Russian Journal of Nondestructive Testing, Vol. 41, No. 8, 2005, pp. 502–513. Translated from Defektoskopiya, Vol. 41, No. 8, 2005, pp. 28–42.

The Field of a Finite Defect in a 3D Semispace

V. V. Dyakin, O. V. Umergalina, and V. Ya. Raevskii

Institute of Metal Physics, Ural Division, Russian Academy of Sciences, ul. Sof'i Kovalevskoi 18, Yekaterinburg, 620219 Russia

Received April 14, 2005

Abstract—The field of an arbitrary finite defect in a 3D semispace is calculated. This problem is solved by reduction to a 2D case that yields an integral equation on the surface of the defect alone. Calculation formulas for a spherical defect are presented. Results of calculations using these formulas for the case of a uniform external field normal or tangential to the surface of a magnetic semispace are presented in a graphical form.

Solving the magnetostatic problem for different models of bodies containing defects remains a challenging task for practical use of magnetic testing methods. The internal defect is approximated mainly by an infinite elliptic cylinder placed either in an infinite magnetic space, in a semispace, or in a plate. (Such an approximation is used owing to its simplicity.). The symmetry of the model enables significant simplification of the problem, which can be reduced to a 2D case. Corresponding results can be found, e.g., in [5].

A solution of the magnetostatic problem for a defect with finite dimensions has only been found for an infinite magnetic space containing a cavity (insert) with an ellipsoidal surface [5]. In continuation of [6], we solve in this study the problem of a semispace containing a defect with finite dimensions. A technique was used in [6] that allows reduction of the problem of finding the field of a defect with finite dimensions located in a semispace to calculation of the value of the normal field component on the defect's surface alone. Then, the case of a spherical defect was considered where the external field was assumed to be uniform and directed normally to the surface of the semispace. Publication [6] contains typographical errors. We apologize to the readers for them and briefly iterate the main stages of the study and the most important formulas since, moreover, they are of importance for this study as well.

1. Let us use an integrodifferential approach to the magnetostatic problem. The equation for the strength vector of the magnetic field has the form

$$\mathbf{H}(\mathbf{r}) - \nabla \mathrm{div}\int_{\Omega}\frac{(\mu-1)\mathbf{H}(\mathbf{r}')}{4\pi R}d\mathbf{r}' = \mathbf{H}^0(\mathbf{r}), \qquad (1)$$

where Ω is the area occupied by a magnetic matter with magnetic permeability μ, $\mathbf{H}^0(\mathbf{r})$ is the strength of the field created by external sources, and $R = |\mathbf{r} - \mathbf{r}'|$.

Using Eq. (1), let us consider the following problem: a defect characterized by a finite diameter and magnetic permeability $\mu_d = \mathrm{const}$ and confined within smooth surface S is located in the semispace $z \le d$ $(d > 0)$ with $\mu = \mathrm{const}$. The frame origin is placed at the center of the defect. In this case, area Ω in (1) is a semispace containing the defect. Let us denote the area occupied by the defect as Ω_d and $\Omega_1 = \Omega/\Omega_d$. The described situation where the defect is approximated by a sphere is illustrated by Fig. 1. It is worth stressing that, for the calculations made in the first part of this study to hold for a defect of any form, it only needs to be finite. The result of the first stage is formulated as Eq. (7) applicable to solution of the problem for a defect of any form. Moreover, Eq. (7) is preferable for practical usage since it requires the normal component of the magnetic-field strength to be known on the defect's surface alone. (To this end, Eq. (8) is presented, which can be deduced easily from (7).) In the second part of this publication, a spherical defect is considered, while this part is devoted to deduction of the promised equations.

If the magnetic permeability is constant, $\mathrm{div}\mathbf{H} = 0$ and can be reduced by Ostrogradskii–Gauss's theorem Eq. (1) to

$$\mathbf{H}(\mathbf{r}) + \frac{\mu-1}{4\pi} = \int_{z'=d}\frac{H_z(\rho', d)d\rho'}{\sqrt{(\rho-\rho')^2+(z-d)^2}} = \mathbf{H}^0(\mathbf{r}) - \frac{\mu-\mu_d}{4\pi\mu_d}\nabla\int_S\frac{H_n(\mathbf{r}')dS'}{R}, \qquad (2)$$

1061-8309/05/4108-0502 © 2005 MAIK "Nauka/Interperiodica"

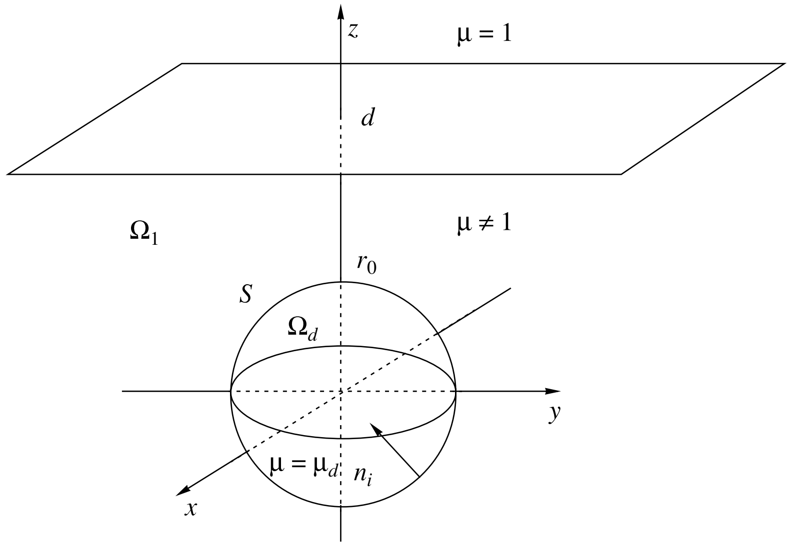

Fig. 1. Semispace with a defect.

where $\boldsymbol{\rho} = (x, y)$ is a 2D radius-vector, $H_n = (\mathbf{H}, \mathbf{n}_i)$ is a projection of the field measured on surface S from the interior of area Ω_1 on normal $\mathbf{n}_i$ set to the same surface also from area Ω_1, and H_z is a normal component of the field on the semispace surface from the interior of Ω_1.

