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Whom should we believe? Aggregation of heterogeneous beliefs

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Abstract We examine the collective risk attitude of a group with heterogeneous beliefs. We prove that the wealth-dependent probability distribution used by the representative agent is biased in favor of the beliefs of the more risk tolerant consumers. Moreover, increasing disagreement on the state probability raises the state probability of the representative agent. It implies that when most disagreements are concentrated in the tails of the distribution, the perceived collective risk is magnified. This can help to solve the equity premium puzzle. We show that the trade volume and the equity premium are positively correlated.

Keywords Aggregation of beliefs · State-dependent utility · Efficient risk sharing · Disagreement · Asset pricing · Portfolio choices

JEL Classification D81 · D83

People have divergent opinions on a wide range of subjects, from the growth rate of the economy next year, the profitability of a new technology to the risk of global warming. Suppose that this heterogeneity of beliefs does not come from asymmetric information but rather from intrinsic differences in how to

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view the world. People agree to disagree. We examine how the group as a whole will behave towards risk. Aggregating beliefs when agents differ on their expectations is useful to solve various economic questions, from asset pricing to cost-benefit analyses of collective risk prevention.

We assume that risks are efficiently shared in the economy. The properties of the preferences of the representative agent are derived from the characteristics of the efficient allocation of risk in the economy, such as the one derived from the competitive allocation with complete Arrow–Debreu markets. Borch (1960, 1962), Wilson (1968) and Rubinstein (1974) were the first to characterize the properties of Pareto-efficient risk sharing. Wilson (1968), Lintner (1969), and more recently Calvet, Grandmont and Lemaire (2001), Jouini and Napp (2007) and Chapman and Polkovnichenko (2006), showed that the standard methodology of the representative agent can still be used when agents have heterogeneous beliefs. Leland (1980) examined the competitive equilibrium asset portfolios when agents have different priors on the distribution of state probabilities. Bossaerts et al. (2007) determine the equilibrium collective attitude towards risk when people have multiple priors and have different degrees of ambiguity aversion.

The cornerstone result of this paper is to compare two states of nature for which the distribution of individual probabilities are different. Consider for example a situation where all agents believe that state s_2 has the same probability of occurrence as another state s_1, except agent θ. Suppose that this agent has a subjective probability for s_2 that is 1 percent larger than for s_1. By how much should we increase the probability of state s_2 with respect to s_1 in the collective decision process? We show that the collective probability should be increased by x percents, where x is the percentage share of the aggregate risk that is borne by agent θ, or the agent θ's tolerance to risk expressed as a percentage share of the group's risk tolerance. More generally, the rate of change of the collective probability is a weighted mean of the rate of change of the individual probabilities. The weights are proportional to the individual risk tolerances. More risk tolerant agents see their beliefs better represented in the collective decision making under uncertainty. At the limit, if an agent is fully insured by other agents, it is intuitive that this agent's beliefs should not affect the social welfare function. This intuitive result has several important consequences.

Observe first that, as initially observed by Hylland and Zeckhauser (1979),[1] there is in general no subjective probabilities, separated from utilities that represent the aggregation of individual beliefs. The representative agent has state-additive preferences as under the standard expected utility model, but the different terms of the sum cannot be written as a product of a unique state probability by a state utility level. Equivalently, this means that the representative agent has a state-dependent utility function, despite the fact

[1] See also Mongin (1995), Gilboa, Samet and Schmeidler (2004), and Gajdos et al. (2005). Our work differs much from this branch of the literature by taking into account risk-sharing opportunities within the group.

that all members of the group have state-independent preferences.[2] However, we show that one can formally disentangle the collective probability and the collective utility if and only if agents have an absolute risk tolerance that is linear with a common slope.

We say that there is a relative increase in disagreement between two states if the distribution of individual *log* probabilities is more dispersed in one state than in the other. If this relative increase in disagreement preserves the mean log probabilities, we show that it raises the collective probability if and only if absolute risk aversion is decreasing (DARA). To illustrate, suppose that Mrs. Jones has a larger subjective probability for a flood to occur this year than Mr. Jones. Compared her own beliefs about floods, Mrs. Jones has a subjective probability for the risk of an earthquake that is k percents larger, whereas Mr. Jones has a subjective probability for an earthquake that is k percents smaller than his estimate of the probability of a flood. Thus, the mean log probabilities in the couple is the same for the two potential damages, but there is more disagreement about the likelihood of an earthquake than for a flood. Under DARA, it implies that, when Mr. and Mrs. Jones decide about their collective prevention efforts and insurance, they should use a larger probability for an earthquake than for a flood.

