \tag{15e}$$

$$[J_k, \pi_l] = i\varepsilon_{klm}\pi_m - i(X_{ab} + Y_{ab} + Z_{ab})r_l\psi_k \,. \tag{15f}$$

Keeping in mind the usual Coulombian problem of electric charges, we except the following manifestly gauge invariant and rotationally symmetric Hamiltonian for the system of unified charges:

$$H = \frac{\pi^2}{2m} - \frac{W_{ab}}{r} + V(r) \,, \tag{16}$$

where π is the gauge invariant linear momentum described by Eq. (15). The additional potential term $V(r)$ may be identified and decided by the symmetry requirements of the system. Using Eq. (14), the value of the operator J^2 may be calculated as

$$J^2 = \boldsymbol{JJ} = (\boldsymbol{r}\times\boldsymbol{\pi})(\boldsymbol{r}\times\boldsymbol{\pi}) + (X_{ab}^2 + Y_{ab}^2 + Z_{ab}^2) + 2(X_{ab} + Y_{ab} + Z_{ab})\boldsymbol{r}(\boldsymbol{r}\times\boldsymbol{\pi}) \,. \tag{17}$$

The third term in Eq. (17) is zero and hence we write

$$\frac{\pi^2}{2m} = \frac{J^2}{2mr^2} - \frac{(X_{ab}^2 + Y_{ab}^2 + Z_{ab}^2)}{2mr^2} + \frac{p^2}{2m} \,, \tag{18}$$

where p is spatial part of linear momentum. It is quite obvious that the Hamiltonian given by Eq. (16) possesses the higher symmetry same as that of the pure Coulomb Hamiltonian provided the additional potential $V(r)$ in Eq. (16) takes the scalar form:

$$V(r) = \frac{(X_{ab}^2 + Y_{ab}^2 + Z_{ab}^2)}{2mr^2} \,. \tag{19}$$

Thus the Hamiltonian given by Eq. (16) may be written as

$$H = \frac{\pi^2}{2m} - \frac{W_{ab}}{r} + \frac{(X_{ab}^2 + Y_{ab}^2 + Z_{ab}^2)}{2mr^2} \,, \tag{20}$$

which leads to the following commutation relations:

$$[J^2, H] = [J, H] = 0. \tag{21}$$

Quaternion valued unified vector field $\boldsymbol{\psi}$ is written as

$$\boldsymbol{\psi} = (\boldsymbol{E}, \boldsymbol{M}, \boldsymbol{G}, \boldsymbol{H}) \,, \tag{22}$$

where $\boldsymbol{E}, \boldsymbol{M}, \boldsymbol{G},$ and $\boldsymbol{H}$ are generalized electric, magnetic, gravitational, and Heavisidian fields, respectively, defined as [15]

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla A_0 + \nabla \times \boldsymbol{B} \,, \tag{23a}$$

$$\boldsymbol{M} = -\frac{\partial \boldsymbol{B}}{\partial t} - \nabla B_0 + \nabla \times \boldsymbol{A}, \tag{23b}$$

$$\boldsymbol{G} = -\frac{\partial \boldsymbol{C}}{\partial t} - \nabla C_0 + \nabla \times \boldsymbol{D}, \tag{23c}$$

$$\boldsymbol{H} = -\frac{\partial \boldsymbol{D}}{\partial t} - \nabla D_0 + \nabla \times \boldsymbol{C}. \tag{23d}$$

These generalized electric, magnetic, gravitational, and Heavisidian fields are described in terms of quaternion valued fields in the following manner:

$$\begin{aligned} F_{0i} &= E_i, \quad F_{ij} = \varepsilon_{ijk} M^k, \\ F_{0i}^d &= M_i, \quad F_{ij}^d = \varepsilon_{ijk} E^k, \\ f_{0i} &= G_i, \quad f_{ij} = \varepsilon_{ijk} H^k, \\ f_{0i}^d &= H_i, \quad f_{ij}^d = \varepsilon_{ijk} G^k, \end{aligned} \tag{24}$$

and thus satisfy the following pair of Maxwell's equations for dyons and gravito-dyons [11–12]:

$$\begin{aligned} \nabla \boldsymbol{E} &= j_0^{(e)}, \\ \nabla \boldsymbol{G} &= j_0^{(m)}, \\ \nabla \boldsymbol{M} &= j_0^{(G)}, \\ \nabla \boldsymbol{H} &= j_0^{(H)}, \end{aligned} \tag{25a}$$