Consider expression (2) while assuming that $\mathbf{r} \in \Omega_1$. Having found the limit of the z component in both parts of formula (2) when $z \longrightarrow d$, we arrive at

$$H_z(\boldsymbol{\rho}, d) = \frac{2}{\mu+1}\left\{H_z^0(\rho, d) + \frac{\mu-\mu_d}{4\pi\mu_d}\int_S \frac{H_n(\mathbf{r}')(d-z')dS'}{((\rho-\rho')^2+(d-z')^2)^{3/2}}\right\}. \tag{3}$$

During deduction of (3), formula

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{1}{2\pi}\int_{R^2} \frac{\varepsilon f(\boldsymbol{\rho}')d\boldsymbol{\rho}'}{((\boldsymbol{\rho}-\boldsymbol{\rho}')^2+\varepsilon^2)^{3/2}} = f(\boldsymbol{\rho}) \tag{4}$$

was used. The latter, in turn, can be proved easily by applying Parseval's equality

$$\int_{R^2} f(\rho)g(\rho)d\rho = \int \hat{f}(\mathbf{k})\hat{g}(\mathbf{k})d\mathbf{k} \tag{5}$$

to functions $f(\boldsymbol{\rho}')$ and $\varepsilon/((\boldsymbol{\rho}-\boldsymbol{\rho}')^2+\varepsilon^2)^{3/2}$, according to which an integral of the product of two functions $f(\boldsymbol{\rho})$ and $g(\boldsymbol{\rho})$ is equal to the integral of their Fourier transforms $\hat{f}(\mathbf{k})$ and $\hat{g}(\mathbf{k})$.

Substituting (3) into the left-hand side of (2), we obtain

$$\mathbf{H}(\mathbf{r}) + \frac{\mu-\mu_d}{4\pi\mu_d}\nabla\left\{\int_S \frac{H_n(\mathbf{r}')}{R}dS' + \frac{\lambda}{2\pi}\int_S dS''H_n(\mathbf{r}'')\int_{R^2}\frac{d\boldsymbol{\rho}'}{\sqrt{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2+(z-d)^2}}\frac{(d-z'')}{((\boldsymbol{\rho}'-\boldsymbol{\rho}'')^2+(d-z'')^2)^{3/2}}\right\}$$
$$= \mathbf{H}^0(\mathbf{r}) - \frac{\lambda}{2\pi}\nabla\int_{R^2}\frac{H_z^0(\boldsymbol{\rho}', d)d\boldsymbol{\rho}'}{\sqrt{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2+(z-d)^2}}, \tag{6}$$

where $\lambda = (\mu-1)/(\mu+1)$. If the notations $\tau = \boldsymbol{\rho}' - \boldsymbol{\rho}$, $\mathbf{t} = \boldsymbol{\rho}'' - \boldsymbol{\rho}$, $a = |z-d|$, and $b = d - z''$ are introduced, the internal integral in the second term in the braces can be written as

$$J = \frac{1}{2\pi}\int_a^{+\infty} da' \int_{R^2} \frac{a'}{(\boldsymbol{\tau}^2 + a'^2)^{3/2}} \frac{b}{((\boldsymbol{\tau} - \mathbf{t})^2 + b^2)^{3/2}} d\boldsymbol{\tau}.$$

The internal integral in the expression above can be calculated analytically with expression (5). Having calculated Fourier transforms of the factors and using tables of integrals [1, pp. 967, 721], we arrive at $J = 1/\sqrt{t^2 + (a+b)^2}$. With allowance for this result, expression (6) can be rewritten in the form

$$\mathbf{H}(\mathbf{r}) + \frac{\mu - \mu_d}{4\pi\mu_d}\nabla\left\{\int_S \frac{H_n(\mathbf{r}')}{R} dS' + \lambda \int_S \frac{H_n(\mathbf{r}')}{R^*} dS'\right\} = \mathbf{H}^0(\mathbf{r}) - \frac{\lambda}{2\pi}\nabla \int_{R^2} \frac{H_z^0(\boldsymbol{\rho}', d)}{R_0} d\boldsymbol{\rho}', \tag{7}$$

where $R^* = \sqrt{(\boldsymbol{\rho} - \boldsymbol{\rho}')^2 + (|d - z| + d - z')^2}$ and $R_0 = \sqrt{(\boldsymbol{\rho} - \boldsymbol{\rho}')^2 + (z - d)^2}$.

A review of the procedure used to deduce (7) shows that the applied technique suits not only the case of a semispace but the case of any other body containing a defect provided that the problem not involving defects can be solved analytically and the integral calculated for (6) can be represented in an analytical form.

Equation (7) is only needed to find the value of normal field component H_n on surface S of a defect. Then, the result obtained is substituted into (7) and the field can be calculated at any point in the space. Let us multiply expression (7) by $\mathbf{n}_i$ and then find the limit of the product for surface S by use of the formulas of the classical potential theory [2]. In this way, we obtain the required equation for normal field component H_{n_i} on S:

$$\begin{aligned} H_{n_i}(\mathbf{r}) + \frac{\lambda_d}{2\pi}\int_S H_{n_i}(\mathbf{r}')\overline{\left(\frac{\partial}{\partial n_i}\frac{1}{R}\right)} dS' + \frac{\lambda\lambda_d}{2\pi}\int_S H_{n_i}(\mathbf{r}')\frac{\partial}{\partial n_i}\frac{1}{R^*} dS' \\ = \frac{2\mu_d}{\mu + \mu_d}\left\{H_{n_i}^0(\mathbf{r}) - \frac{\lambda}{2\pi}\int_{R^2} H_z^0(\boldsymbol{\rho}', d)\frac{\partial}{\partial n_i}\frac{1}{R_0} d\boldsymbol{\rho}'\right\}, \end{aligned} \tag{8}$$

where $\mathbf{r} \in S$, $\lambda_d = (\mu - \mu_d)/(\mu + \mu_d)$. The second term in the left-hand side of (8) is the value of the normal derivative of the potential of a simple layer on surface S [2]. Kernels $1/R^*$ and $1/R_0$ are regular on S, and $R^* = \sqrt{(\boldsymbol{\rho} - \boldsymbol{\rho}')^2 + (2d - z - z')^2}$.

2. Continuing further, let us now specify the shape of the defect. Let us consider a spherical defect that has radius r_0 and $\mu_d = 1$. Let d be the distance from the center of the sphere, where the frame origin is located, to the surface of the semispace (Fig. 1). In the spherical system of coordinates, $H_{n_i} = -H_r$, $\frac{\partial}{\partial n_i} = -\frac{\partial}{\partial r}$, and Eq. (8) takes the form

$$\begin{aligned} H_r(\mathbf{r}) - \frac{\lambda}{2\pi}\int_S H_r(\mathbf{r}')\overline{\left(\frac{\partial}{\partial r}\frac{1}{R}\right)}_{r = r' = r_0} dS' - \frac{\lambda^2}{2\pi}\int_S H_r(\mathbf{r}')\left(\frac{\partial}{\partial r}\frac{1}{R^*}\right)_{r = r' = r_0} dS' \\ = (1 - \lambda)\left\{H_r^0(\mathbf{r}) - \frac{\lambda}{2\pi}\int_{R^2} H_z^0(\boldsymbol{\rho}', d)\left(\frac{\partial}{\partial r}\frac{1}{R_0}\right)_{r = r_0} d\boldsymbol{\rho}'\right\}, \quad \mathbf{r} \in S. \end{aligned} \tag{9}$$

The last equation can be presented in the following shorthand form:

$$(\hat{I} + \lambda\hat{W}_1)H_r - \lambda^2\hat{W}_2 H_r = (1 - \lambda)\{H_r^0 - \lambda\hat{W}_3 H_z^0\}, \tag{10}$$

where $\hat{I}$ is the unit operator, and the form of operators $\hat{W}_k$ ($k = 1, 2, 3$) becomes clear when comparing expressions (10) and (9).