These results describe how the heterogeneity of beliefs affects the difference in collective probabilities for any pair of states. Going from this partial analysis to a more global one, it is necessary to describe the structure of disagreements across states. More precisely, it would be useful to determine whether conflicts of opinion in the population raise the risk perceived by the representative agent, in the sense of the first or second stochastic dominance order. If the answer to this question is positive, this could help to solve the equity premium puzzle, as explained by Cecchetti, Lam and Mark (2000) and Abel (2002). If the relative degree of disagreement is increasing towards the extreme states, DARA implies that the representative agent would indeed use probabilities that are relatively larger in the tails. Because the representative agent perceives a riskier macroeconomic risk, the equity premium is increased. In a plausible simulation, the conflict of opinions multiplies the equity premium by 4. Moreover, we show that the volume of trade and the equity premium are positively related in that case, because they are both positively affected by the degree of disagreement. Calvet, Grandmont and Lemaire (2001) also examine the effect of heterogeneous beliefs on the equity premium. They are able to sign this effect when the relative risk aversion of the representative agent is decreasing with average wealth. Jouini and Napp (2007) examine a continuous-time model with constant relative risk aversion, allowing them to discuss the effect of the heterogeneity of beliefs on the risk-free rate.

[2]Karni (1993), Karni and Schmeidler (1993) and Nau (1995) examined the problem of disentangling beliefs and tastes with state-dependent preferences. Drèze (2001) and Drèze and Rustichini (2001) examine the effect of the state dependency of the utility function for risk management and risk transfers.

The structure of the paper is as follows. Section 1 is devoted to the description of the aggregation problem when agents have heterogeneous preferences and beliefs. We show how to aggregate individual risk tolerances and individual beliefs in this framework in Section 2. In Section 3, we define our concept of increasing disagreement, and we determine its effect on the perception of risk by the representative agent. Section 4 is devoted to the analysis of the effect of the heterogeneity of beliefs on the equity premium. Finally, we present concluding remarks in Section 5.

1 The aggregation problem

We consider an economy of N heterogeneous agents indexed by $\theta = 1, ..., N$. Agents extract utility from consuming a single consumption good. The model is static with one decision date and one consumption date. At the decision date, there is some uncertainty about the state of nature s that will prevail at the consumption date. There are S possible states of nature, indexed by $s = 1, ..., S$. Agents are expected-utility maximizers with a state-independent utility function $u(., \theta) : R \to R$ where $u(c, \theta)$ is the utility of agent θ consuming c. We assume that $u_c = \partial u/\partial c$ is continuously differentiable and concave in c. As in Calvet et al. (2001), we focus on interior solutions. To guarantee this, we assume that $\lim_{c\to 0} \partial u/\partial c = +\infty$ and that $\lim_{c\to+\infty} \partial u/\partial c = 0$.We also assume that each agent θ has beliefs that can be represented by a vector $(p(1, \theta), ..., p(S, \theta))$, where $p(s, \theta) > 0$ is the probability of state s assumed by agent θ, with $\sum_{s=1}^{S} p(s, \theta) = 1$.

There is an aggregate risk in this economy, which is characterized by the state-contingent endowment $z(s)$ per capita. The crucial assumption of this paper is that the group can allocate risks efficiently among its members. An allocation C is Pareto-efficient if it is feasible and if there is no other feasible allocation that raises the expected utility of at least one member without reducing the expected utility of the others. A special case is the competitive solution, which will be examined in Section 4. In this paper as in Wilson (1968), we characterize the properties of *all* Pareto-efficient allocations. For a given vector of positive Pareto weights $\Lambda = (\lambda(1), ..., \lambda(N))$, normalized in such a way that $N^{-1} \sum_{\theta=1}^{N} \lambda(\theta) = 1$, the group would select the allocation of risk that maximizes the weighted sum W of the members' expected utility under the feasibility constraint:

$$W = \max_{C} \sum_{\theta=1}^{N} \lambda(\theta) \sum_{s=1}^{S} p(s, \theta) u(C(s, \theta), \theta) \tag{1}$$

$$s.t. \quad \frac{1}{N} \sum_{\theta=1}^{N} C(s, \theta) = z(s) \;\text{ for all } s = 1, ..., S. \tag{2}$$

Obviously, this problem can be decomposed into a sequence of S cake-sharing problems. Consider a specific state of nature s with wealth per capita

$z(s) = z$ and with a vector $P(s) = P = (p(s, 1), ..., p(s, N))$ of individual-specific state probability. For this pair (z, P), define the following cake-sharing problem:

$$v(z, P) = \max_{x(.)} \sum_{\theta=1}^{N} \lambda(\theta) p(\theta) u(x(\theta), \theta) \ \ s.t. \ \frac{1}{N} \sum_{\theta=1}^{N} x(\theta) = z. \tag{3}$$

The solution of this program is denoted $x^*(.) = c(z, P, .)$. The interpretation of this program is straightforward. A cake of size Nz must be shared among the N members of the group. The cake-sharing rule is selected in order to maximize a weighted sum of the individual utility functions. In this well-behaved cake-sharing problem, z represents the consumption per capita, and $v(z, P)$ is the maximum sum of the members' utility weighted by the product of the Pareto weights $(\lambda(1), ..., \lambda(N))$ and the vector P. Notice that by construction, v is homogeneous of degree 1 with respect to P.

The following proposition, whose proof is skipped, states that for any vector Λ, there exists a representative agent associated to this economy.