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -\frac{\partial \vec{M}}{\partial t} - \boldsymbol{j}^{(G)}, \\ \nabla \times \boldsymbol{G} &= -\frac{\partial \vec{H}}{\partial t} - \boldsymbol{j}^{(H)}, \\ \nabla \times \boldsymbol{M} &= -\frac{\partial \boldsymbol{E}}{\partial t} - \boldsymbol{j}^{(e)}, \\ \vec{\nabla} \times \boldsymbol{H} &= -\frac{\partial G}{\partial t} - \boldsymbol{j}^{(m)}, \end{aligned} \tag{25b}$$

where $\left(j_0^{(e)}, j_0^{(m)}, j_0^{(G)}, j_0^{(H)}\right)$ are respectively charge densities for electric, magnetic, gravitational, and Heavisidian charges while $\left(\boldsymbol{j}^{(e)}, \boldsymbol{j}^{(m)}, \boldsymbol{j}^{(G)}, \boldsymbol{j}^{(H)}\right)$ are the corresponding current source densities of these charges. Quaternion unified potential $\boldsymbol{V}$ given by Eq. (7a) and unified field $\boldsymbol{\psi}$ are then related as

$$\boldsymbol{\psi} = -\frac{\partial \boldsymbol{V}}{\partial t} - \nabla V_0 + \nabla \times \boldsymbol{V}. \tag{26a}$$

Operating the quaternionic differential operator ∂ to Eq. (7a), we get the following unified form of potential equation:

$$\partial \boldsymbol{V} = \boldsymbol{\psi}, \tag{26b}$$

which is quaternion valued unified potential equation for generalized charges of dyons. Unified quaternion valued current may then is written as

$$\boldsymbol{J} = \left(j^{(e)}, j^{(m)}, j^{(G)}, j^{(H)} \right),$$

where

$$\partial\bar{\partial}\, A = []A = j^{(e)}, \tag{27a}$$

$$\partial\bar{\partial}\, B = []B = j^{(m)}, \tag{27b}$$

$$\partial\bar{\partial}\, C = []C = j^{(G)}, \tag{27c}$$

$$\partial\bar{\partial}\, D = []D = j^{(H)}, \tag{27d}$$

$$\partial\bar{\partial} = [] = \partial_{\mu}\partial_{\mu} = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{28}$$

Quaternion field equation associated with unified quaternion charge may then be written as

$$\partial\bar{\partial} V = []V = J. \tag{29}$$

Quaternion valued field tensor density is defined as

$$Q_{\mu\nu} = (A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}, D_{\mu\nu}) \quad (\mu, \nu = 0,1,2,3), \tag{30}$$

where

$$Q_{\mu\nu} = V_{\mu,\nu} - V_{\nu,\mu}, \tag{30a}$$

$$A_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}, \tag{30b}$$

$$B_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu}, \tag{30c}$$

$$C_{\mu\nu} = C_{\mu,\nu} - C_{\nu,\mu}, \tag{30d}$$

$$D_{\mu\nu} = D_{\mu,\nu} - D_{\nu,\mu}, \tag{30e}$$

are respectively field tensors of electric, magnetic, gravitational, and Heavisidian charges and comma (,) denotes partial differentiation; A_{μ} and B_{μ} are dual invariant for generalized electromagnetic fields of dyons while C_{μ} and D_{μ} are dual invariant under duality transformations for generalized fields of gravito-dyons. Unified quaternion valued current and field tensor $Q_{\mu\nu}$ are then related in the following manner:

$$Q_{\mu\nu,\nu} = J_{\mu}, \tag{31}$$

which is equivalent to the following quaternionic form:

$$\bar{\partial}\psi = J. \tag{32}$$

A unified field theory of generalized electromagnetic and Heavisidian fields have been developed in view of the fact that these fields essentially possess the structural symmetry. Quaternion charge defined by Eq. (1) represents the theory of particles carrying simultaneously electric, magnetic, gravitational, and Heavisidian charges. Equation (1) has been described as the combination of two complex charges of dyons (e, g, o, o) and gravito-dyons (o, o, m, h). Equation (6) represents the coupling parameters W_{ab}, X_{ab}, Y_{ab}, and Z_{ab} and this equation shows that in the presence of only electric charge the coupling becomes $W_{ab} = e_a e_b$ while the other parameters X_{ab}, Y_{ab}, and Z_{ab} vanish. If the particles are considered as dyons of electromagnetic fields, we have $W_{ab} = e_a e_b + g_a g_b$ and $X_{ab} = e_a g_b - g_b e_a$ only respectively named as electric and magnetic coupling parameters. For gravito-dyons (o, o, m, h) we have only $W_{ab} = m_a m_b + h_a h_b$ and $Y_{ab} = e_a h_b - h_a e_b$. For other dyon like ($e, o, o, h$), i.e., electric and Heavisidian charges we get $W_{ab} = e_a e_b + h_a h_b$ and $Z_{ab} = e_a h_b - h_a e_b$. For other dyon like ($0, g, m, o$), i.e., gravitational charge and magnetic monopole, we get $W_{ab} = m_a m_b + g_a g_b$ and $Z_{ab} = g_a m_b - m_b g_a$. Similarly for purely hypothetical dyons (o, g, o, h), i.e., Heavisidian and magnetic charge we get $W_{ab} = h_a h_b + g_a g_b$ and $Z_{ab} = h_a g_b - g_a h_b$. As such the unified theory of quaternionic charge characteristics is the study of fields associated with dyons.

Equation (7) represents unified quaternion valued potential of dyons. It is to be noted that here we have combined four-charges (electric, magnetic, gravitational, and Heavisidian) in terms of tetrad combination in the context of special relativity (not in terms of generalized coordinates) and maintained the Abelian gauge structure $U^e(1) \times U^m(1) \times U^g(1) \times U^h(1)$. While the quaternion structure leads to rotation charge space and we shall try to find out these investigation of quaternion rotation shortly to mix up, the gravitational charge m can be either positive or negative; this should imply that gravitational and Heavisidian charges may be either positive or negative dealing with the concept of negative masses. On the other hand, properties of quantum algebra are incorporated to derive field equation (8a) as a quaternion and the coefficients of its basis elements representing the algebraic gauge structure field equations. Lagrangian density (10) has been shown to yield the compact and simpler representation of combined dyonic field equation (8b) and equation of motion (12). The gauge invariant and rotationally symmetric angular momentum in the unified quaternionic fields of dyons and gravito-dyons shown by Eq. (14) contains a residual angular momentum $J_{res} = (X_{ab} + Y_{ab} + Z_{ab})\frac{\boldsymbol{r}}{r}$ in addition to the usual kinetic angular momentum. The rotationally symmetric nature of angular momentum is reflected in the commutation relation given by Eq. (15c). The commutation relations given by Eqs. (15b) and (21) show that the operators J^2 and J_z are the constant of motion. The commutation relation (21) demands an additional term $\frac{(X_{ab}^2 + Y_{ab}^2 + Z_{ab}^2)}{2mr^2}$ in the Hamiltonian, so that it possesses the higher symmetry as the pure Coulomb Hamiltonian. From the above analysis it may be concluded that besides the potential importance of monopole as intrinsic part of grand unified theories, monopoles and dyons may provide even more ambitious model to purport the unification of gravitation with strong and electro-weak forces. It is the question of importance that there must be a dyon where e, g, m, and h are nonzero to yield the consistent, unified, symmetrical, and dual invariant concepts of electromagnetic and gravito-Heavisidian fields. If one has to consider the unified representation for dealing the theories of electron, monopole, gravitational, and Heavisidian charges together, one must have quaternion representation of four charges (e, g, m, h) instead of two charges (dyons) (e, g, o, o), (e, o, m, o), (e, o, m, o), (o, g, o, h), etc.

The electric, magnetic, gravitational, and Heavisidian fields defined by Eq. (23) describe the pair of Maxwell's equations associated with generalized fields of dyons (e, g) and gravito-dyons (m, h). Equation (32) describes the unified forms of quaternion four-current density. These quaternion field equations are invariant under quaternion and duality transformations.

We are highly thankful to the referee for his valuable suggestions and critical analysis of the manuscript.

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