Below, the well-known expansion is used repeatedly:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{4\pi}{r}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(\frac{r'}{r}\right)^l\sum_{m=-l}^{l}Y_{lm}(\theta, \varphi)Y^*_{lm}(\theta', \varphi'), \tag{11}$$

where r, θ, and φ are spherical coordinates of radius-vector $\mathbf{r}$ (primed spherical coordinates refer to vector $\mathbf{r}'$), $r' < r$, and spherical harmonics $Y_m(\theta, \varphi)$ are defined as follows [3]:

$$\begin{aligned} Y_{lm}(\theta, \varphi) &= \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}, \\ P_l^m(\cos\theta) &= (-\sin\theta)^m d^m P_l(t)/dt^m\big|_{t=\cos\theta}, \quad m \ge 0, \\ P_l^{-|m|}(\cos\theta) &= (-1)^{|m|}\frac{(l-|m|)!}{(l+|m|)!}P_l^{|m|}(\cos\theta), \quad m < 0, \end{aligned} \tag{12}$$

$P_l(t)$ are Legendre polynomials, and $l = 0, 1, \ldots, |m| \le l$. Each spherical harmonic depends on θ and φ (angular coordinates of a vector). Below, if notation $Y_{lm}(\mathbf{r})$ is used, then, of course, we will obtain the angular coordinates of a given vector in a form that uses independent variables.

Let us consider the action of operators contained in (10).

(a)
$$\hat{W}_1 H_r = -\frac{1}{2\pi}\int_S H_r(\mathbf{r}')\overline{\left(\frac{\partial}{\partial r}\frac{1}{R}\right)}_{r=r'=r_0} dS'. \tag{13}$$

Calculating the derivative and using (11), we obtain

$$\hat{W}_1 H_r = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}(\theta, \varphi)}{2l+1}\int_{S_1} H_r(r_0, \theta', \varphi')Y^*_{lm}(\theta', \varphi')dS'_1, \tag{14}$$

where S_1 is a unit sphere. Expression (14) takes into account that, if $\mu = \text{const}$, $\int_S H_n(\mathbf{r})dS' = 0$.

(b)
$$\hat{W}_2 H_r = \frac{1}{2\pi}\int_S H_r(\mathbf{r}')\left(\frac{\partial}{\partial r}\frac{1}{R^*}\right)_{r=r'=r_0} dS'. \tag{15}$$

To calculate this integral, notations $\mathbf{r}_1 = (x, y, -z)$ and $\mathbf{D} = (0, 0, 2d)$ are adopted; in that case, $R^* = |\mathbf{r}' - \mathbf{r}_1 - \mathbf{D}|$. When expanding kernel $\frac{1}{4\pi R^*}$ using formula (11), it is taken into account that $|\mathbf{r}' - \mathbf{r}_1| < D$, $Y_{lm}(\mathbf{D}) = \delta_{m,0}\sqrt{(2l+1)/4\pi}$, and the expansion theorem [3] is applied

$$|\mathbf{r}'-\mathbf{r}_1|^l Y_{l0}(\mathbf{r}'-\mathbf{r}_1) = 2^l\sqrt{4\pi(2l+1)}\sum_{\alpha=0}^{l}\sum_{\beta=-\sigma}^{\sigma}\frac{1}{\alpha}T^l_{\alpha,\beta;\, l-\alpha,\beta}r^{\alpha}r'^{\,l-\alpha}Y_{\alpha\beta}(\mathbf{r})Y^*_{l-\alpha,\beta}(\mathbf{r}'),$$

$$T^l_{\alpha,\beta;\, l-\alpha,\beta} = \frac{\alpha}{2^l}\sqrt{\frac{C_l^{\alpha-\beta}C_l^{\alpha+\beta}}{(2\alpha+1)(2l-2\alpha+1)}}, \quad C_n^m = \frac{n!}{m!(n-m)!}, \quad \sigma = \min(\alpha, l-\alpha).$$

We obtain

$$\hat{W}_2 H_r = \sum_{l=2}^{\infty}\varepsilon^{l+1}\sum_{\alpha=1}^{l-1}\sum_{\beta=-\sigma}^{\sigma}T^l_{\alpha,\beta;\, l-\alpha,\beta}Y_{\alpha\beta}(\theta, \varphi)\int_{S_1} H_r(r_0, \theta', \varphi')Y^*_{l-\alpha,\beta}(\theta', \varphi')dS'_1. \tag{16}$$

(c)
$$\hat{W}_3 H_z^0 = \left\{\frac{1}{2\pi}\frac{\partial}{\partial r}\int_{R^2}\frac{H_z^0(\boldsymbol{\rho}', d)}{R_0}d\boldsymbol{\rho}'\right\}_{r=r_0}.$$

Kernel $1/R_0$ also can be handled using expansion (11) and taking into account that $r' > r = r_0$ if $\mathbf{r}' = (x', y', d) = (\boldsymbol{\rho}', d)$. We arrive at the following expression

$$\hat{W}_3 H_z^0 = 2\sum_{\substack{l=1\\|m|\le l}}^{\infty} \frac{l r_0^{l-1} Y_{lm}(\theta,\varphi)}{2l+1}\int_{R^2}\frac{H_z^0(\boldsymbol{\rho}', d)Y_{lm}^*(\theta',\varphi')}{r'^{l+1}}d\boldsymbol{\rho}', \tag{17}$$

where $r' = \sqrt{\rho'^2 + d^2}$ and $\cos\theta' = d/r'$.

Results obtained in this section can be used to solve the problem of a semispace with a spherical cavity wherein different models of the external-field sources can be chosen.