Proposition 1 *Consider a given Pareto-weight vector Λ. The corresponding efficient risk allocation that solves Eqs. 1–2 is such that $C(s, \theta)$ equals $c(z(s), P(s), \theta)$ for all (s, θ), where $c(z, P, .)$ solves Eq. 3. The social welfare W equals $\sum_{s=1}^{S} v(z(s), P(s))$, where $v(z, P)$ is the maximum of Eq. 3.*

The representative agent's welfare ex ante is measured by the sum of the $v(z(s), P(s))$. The characterization of v would be very useful since, from the above proposition, it would allow us to compare and to rank different aggregate risks on which we could build either collective risk policy recommendations or asset pricing formulas. We will link the properties of the v function to the primitive characteristics of individual preferences and beliefs. This will be done by focusing on the cake-sharing problem that defines the v function. The collective valuation function v is linked to the individual-specific utility functions and beliefs through the cake-sharing program (3). Its first-order condition is written as

$$\lambda(\theta) p(\theta) u_c(c(z, P, \theta), \theta) = \psi(z, P) = v_z(z, P), \tag{4}$$

for all (z, P), and for all $\theta = 1, ..., N$, where ψ is the Lagrange multiplier associated to the feasibility constraint of program (3). The second equality comes from the envelop theorem.

The remainder of the paper focuses on the characterization of function v_z, where $v_z(z(s), P(s))$ is referred to as the willingness to consume in state s. The willingness to consume is central for the determination of optimal collective choices under uncertainty. For example, if society has the opportunity ex ante to transform one unit of wealth in state s into π units of wealth in state s', it would be socially efficient to do so at the margin if $v_z(z_{s'}, P_{s'})$ is larger than $v_z(z_s, P_s)/\pi$. It would be nice if the willingness to consume would be multiplicatively separable, as in the standard expected utility model. Indeed, if there would exist two functions $p^v : R^S \to R^+$ and $h : R \to R^+$

such that $v_z(z, P)$ would equal $p^v(P)h(z)$ for all (z, P), we could refer to $(p^v(P(1)), ..., p^v(P(S)))$ as the vector of state-probabilities of the representative agent (up to a normalizing constant). In the following proposition, we show that this separability does not hold in general, and there exists no aggregate beliefs in the classical sense. Define the absolute risk tolerance of agent θ as $T^u(c, \theta) = -u_c(c, \theta)/u_{cc}(c, \theta)$. We say that the economy has the Identically Sloped Harmonic Absolute Risk Aversion (ISHARA) property if T^u is linear in c with a slope identical for all consumers.[3]

Proposition 2 *The willingness to consume $v_z(z, P)$ is multiplicatively separable in (z, P) if and only if consumers have ISHARA preferences:*

$$\frac{\partial^2 \ln v_z(z, P)}{\partial \ln p(\theta)\partial z} \equiv 0 \quad \Longleftrightarrow \quad \frac{\partial T^u(c, \theta)}{\partial c} \text{ is independent of } c \text{ and } \theta.$$

Proof Because the proof of this proposition relies on results proved in the next section, it is relegated to the Appendix.

This implies that in the special case of ISHARA, the representative agent is an expected-utility-maximizer with a well-defined vector $(p^v(P(1)), ..., p^v(P(S)))$ of subjective probabilities, and a well-defined marginal utility function $h(.)$. In all other cases, things are more complex, but the complexity is linked to the terminology to be used rather than to some more fundamental aspects of the problem. Following Karni and Schmeidler (1993) and Nau (1995), the collective probabilities p^v could be defined by the following conditions:

$$\frac{p^v(s)}{p^v(s')} = \frac{v_z(z_s, P(s))}{v_z(z_{s'}, P(s'))}, \tag{5}$$

together with $\Sigma_s p^v(s) = 1$.

2 The aggregation rules

In this section, we characterize the group's degree of tolerance to risk on the wealth per capita z and the group's beliefs as functions of the primitives of the model, i.e., the set of individual utility functions $u(., \theta)$ and beliefs $p(., \theta)$.

The collective attitude towards risk depends upon how this collective risk is allocated to the members' risk on consumption. This is characterized by $\partial c/\partial z$. Fully differentiating first-order condition (4) with respect to z and using the

[3] In an earlier version of this paper (Gollier 2003), we presented various results on this aspect. We explored the problem of aggregating beliefs when risk aversion and pessimism are two correlated treats of individual consumers.

feasibility constraint $\sum_{\theta=1}^{N} c(z, P, \theta)/N = z$ yields the following well-known Wilson's (1968) result:

$$\frac{\partial c}{\partial z}(z, P, \theta) = \frac{T^u(c(z, P, \theta), \theta)}{N^{-1}\sum_{\theta'=1}^{N} T^u(c(z, P, \theta'), \theta')}. \tag{6}$$

One can interpret this property of the efficient risk-sharing rule as follows: suppose that there are two states of nature that are perceived to be equally likely by all agents ($p(s, \theta) = p(s', \theta)$ for all θ), but that yield different wealth levels ($z(s) \neq z(s')$). Equation 6 shows how to allocate the collective wealth differential in the two states. Observe that the right-hand side of Eq. 6 being positive implies that individual consumption levels are all comonotone, but more risk-tolerant agents should bear a larger fraction of the collective risk.