3. Let us now specify the problem. Let us assume that an external field is directed normally to the semispace surface; i.e. $\mathbf{H}^0 = (0, 0, H^0)$. Then, $H_r^0|_S = H^0\cos\theta$ and the calculations using expression (17) yield $\hat{W}_3 H_z^0 = H^0\cos\theta$. In the considered case, the solution to Eq. (10) does not depend on angle φ as a result of the symmetry of the problem. Let us introduce new function $h(\theta) = \dfrac{H_r(\theta)}{(1-\lambda)^2 H^0}$ and rewrite Eq. (10) for this function:

$$(\hat{I} + \lambda\hat{W}_1)h - \lambda^2\hat{W}_2 h = \cos\theta. \tag{18}$$

Solution of (18) can be found in the form of a series in spherical harmonics. Since $h = h(\theta)$, this series reduces to a series of the Legendre polynomials

$$h(\theta) = \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)h_n P_n(\cos\theta), \quad h_l = \int_{-1}^{1} h(t)P_l(t)dt. \tag{19}$$

Formulas (14) and (16) yield, respectively, $\hat{W}_1 h = \frac{1}{2}\sum_{l=1}^{\infty} h_l P_l(\cos\theta)$ and $\hat{W}_2 h = \sum_{l=2}^{\infty}\varepsilon^{l+1}\sum_{\alpha=1}^{l-1} h_{l-\alpha}T_{l,\alpha}P_\alpha(\cos\theta)$, where $T_{l,\alpha} = \dfrac{\alpha}{2^{l+1}}C_l^\alpha$. Therefore, having substituted obtained expressions into (18), we arrive at an infinite set of equations for coefficients h_l, $l = 1, 2, \ldots$:

$$\frac{1}{2}\sum_{l=1}^{\infty}(2l+1+\lambda)h_l P_l(\cos\theta) - \lambda^2\sum_{l=2}^{\infty}\varepsilon^{l+1}\sum_{\alpha=1}^{l-1}T_{l,\alpha}h_{l-\alpha}P_\alpha(\cos\theta) = P_1(\cos\theta). \tag{20}$$

Multiplying both sides of (20) by $P_n(\cos\theta)$, $n = 1, 2, \ldots$, integrating by $t = \cos\theta$ from –1 to 1, and taking into account the orthogonality of the Legendre polynomials and their normalization $\|P_n\|^2 = 2/(2n+1)$, we obtain a set of equations

$$\sum_{m=1}^{\infty}\left\{\frac{2n+1+\lambda}{n}\delta_{nm} - \lambda^2\varepsilon A_{nm}\right\}h_m = 2\delta_{n1}; \quad n = 1, 2, \ldots;$$
$$A_{nm} = \left(\frac{\varepsilon}{2}\right)^{n+m}C_{n+m}^n. \tag{21}$$

By introducing new variables $x_m = h_m\sqrt{2 + \dfrac{1+\lambda}{m}}$, $m = 1, 2, \ldots$, system (21) can be reduced to a form that is easier to comprehend. Then,

$$\sum_{m=1}^{\infty}\{\delta_{nm} - K_{nm}\}x_m = 2\delta_{n1}/\sqrt{3+\lambda}, \quad n = 1, 2, \ldots;$$
$$K_{nm} = \frac{\varepsilon\lambda^2 A_{nm}}{\sqrt{2+\dfrac{1+\lambda}{n}}\sqrt{2+\dfrac{1+\lambda}{m}}}. \tag{22}$$

Thus, the solution to Eq. (22) yields coefficients $\{x_n\}_{n=1}^{\infty}$ and, hence, $\{h_n\}_{n=1}^{\infty}$. In that case, the field on the defect's surface is determined using (19):

$$H_r(\theta) = \frac{1}{2}(1-\lambda)^2 H^0 \sum_{n=1}^{\infty}(2n+1)h_n P_n(\cos\theta). \tag{23}$$

This result can be used to find field $\mathbf{H}^d$ created by the defect above a metal: $\mathbf{H}^d = \mathbf{H} - \mathbf{H}^e$, where $\mathbf{H}$ is the field created by the semispace with a defect and $\mathbf{H}^e$ is the field created by the same semispace without it. After the normal component has been found using formula (23), both vectors can be calculated using expression (7): $\mathbf{H}$ corresponds to $\mu_d = 1$ and $\mathbf{H}^e$ to $\mu_d = \mu$ (no defect). Since $R^* = R$ for $z > d$, we obtain the following expression from (7):

$$\mathbf{H}^d(\mathbf{r}) = \frac{\lambda(1+\lambda)}{2\pi(1-\lambda)}\nabla\int_S \frac{H_r(\mathbf{r}')}{R}dS'. \tag{24}$$

Then, this formula is used to deduce expressions for the normal and tangential components of $\mathbf{H}^d$ on the outer surface of semispace $z = d$. Owing to the symmetry of resulting field $\mathbf{H}^d$ in the considered case, it is sufficient to provide formulas for $y = 0$:

$$\begin{aligned} H_z^d(\xi, y=0) &= \lambda(1-\lambda^2)H^0\sum_{l=1}^{\infty}(\varepsilon\eta)^{l+2}h_l\{(l+1)\eta P_l(\eta) - (1-\eta^2)P'_l(\eta)\},\\ H_x^d(\xi, y=0) &= \lambda(1-\lambda^2)H^0\sum_{l=1}^{\infty}(\varepsilon\eta)^{l+2}\sqrt{1-\eta^2}\,h_l\{(l+1)P_l(\eta) + \eta P'_l(\eta)\}, \end{aligned} \tag{25}$$

where $\xi = x/d$, $\eta = 1/\sqrt{1+\xi^2}$, and $\varepsilon = r_0/d$. During performance of the numerical calculations based on (25), formula $P'_l(\eta) = l(1-\eta^2)^{-1}\{P_{l-1}(\eta) - \eta P_l(\eta)\}$ may be of use and the values of the Legendre polynomials can be computed reliably using recursion relations [1].