The concavity of v with respect to z is essential to determine the collective attitude towards the aggregate risk. By analogy to the individual risk to the individual risk tolerance, we define the representative agent's risk tolerance as

$$T^v(z, P) = \left[-\frac{\partial \ln v_z(z, P)}{\partial z}\right]^{-1} = -\frac{v_z(z, P)}{v_{zz}(z, P)}.$$

From the efficient collective risk-sharing rule characterized by Eq. 6, it is easy to derive the degree of risk tolerance of the representative agent. As in Wilson, we obtain that

$$T^v(z, P) = N^{-1}\sum_{\theta'=1}^{N} T^u(c(z, P, \theta'), \theta'). \tag{7}$$

The representative agent's absolute risk tolerance is the mean of individual tolerances. We conclude that this rule already valid in the simpler Wilson's model is robust to the introduction of heterogeneous expectations. In the special case of ISHARA, T^v is independent of P, as shown by Proposition 2, and of λ, which is not true in general.

In the classical case with homogeneous beliefs, an important property of any Pareto-efficient allocation of risk is the so-called mutuality principle. It states that efficient individual consumption levels depend upon the state only through the wealth per capita z. Its economic interpretation is that all diversifiable risks are eliminated through exchanges. In this classical case, the wealth level per capita z is a sufficient statistic for efficient individual consumption levels: $z(s) = z(s')$ implies that $C(s, \theta) = C(s', \theta)$ for all θ. The mutuality principle is obviously not robust to the introduction of heterogeneous beliefs because efficient allocation plans $c(z, P, \theta)$ depend also upon the distribution of individual subjective probabilities associated to the state.

The effect of the heterogeneity of beliefs cannot be disentangled from how it affects the allocation of risk in the group. In the following proposition, we derive jointly the aggregation rule of beliefs and the allocation of risks. The comparative exercise there and in the remainder of the paper consists in comparing two states of nature s and s' with $P(s') = P(s) + \Delta P$. In particular, we are interested in determining how does a difference in agent θ''s probability affect the share of the cake, and the representative agent's willingness to consume, which is measured by v_z.

Proposition 3 *The elasticity of v_z to the subjective state probability of agent θ is proportional to agent θ's risk tolerance. More precisely, we have that*

$$R(z, P, \theta) = \frac{\partial \ln v_z(z, P)}{\partial \ln p(\theta)} = \frac{T^u(c(z, P, \theta), \theta)}{T^v(z, P)}, \tag{8}$$

where function T^v is defined in Eq. 7. The efficient allocation of consumption satisfies the following condition:

$$\frac{\partial c(z, P, \theta')}{\partial \ln p(\theta)} = \begin{cases} T^u(c(z, P, \theta'), \theta')\left[1 - \frac{T^u(c(z, P, \theta'), \theta')}{T^v(z, P)}\right] & if\ \theta = \theta' \\ -\frac{T^u(c(z, P, \theta'), \theta')T^u(c(z, P, \theta), \theta)}{T^v(z, P)} & if\ \theta \neq \theta'. \end{cases} \tag{9}$$

Proof Fully differentiating the first-order condition (4) with respect to $p(\theta)$ and dividing both sides of the equality by $\lambda p u_c = \psi$ yields

$$dc(z, P, \theta') = -T^u(c(z, P, \theta'), \theta')\frac{d\psi}{\psi} \tag{10}$$

for all $\theta \neq \theta'$, and

$$dc(z, P, \theta') = T^u(c(z, P, \theta'), \theta')\left[\frac{dp(\theta)}{p(\theta)} - \frac{d\psi}{\psi}\right]. \tag{11}$$

By the feasibility constraint, it must be that $\sum_{\theta'=1}^{N} dc(z, P, \theta') = 0$. Replacing $dc(z, P, \theta')$ by its expression given above allows us to rewrite this equality as

$$\frac{d\psi}{\psi} = \frac{T^u(c(z, P, \theta), \theta)}{T^v(z, P)}\frac{dp(\theta)}{p(\theta)}. \tag{12}$$

Combining Eqs. 10, 11 and 12 yields Eq. 9. By the envelop theorem, we also know that $v_z(z, P) = \psi(z, P)$. It implies that

$$d \ln v_z(z, P) = \frac{d\psi}{\psi}. \tag{13}$$

Combining Eqs. 12 and 13 yields property (8). □

Let us first focus on property (9). Ceteris paribus, an increase in the state probability by agent θ increases his efficient consumption and it reduces the consumption by all other consumers. Ex-ante, this means that consumers take risks on their consumption even when there is no social risk. Agents take a long position on states that they perceive to have a relatively larger probability of occurrence relative to the other members of the group. This illustrates the violation of the mutuality principle. Notice that the size of these side bets is proportional to the members' risk tolerance. At the limit, if agent θ has a zero tolerance to risk, it is not efficient for him to gamble with others in spite of the divergence of opinions in the group.