4. Let us now consider the case of a uniform external field tangential to the semispace's surface. If the ox axis is directed along this field, $\mathbf{H}^0 = (H^0, 0, 0)$; then $\hat{W}_3 H_z^0 = 0$ and $H_r^0 = H^0\sin\theta\cos\varphi$. Let us introduce new function $h(\theta, \varphi) = \dfrac{H_r(\theta, \varphi)}{(1-\lambda)H^0}$ and rewrite Eq. (10) for this function:

$$(\hat{I} + \lambda\hat{W}_1)h - \lambda^2\hat{W}_2 h = \sin\theta\cos\varphi. \tag{26}$$

The solution to this equation is sought in the form of a series in spherical harmonics:

$$h(\theta, \varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} h_{lm}Y_{lm}(\theta, \varphi), \quad h_{lm} = \int_{S_1} h(\theta, \varphi)Y_{lm}^*(\theta, \varphi)dS'_1. \tag{27}$$

Formulas (14) and (16) yield, respectively, $\hat{W}_1 h = \sum_{l=1}^{\infty}\frac{1}{2l+1}\sum_{m=-l}^{l} h_{lm}Y_{lm}(\theta, \varphi)$ and $\hat{W}_2 h = \sum_{l=2}^{\infty}\varepsilon^{l+1}\sum_{\alpha=1}^{l-1}\sum_{\beta=-\sigma}^{\sigma} T^l_{\alpha,\beta;\,l-\alpha,\beta}h_{l-\alpha,\beta}Y_{\alpha,\beta}(\theta, \varphi)$. Substituting these expressions into (26), we arrive at

$$\sum_{l=1}^{\infty}\left(1 + \frac{\lambda}{2l+1}\right)\sum_{m=-l}^{l} h_{lm}Y_{lm}(\theta, \varphi) - \lambda^2\sum_{l=2}^{\infty}\varepsilon^{l+1}\sum_{\substack{\alpha=1\\|\beta|\le\sigma}}^{l-1} T^l_{\alpha,\beta;\,l-\alpha,\beta}h_{l-\alpha,\beta}Y_{\alpha,\beta}(\theta, \varphi)$$

$$= -\sqrt{\frac{2\pi}{3}}(Y_{11}(\theta, \varphi) - Y_{1,-1}(\theta, \varphi)).$$

This formula shows that $h_{lm} \neq 0$ only when $m = \pm 1$. Therefore, multiplying this equation by $Y^*_{l,\pm 1}(\theta, \varphi)$, integrating it over S_1, and taking into account the orthogonality relations between spherical harmonics $\int_{S_1} Y_{lm}(\theta, \varphi) Y^*_{pq}(\theta, \varphi) dS'_1 = \delta_{lp}\delta_{mq}$, we again arrive at an infinite set of equations for coefficients $h_{n,\pm 1}$, $n = 1, 2, \ldots$:

$$\sum_{m=1}^{\infty}\left(\frac{2n+1+\lambda}{n(2n+1)}\delta_{nm} - \lambda^2 \varepsilon A_{nm}\right) h_{m,\pm 1} = \mp\sqrt{\frac{2\pi}{3}}\delta_{n1}, \quad n = 1, 2, \ldots;$$
$$A_{nm} = \left(\frac{\varepsilon}{2}\right)^{n+m} C^n_{n+m}\sqrt{\frac{nm}{(n+1)(m+1)(2n+1)(2m+1)}}. \tag{28}$$

The form of system (27) allows the conclusion that it is sufficient to find only $h_{m,1}$ since $h_{m,-1} = -h_{m,1}$. Introducing new variables $x_m = h_{m,1}\sqrt{\frac{2m+1+\lambda}{m(2m+1)}}$, we find after some transformations that

$$\sum_{m=1}^{\infty}(\delta_{nm} - K_{nm})x_m = -\sqrt{\frac{2\pi}{3+\lambda}}\delta_{n1}, \quad n = 1, 2, \ldots,$$
$$K_{nm} = \frac{\lambda^2 \varepsilon^{n+m+1} C^n_{n+m}}{2^{n+m}\sqrt{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{m}\right)\left(2+\frac{1+\lambda}{n}\right)\left(2+\frac{1+\lambda}{m}\right)}}. \tag{29}$$

Solution of this set yields $\{h_{m,1}\}^{\infty}_{n=1}$ and, with allowance for (12), the field on the defect's boundary as well:

$$H_r(r_0, \theta, \varphi) = (1-\lambda)H^0 \sum_{l=1}^{\infty} h_{l,1}\sqrt{\frac{2l+1}{\pi l(l+1)}} P^1_l(\cos\theta)\cos\varphi. \tag{30}$$

To calculate the defect's field on the surface of semispace $z = d$, expressions (30) should be substituted into (24):

$$\mathbf{H}^d(\mathbf{r}) = 2r_0\mu\lambda(1-\lambda)H^0 \nabla \sum_{l=1}^{\infty}\left(\frac{r_0}{r}\right)^{l+1}\frac{h_{l1}P^1_l(\cos\theta)\cos\varphi}{\sqrt{\pi l(l+1)(2l+1)}}. \tag{31}$$

In formula (31), we introduce Cartesian coordinates (then, $\cos\theta = z/r = \eta$, $\cos\varphi = x/\rho$, $r = \sqrt{\rho^2 + z^2}$, and $\rho^2 = x^2 + y^2$); calculate the gradient; and, after having made some transformations, obtain formulas for the component of the defect's field on the outer surface of semispace $z = d$:

$$H^d_x(\xi, \zeta) = \begin{cases} \dfrac{2\lambda(1+\lambda)H^0}{\eta(\xi^2+\zeta^2)^{3/2}} \displaystyle\sum_{l=1}^{\infty} G_l\{l\eta\xi^2 P^1_{l+1} - [(l+1)\xi^2 - \zeta^2]P^1_l\}, & \xi \neq 0,\ \zeta \neq 0, \\ -\lambda(1+\lambda)H^0 \displaystyle\sum_{l=1}^{\infty} h_{l1}\varepsilon^{l+2}\sqrt{\frac{l(l+1)}{\pi(2l+1)}}, & \xi = \zeta = 0, \end{cases}$$

$$H^d_y(\xi, \zeta) = \begin{cases} \dfrac{2\lambda(1+\lambda)\xi\zeta}{\eta(\xi^2+\zeta^2)^{3/2}} H^0 \displaystyle\sum_{l=1}^{\infty} G_l\{l\eta P^1_{l+1} - (l+2)P^1_l\}, & \xi \neq 0,\ \zeta \neq 0, \\ 0, \quad \xi = \zeta = 0, \end{cases} \tag{32}$$

$$H_z^d(\xi, \zeta) = \begin{cases} -\dfrac{2\lambda(1+\lambda)\xi}{(\xi^2+\zeta^2)^{1/2}} H^0 \sum_{l=1}^{\infty} G_l l P_{l+1}^1, & \xi \neq 0,\ \zeta \neq 0, \\ 0, & \xi = \zeta = 0, \end{cases}$$

where $P_l^1 = P_l^1(\eta)$ is a first-order associated Legendre function. In (32), notation $G_l = \dfrac{h_{l,1}(\varepsilon\eta)^{l+2}}{\sqrt{\pi l(l+1)(2l+1)}}$ is used and new variables $\xi = x/d$ and $\zeta = y/d$ are introduced. When performing numerical calculations with (32), it is convenient to use the recurrence formula $(l+1)P_{l+2}^1(\eta) = (2l+3)\eta P_{l+1}^1(\eta) - (l+2)P_l^1(\eta)$.