Equation 8 states that the elasticity of v_z to $p(\theta)$ is proportional to that agent θ's degree of absolute risk tolerance. Thus, the determination of the collective willingness to consume in a given state is biased in favor of the probability of those agents who are more risk tolerant in that state. Combining properties (8) and (6), we obtain that

$$R(z, P, \theta) = \frac{1}{N}\frac{dc(z, P, \theta)}{dz}. \tag{14}$$

The distribution of the collective willingness to consume is biased towards those who actually bear the collective risk in the group. In the special case of ISHARA, R is independent of z, and $R(z, P, \theta)$ represents the elasticity of the representative agent's probability with respect to agent θ's probability. In that case, result (8) tells us *the aggregate beliefs p^v are biased towards the beliefs of more risk tolerant consumers*, i.e., towards agents who bear a larger share of the aggregate risk.

3 The effect of increasing disagreement on v_z

In this section, we want to determine the effect of the divergence of opinions on the collective attitude towards risk. More specifically, we want to compare $v_z(z, P')$ to $v_z(z, P)$ in the large. In the ISHARA case, $v_z(z, P(s')) > v_z(z, P(s))$ would implicitly mean that the representative agent believes that state s' is more likely to occur than state s: $p^v(s') > p^v(s)$.

As a benchmark, consider the proportional shift in distribution with $P' = kP$. As stated earlier, v and its partial derivatives with respect to z are homogeneous of degree 1 in the vector of individual probabilities P: $v_z(z, kP) = kv_z(z, P)$, for all z and P. In the ISHARA case, this means that when all consumers believe that state s' is k times more likely than state s, the representative agent will have the same relative evaluation of relative probabilities. In the following, we define a family of shifts in P that are not proportional.

3.1 Relative increase in disagreement

In this subsection, we want to define a notion of increasing disagreement. We hereafter define a concept of increasing disagreement that is based on the Monotone Likelihood Ratio (MLR) order. However, the main ingredient in this section is not the individual subjective probabilities $p(\theta)$, but rather the Pareto-weighted ones $q(\theta) = \lambda(\theta)p(\theta)$. We say that a marginal shift dP from an initial vector of individual probabilities P yields increasing disagreement if those agents with a larger initial $q(\theta)$ also have a larger *rate* of increase $d\ln q(\theta)$. Compared to a proportional increase, the distribution of individual probabilities becomes more dispersed. Thus, there is an increase in disagreement relative to a proportional shift in individual probabilities.

Definition 1 Consider a specific distribution of individual probabilities $P = (p(1), ..., p(N))$ and a specific Pareto-weight vector Λ. We say that a marginal shift $dP = (dp(1), ..., dp(N))$ yields a relative increase in disagreement if $q(\theta) = \lambda(\theta)p(\theta)$ and $d\ln q(\theta)$ are comonotone: for all (θ, θ') :

$$\left[q(\theta') - q(\theta)\right]\left[d\ln q(\theta') - d\ln q(\theta)\right] \geq 0. \tag{15}$$

In an earlier version of this paper (Gollier 2003), we show how to generalize our definition of increasing disagreement "in the small" to non-marginal changes of individual probabilities. If we assume without loss of generality that q is increasing in θ, this is equivalent to require that $q(\theta')/q(\theta)$ be increased by the shift whenever $\theta' > \theta$.

3.2 Our main results

In order to isolate the effect of heterogeneous beliefs, we hereafter assume that preferences are homogeneous in the population in the sense that all agents have the same utility function. Consider an initial distribution $P = (p(1), ..., p(N))$ of individual probabilities, and a shift $dP = (dp(1), ..., dp(N))$ in this distribution. Using Proposition 3 together with the assumption that all agents have the same utility function, this can be rewritten as

$$d\ln v_z(z, P) = \sum_{\theta=1}^{N} \frac{T^u(c(z, P, \theta))}{T^v(z, P)} d\ln p(\theta), \tag{16}$$

assuming $dz = 0$. In the ISHARA case, the left-hand side of this equality can be interpreted as the rate of increase in the collective probability. In the special case of identical constant absolute risk aversion (ICARA), Eq. 16 can be rewritten as

$$d\ln v_z(z, P) = d\ln p^v(P) = N^{-1} \sum_{\theta=1}^{N} d\ln p(\theta). \tag{17}$$

This means that relative increases in disagreement have no effect on the collective probability in the ICARA case: only the mean increase in the individual log probability matters to determine the collective willingness to consume. It implies that the collective probability of state s is proportional to the geometric mean of the individual probabilities associated to that state. Our main proposition below characterizes the condition under which increasing relative disagreement has a positive impact on the willingness to consume v_z, or on the collective probability in the ISHARA case.

Proposition 4 *Suppose that the individual utility functions are identical. The following two conditions are equivalent:*

1. *For any wealth z, any initial distribution of individual probabilities P and any shift dP yielding a relative increase in disagreement, the rate of increase in v_z is larger than the mean rate of increase of individual probabilities:*

$$d\ln v_z(z, P) \geq \frac{1}{N} \sum_{\theta=1}^{N} d\ln p(\theta). \tag{18}$$

2. *Absolute risk aversion is decreasing (DARA):* $\partial T^u/\partial c \geq 0$.