In this case, formulas (32) that determine components of the defect's field (as well as (25)) contain the ratio of defect's radius r_0 and depth d at which its center is located ($\varepsilon = r_0/d$). One of the problems to be solved by nondestructive testing is the separate determination of each of these parameters' values. The next section is devoted to this problem.

5. Initial stages of the numerical calculations in both problems involve solving an infinite inhomogeneous system of equations of the form

$$\sum_{m=1}^{\infty} (\delta_{nm} - K_{nm}) x_m = b_n, \quad n = 1, 2, \ldots. \qquad (33)$$

A reduction method can be applied to such systems [4], which assumes replacement of the infinite system with a finite system comprising N equations. The value of N is increased until the required accuracy of the solution is attained. Both existence of a solution and convergence of the procedure are ensured if the twofold series $\sum_{n,m=1}^{\infty} K_{nm}^2$ converges. It is not difficult to show that systems (22) and (29) can be majorized as

$$\sum_{n,m=1}^{\infty} K_{nm}^2 \le \frac{\lambda^4 \varepsilon^6}{4(1-\varepsilon^2)^2},$$

thus allowing the conclusion that series $\sum_{n,m=1}^{\infty} K_{nm}^2$ converges.

Codes have been developed that find a numerical solution of systems (22) and (29) and then calculate the defect's field at the semispace's boundary from formulas (25) and (32), respectively. A numerical experiment has shown that four significant digits are stabilized for $N \approx 200$ when $0 \le \varepsilon \le 0.99$; $N \approx 10$ suffices for smaller values of ε.

The plots presented below show components of the defect's field that are normalized to H^0. Figures 2a and 2b present plots of the coordinate dependence of the normal (H_z^d) and tangential (H_x^d) components of the defect's field for $\lambda = 0.98$ in plane $\zeta = 0$. These fields are calculated for the problem when external field $\mathbf{H}^0 = (0, 0, -H^0)$ is transverse to the semispace surface. 3D figures are obtained by rotating the obtained curves around the oz axis.

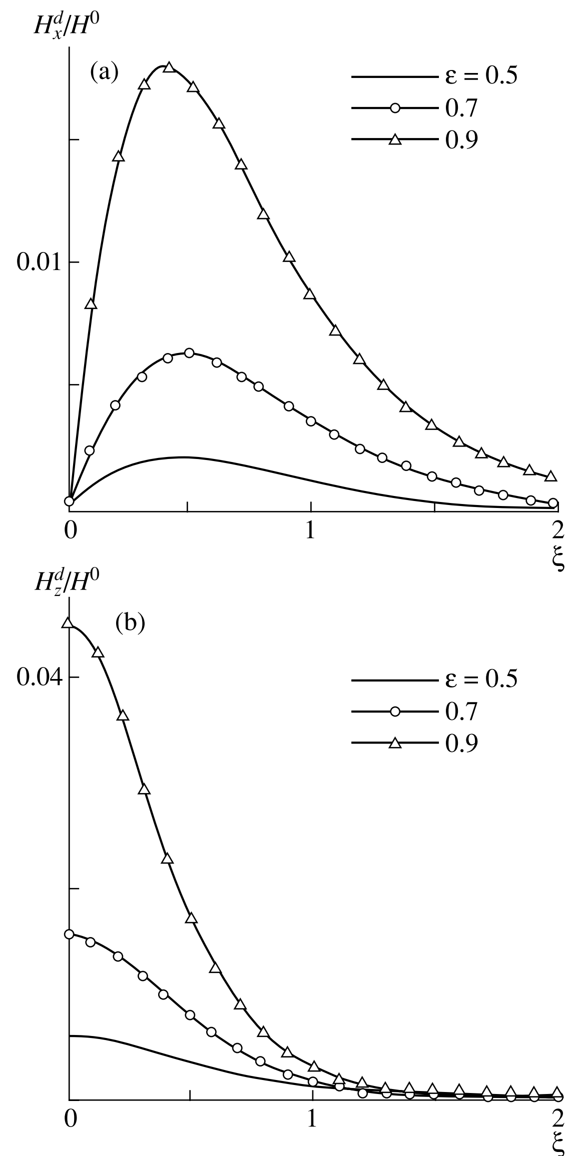

Fig. 2. Tangential component of a defect's field in an external field transverse with respect to the semispace's surface (a) and the normal component in an external field parallel to the semispace's surface (b) for $\lambda = 0.98$.

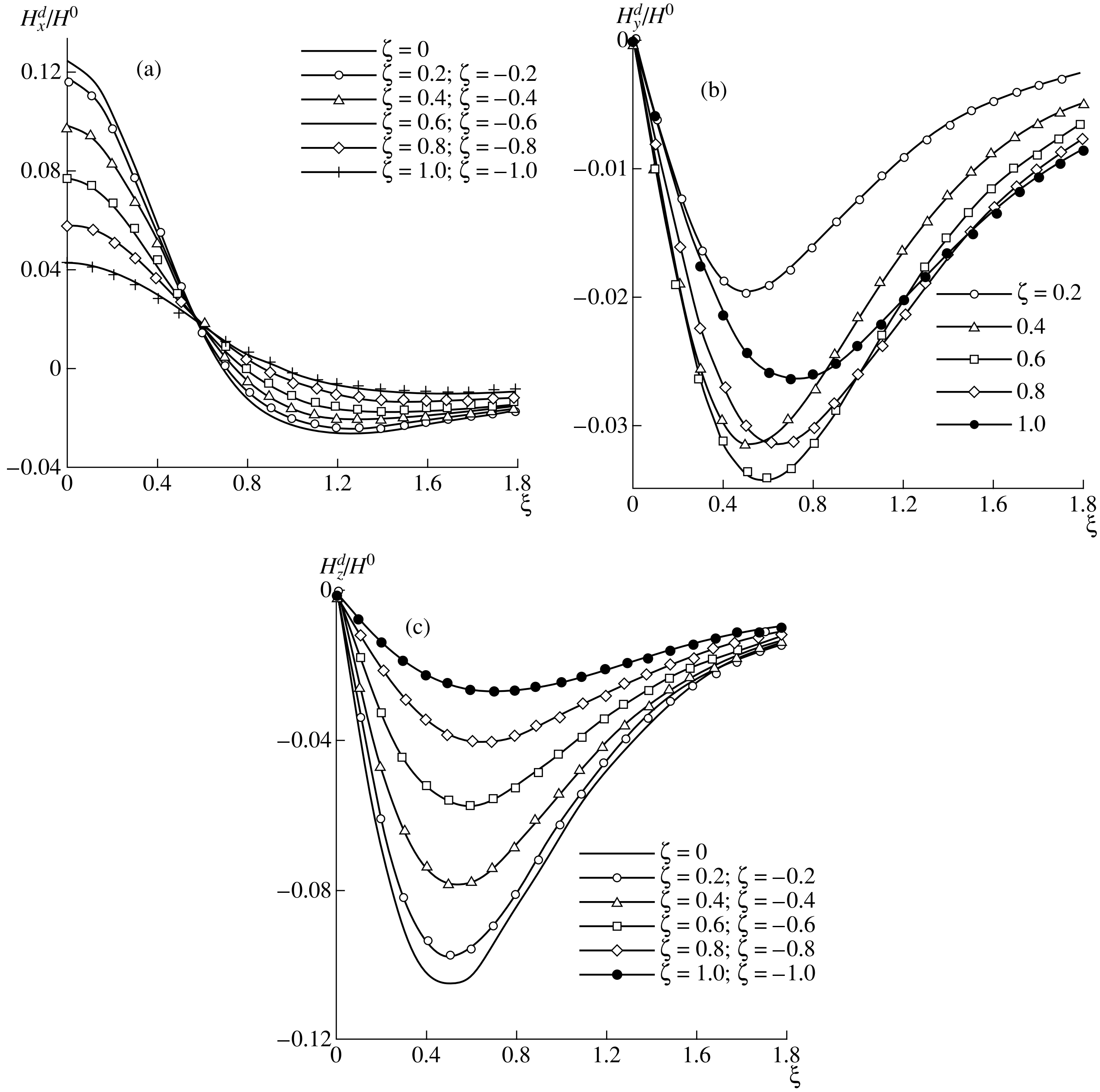

Fig. 3. x, y, and z components ((a) to (c), respectively) of the defect's field in an external field parallel to the semispace's surface for $\varepsilon = 0.5$ and $\lambda = 0.98$.

The next group of plots refers to the case of the external field $\mathbf{H}^0 = (H^0, 0, 0)$ parallel to the semispace surface. Figures 3a–3c show the dependences of the three Cartesian components of the defect's field on coordinates for $\lambda = 0.98$ and $\varepsilon = 0.5$ in the different planes $\zeta = \text{const}$. The 3D distribution can be restored easily with allowance for the symmetry of the solution that follows from formulas (32); namely,

$$H_x^d(-\xi, \zeta) = H_x^d(\xi, \zeta) = H_x^d(\xi, -\zeta), \quad H_y^d(-\xi, \zeta) = -H_y^d(\xi, \zeta) = H_y^d(\xi, -\zeta),$$
$$H_z^d(-\xi, \zeta) = -H_z^d(\xi, \zeta), \quad H_z^d(\xi, -\zeta) = H_z^d(\xi, \zeta).$$

It is worth noting that the y component of the defect's field vanishes on the $o\xi$ and $o\zeta$ axes, while the z component of the field vanishes on the $o\zeta$ axes; the x component of the defect's field attains a maximum at the frame origin; the y component has equal maxima in the second and fourth quadrants of the $\xi o\zeta$ frame. (The minima have, in this case, the same absolute values as the maxima and are located in the first and third

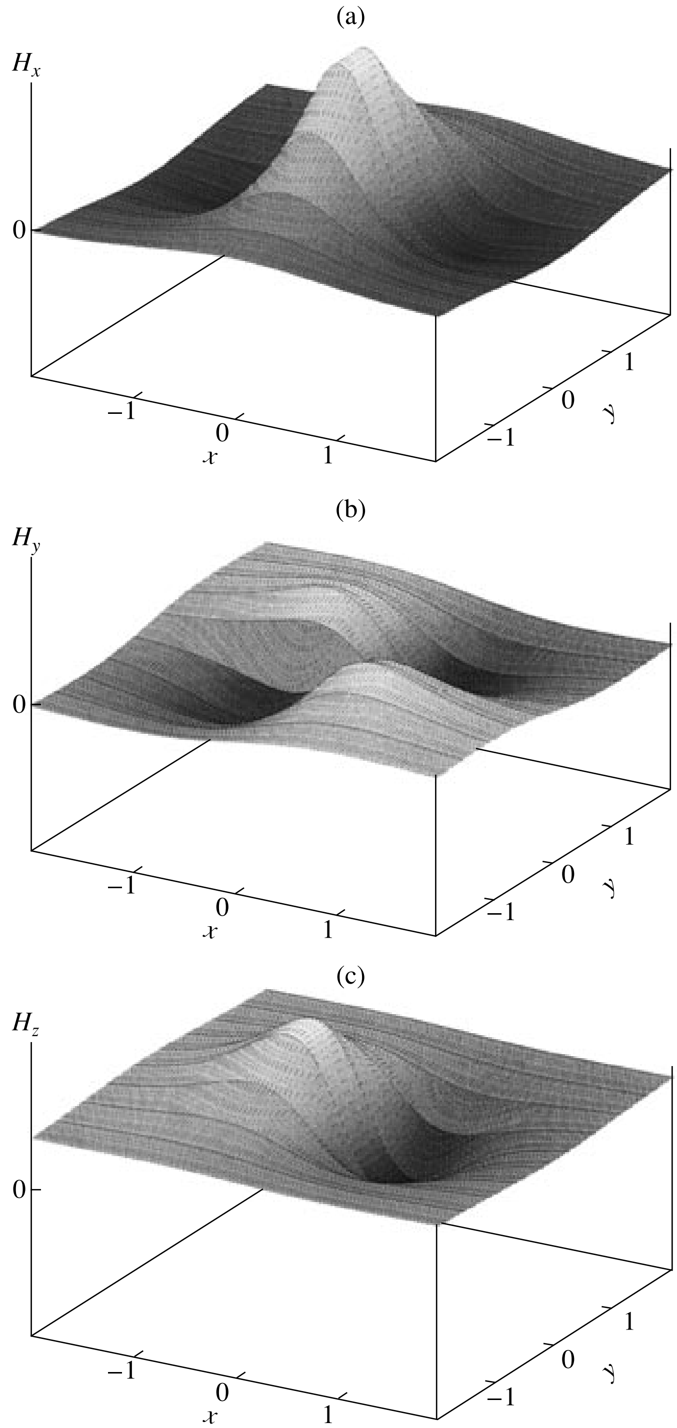

Fig. 4. x, y, and z components ((a) to (c), respectively) of the defect's field.

quadrants; a maximum of the z component is located on the $o\zeta$ axis at $\xi < 0$ and a minimum, which has the same absolute value as this maximum, is located symmetrically with respect to the frame origin.) These conclusions are illustrated more clearly by Fig. 4, which shows the topography of the components of a spherical defect's field schematically.