Proof (2)⇒(1): By Eqs. 16 and 7, condition (18) is equivalent to

$$N^{-1}\sum_{\theta=1}^{N} T^u(c(z,P,\theta))d\ln p(\theta) \geq \left[N^{-1}\sum_{\theta=1}^{N} T^u(c(z,P,\theta))\right]\left[N^{-1}\sum_{\theta=1}^{N} d\ln p(\theta)\right]. \tag{19}$$

Suppose without loss of generality that $q(\theta)=\lambda(\theta)p(\theta)$ is increasing in θ. Combining this with the first-order condition (4) implies that $c(z,P,\theta)$ is increasing in θ, because of risk aversion. Under decreasing absolute risk aversion (DARA), it implies in turn that $T^u(c(z,P,\theta))$ is also increasing in θ. By definition of increasing disagreement, we get that T^u and $d\ln p$ be comonotone under DARA. Applying the covariance rule to $E\left[T^u d\ln p\right]$ directly implies Eq. 19.

(1)⇒(2): Suppose now by contradiction that T^u is locally decreasing in the neighborhood B of c_0. Then, take $z=c_0$ and an initial distribution $P(\varepsilon)$ such that $\lambda(\theta)p(\theta)=k+\varepsilon\theta$ for all θ. When $\varepsilon=0$, $c(z,P(0),\theta)=c_0$ for all θ. Take a small ε such that $c(z,P(\varepsilon),\theta)$ remains in B for all θ. By assumption, the shift dP exhibits increasing disagreement, which means that $c(z,P(\varepsilon),\theta)$ and $d\ln p(\theta)$ are comonotone. This implies that $T^u(c(z,P(\varepsilon),\theta))$ and $d\ln p(\theta)$ are anti-comonotone, thereby reversing the inequality in Eq. 19. This implies that DARA is necessary for property 1. □

Under DARA, a mean-preserving spread in log probabilities always raises the collective probability. The intuition of this result is easy to derive from the central property (8) of the aggregation of heterogeneous beliefs. Under DARA, this property states that those who consume more see their beliefs better represented in the aggregation. But by definition of an increase in disagreement, those who consume more are also those who have a larger rate of increase in their subjective probability. We conclude that, because of the bias in favor of those who consume more, an increase in disagreement raises the collective probability even when the mean rate of increase in individual probabilities is zero.

In Gollier (2003), I proved the following proposition using the efficient aggregation rule (16).

Proposition 5 *Suppose that the individual utility functions are identical. The following two conditions are equivalent:*

1. *For any wealth z, any initial distribution of individual probabilities P and any shift dP yielding a relative increase in disagreement, the rate of increase in v_z is larger than the rate of increase of the mean individual probabilities:* $d\ln v_z(z,P) \geq d\ln\left(N^{-1}\sum_{\theta=1}^{N} q(\theta)\right)$;

2. *The derivative of absolute risk tolerance with respect to consumption is larger than unity:* $T_c^u(c) \geq 1$ *for all c.*

Varian (1985) and Ingersoll (1987) proved that 2 implies 1. Comparing this proposition with Proposition 4 clarifies the main difference between our work with the existing literature. Whereas Varian and Ingersoll compared the rate of increase of the collective probability to the rate of increase of the mean individual probabilities, we compare it to the mean rate of increase in individual probabilities.

To illustrate, let us consider the following two examples. We consider an economy with two agents and a continuum of states. We assume that the above results that have been obtained for a finite number of states also hold in the case of a continuum of states.

3.3 Example 1: Negative exponential beliefs

Suppose that the beliefs of the two agents $\theta = 1, 2$ are distributed according to the negative exponential law $p(s, \theta) = \lambda_\theta Exp(-\lambda_\theta s)$, $s \in S \equiv R_+$. This is typical of an insurance problem where s represents an aggregate loss of a potentially catastrophic event. The two agents agree to disagree on the expected loss $E_\theta \widetilde{s} = 1/\lambda_\theta$.

If the two agents have the same constant absolute risk aversion, the collective probability of state s is proportional to the geometric mean of the individual probabilities of that state s. In this case, it is immediate that this implies that $p^{CARA}(s) = \overline{\lambda} Exp(-\overline{\lambda} s)$, $s \in R_+$, where $\overline{\lambda}$ equals $(\lambda_1 + \lambda_2)/2$. In the CARA case, the representative agent believes that losses are exponentially distributed with parameter $\overline{\lambda}$.

If the two agents have the same relative risk aversion γ, the shape of the collective density compared to p^{CARA} depends upon where do the agents disagree the most. For two negative exponential density functions, it is easy to check that there is no disagreement at the crossing point $s_0 = (\lambda_1 - \lambda_2)\ln(\lambda_1/\lambda_2)$, and that the degree of disagreement increases when moving from this point towards the extremes. By Proposition 4, it implies that the collective density function will be convexified compared to p^{CARA}. More probability masses are put on very small and very large losses. In other words, the heterogeneity of beliefs increases the perceived risk of loss. This implies that, compared to the p^{CARA} beliefs, one should invest relatively more in preventive actions that mitigate losses of very frequent or very unfrequent events.

3.4 Example 2: Normal beliefs

We assume alternatively that S is the real line, and the two agents believe that the states $s \in R$ are normally distributed with variance σ^2, but their beliefs differ on the mean $\mu(\theta)$, with $\mu(2) > \mu(1)$. In the numerical example depicted in Figure 1, the two densities are dashed, with $\mu(2) = -\mu(1) = 2$, and $\sigma = 1$. Considering for simplicity the Pareto-allocation with $\lambda(1) = \lambda(2)$, we have that

$$q(s,2) - q(s,1) = \frac{1}{\sqrt{2\pi}\sigma}\left[e^{-\frac{(s-\mu(2))^2}{2\sigma^2}} - e^{-\frac{(s-\mu(1))^2}{2\sigma^2}}\right]\begin{cases} < 0 \; if \; s < 0.5(\mu(1)+\mu(2)) \\ > 0 \; if \; s > 0.5(\mu(1)+\mu(2)) \end{cases}$$

and

$$d\ln q(s,2) - d\ln q(s,1) = \frac{\mu(2)-\mu(1)}{\sigma^2}ds > 0.$$

It implies that a marginal increase in s yields a *relative* decrease in disagreement whenever $s < 0.5(\mu(1)+\mu(2)) = 0$, otherwise it yields a *relative* increase in disagreement. Relative disagreement is increased towards the extreme states.

Suppose first that the two agents have ICARA preferences. In that case, the disagreement in beliefs does not matter. As stated by Eq. 17, the efficient collective beliefs simply consist in the geometric mean of the individual densities. For normal individual distributions, this yields a normal collective distribution whose mean is equal to the average of the individual means (see Appendix for more details). In the numerical example depicted in Figure 1, the collective density in the CARA case is normal with a zero mean.

Suppose alternatively that agents have identical and constant *relative* risk aversion, which is a special case of ISHARA and DARA. By Proposition 4, we know that increased disagreement has an effect on the collective probability which is larger than in the CARA case. Because relative disagreement is increased towards the extreme states in our example, we should see more

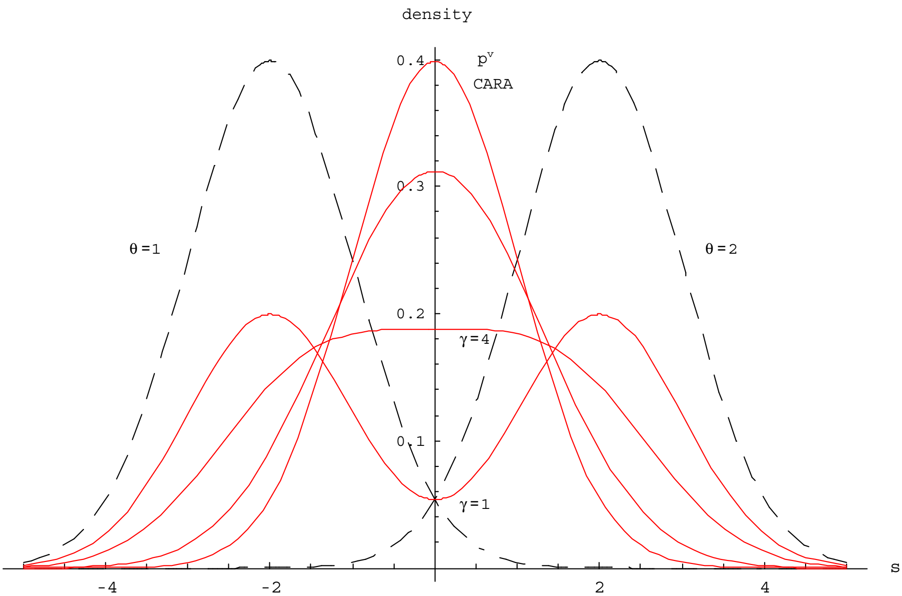

Fig. 1 Efficient collective density functions when relative risk aversion γ is constant. Both agents have normal beliefs (*dashed curves*) with the same variance $\sigma^2 = 1$, assuming $\mu(1) = -2$, $\mu(2) = 2$

probability weights in the tails. In Figure 1, we have drawn the collective density for $\gamma = 1$, 4 and 10, and we see that tails are thicker than in the CARA case. In the extreme case where γ tends to zero, we show in the Appendix that the collective density becomes just proportional to the function $\max\{q(s,1), q(s,2)\}$. There is a simple reason for that. Under risk neutrality above a minimum level of subsistence (which is zero in the CRRA case), agent 1 (resp. 2) will consume the entire cake when $s < 0$ (resp. $s > 0$). So the collective attitude towards a transfer of wealth from one state $s < 0$ (> 0) to another state $s' < 0$ (> 0) will only depend upon the subjective probability of these states for agent 1 (2).