The absolute value of the defect's field in a configuration where an external field is longitudinal with respect to the semispace's surface is about 20 times larger than that in the configuration with a transverse

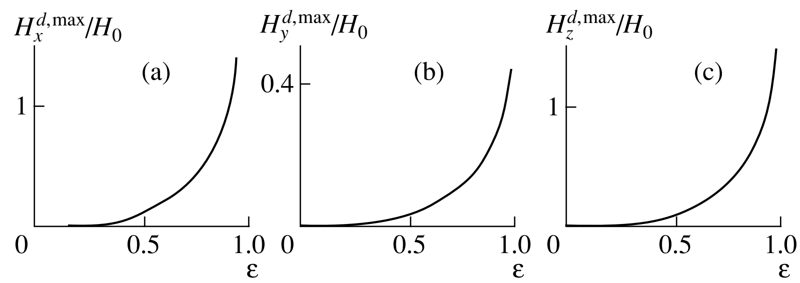

Fig. 5. Dependences of the maximum values of *x*, *y*, and *z* components ((a) to (c), respectively) of the defect's field on ε in an external field parallel to the semispace's surface for λ = 0.98.

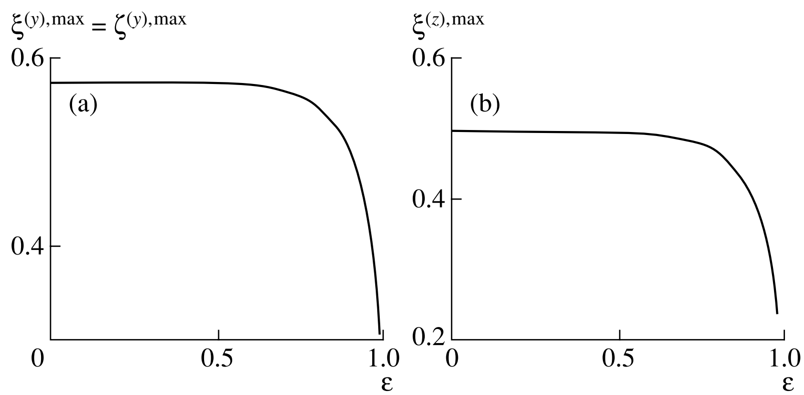

Fig. 6. Dependences of the maximum values of *y* and *z* components ((a) and (b), respectively) of the defect's field on ε in an external field parallel to the semispace's surface for λ = 0.98.

field. (This can be seen clearly by comparing at least the absolute values of the field vectors at the frame origin.) Therefore, a defect can be identified more easily in the case of a longitudinal field. This conclusion is also supported by experimentalists.

A general scheme for solving the inverse problem (determination of a defect's radius and location depth) is proposed in [6]. No significant differences arise if the external field is a longitudinal. An actual defect is not necessarily a sphere, but it may be approximated with a sphere if field components have a topography similar to that described above (and illustrated in Fig. 4). Let us assume that such is the case. Apart from this, it is necessary to somehow specify the value of the magnetic permeability since the problem has been solved in a linear approximation. Therefore, some value adequate for experimental conditions should be adopted as μ.

A general scheme for the separate determination of defect's radius r_0 and depth d at which its center is located comprises the stages described below.

(i) With the use of experimentally found maximum values of the components of the defect's field (denoted as $H_x^{d,\max}$, $H_y^{d,\max}$, and $H_z^{d,\max}$) and the theoretical curves plotted in Fig. 5, the value of parameter ε is found. If maximum values are measured for all three components of the defect's field, ε can be determined first for all three curves and then its average value may be calculated.

(ii) Theoretical extrema of the *y* component of the defect's field are located symmetrically with respect to the frame origin (see Fig. 4b) and $\xi^{y,\max} = \zeta^{y,\max}$. (These are the notations used for coordinates of the *y*-component extrema; the location in the first quadrant alone is sufficient for this scheme.) In that case, $\xi^{y,\max}$ is found from Fig. 6a for a given value of ε.

(iii) Let $x^{y,\max}$ and $y^{y,\max}$ be experimentally measured coordinates of the point where the maximum of the *y* component of the defect's field is located. These values may differ, but the defect may be approximated with a sphere if the difference is not very significant. The ratios $d_1 = x^{y,\max}/\xi^{y,\max}$ and $d_2 = y^{y,\max}/\zeta^{y,\max}$ are then calculated.

(iv) Accuracy of the solution can be improved further by using the measured location of the point where the extremum of the defect's field z component is located. In this case, extrema are located on the ox axis (or, otherwise, on the $o\zeta$ axis), so that it is sufficient to determine $\zeta^{z,\max}$ alone using Fig. 6b. Ratio $d_3 = x^{z,\max}/\xi^{z,\mathrm{ma}}$ is calculated.

(v) d is calculated as an average of d_1, d_2, and d_3.

(vi) Finally, r_0 is determined: $r_0 = \varepsilon d$.

Figure 6 also shows that the location of the maximum can be found theoretically most efficiently if $0.5 < \varepsilon < 1$.

ACKNOWLEDGMENTS

This study was conducted as part of state contract no. 1002–251/П–16/097–348/220704–641 and project no. 12 of program no. 16 of the Presidium of the Russian Academy of Sciences "Mathematical Simulation and Intelligent Systems."

REFERENCES

1. Gradshtein, I.S. and Ryzhik, I.M., *Tablitsy integralov, summ, ryadov i proizvedenii* (Integrals, Sums, Series, and Products Tables), Moscow: GIFML, 1962.
2. Mikhlin, S.G., *Lineinye uravneniya v chastnykh proizvodnykh* (Linear Partial Differential Equations), Moscow: Vysshaya shkola, 1977.
3. Varshalovich, D.A., Moskaleva, A.N., and Khersonskii, V.K., *Kvantovaya teoriya uglovogo momenta* (Quantum Theory of Angular Momentum), Leningrad: Nauka, 1975.
4. Kantorovich, L.V. and Krylov, V.I., *Priblizhennye metody vysshego analiza* (Approximate Methods of Higher Analysis), Moscow: GIMFL, 1962.
5. Sapozhnikov, A.B., *Teoreticheskie osnovy elektromagnitnoi defektoskopii metallicheskikh tel* (Theoretical Foundations of Electromagnetic Flaw Detection of Metallic Bodies), Tomsk: Izd. Tomsk. Gos. Univ., 1980.
6. Dyakin, V.V. and Umergalina, O.V., Calculation of the Field of a Flaw in Three-Dimensional Half-Space, *Defektoskopiya*, 2003, no. 4, pp. 52–66 [*Rus. J. of Nondestructive Testing* (Engl. Transl.), 2003, vol. 39, no. 4, pp. 297–309].