This example also illustrates another important feature of the aggregation of beliefs. Contrary to the intuition, the collective probability of any state s needs not to belong to the interval bounded by $\min_{\theta\in\Theta} p(s,\theta)$ and $\max_{\theta\in\Theta} p(s,\theta)$. This is in sharp contrast with the rate of increase in the collective probability, which is a weighted mean of the rate of increase in the individual probabilities, as stated by Eq. 16.

4 Asset pricing with heterogeneous beliefs

What are the implications of these results on the equity premium? Consider an economy in which the endowment of the single consumption per capita is ω_s in state s, $s = 1, ..., S$. Suppose that consumers have access to markets for Arrow–Debreu (AD) securities. If the price of the contingent claim associated to state s, the aggregate demand for AD securities solves the following program:

$$\max_{(z_1,\dots,z_S)} \sum_{s=1}^{S} v(z_s, P_s) \quad \text{subject to} \quad \sum_{s=1}^{S} \pi_s (z_s - \omega_s) = 0. \tag{20}$$

Because it is a closed economy, it must be that the optimal solution is such that $z_s = \omega_s$ for all s. The first-order condition of program (20) can be rewritten as an equilibrium condition as follows:

$$v_z(\omega(s), P(s)) = \pi(s),$$

where $\pi(s)$ is the price of the Arrow–Debreu security associated to state s. Proposition 4 directly implies that the price of this asset is increasing in the relative disagreement of individual probabilities associated to the corresponding state. If there are two states with the same average log probability, the Arrow–Debreu security associated to the state with the larger degree of relative disagreement has a larger equilibrium price.

The price of equity equals

$$P^e = \frac{\sum_{s=1}^{S} \omega(s)\pi(s)}{\sum_{s=1}^{S} \pi(s)} = \frac{\sum_{s=1}^{S} \omega(s) v_z(\omega(s), P(s))}{\sum_{s=1}^{S} v_z(\omega(s), P(s))}.$$

Suppose that the ISHARA condition holds, so that $v_z(z, P) = p^v(P)h(z)$. The representative agent then perceives an equity premium equaling

$$\phi = -1 + \frac{\left[\sum_{s=1}^{S} p^v(P(s))\omega(s)\right] \quad \left[\sum_{s=1}^{S} p^v(P(s))h(\omega(s))\right]}{\sum_{s=1}^{S} p^v(P(s))\omega(s)h(\omega(s))}.$$

We are interested in determining the effect of the heterogeneity of beliefs on this price of macroeconomic risk. Proposition 4 implies that when relative disagreements are concentrated in the tails of aggregate consumption, the representative agent perceives a distribution of states that is more dispersed than the distribution generated by using the intuitive geometric aggregation rule. Thus, the heterogeneity of beliefs implies in this case an increase of macroeconomic risk perceived by the representative agent. Because of risk aversion, this should induce a reduction in the demand for equity. Eventually, in equilibrium, this should yield a reduction in the price of equity, and to an increase in the equity premium ϕ.[4]

In the remainder of this section, we examine a purely hypothetical situation of conflicts in beliefs. Our aim is to show that conflicts in beliefs have the potential to have a sizeable effect on the equity premium. Suppose that all agents have the same constant relative risk aversion γ. Consider as a benchmark that all agents believe that the growth rate ω of consumption is lognormally distributed, i.e., that

$$\omega(s) = \exp s \quad \text{and} \quad s \sim N(\mu, \sigma^2).$$

In order to fit the historical U.S. data with an expected growth rate of consumption $E\omega - 1 = 1.8\%$ per year and a standard deviation equaling 3.56% per year, we take $\mu = 0.017$ and $\sigma = 0.035$. Such a specification implies that the equity premium equals $\phi = -1 + \exp(\gamma\sigma^2)$. For the reasonable relative risk aversion $\gamma = 2$, the equity premium is only 0.25% per year, far below the 6% observed equity premium observed during the last century. Hence the equity premium puzzle.

Suppose alternatively that there are two equally-sized groups with heterogeneous beliefs. The two groups believe that s is normally distributed with the true variance, but the optimistic group believes that the mean is $\mu_o = \mu + k\sigma > \mu$, whereas the pessimistic group believes that the mean is $\mu_p = \mu - k\sigma < \mu$. Parameter k is a measure of the heterogeneity of beliefs. Notice that the two groups have wrong beliefs when $k > 0$, *but they are right on average.* This work thus differs much from Abel (2002) who examines how a systematic bias in beliefs affects asset prices. In Figure 2, we show how the equity premium varies with the degree k of heterogeneity of beliefs. This relationship is convex. Observe in particular that introducing heterogeneous beliefs has no effect on the equity premium at the margin. When the difference

[4]This is not true in general, as shown by Rothschild and Stiglitz (1971). Gollier (1995) derives the necessary and sufficient condition for a change in risk to reduce the demand for this risk by risk-averse investors. See also Jouini and Napp (2005).

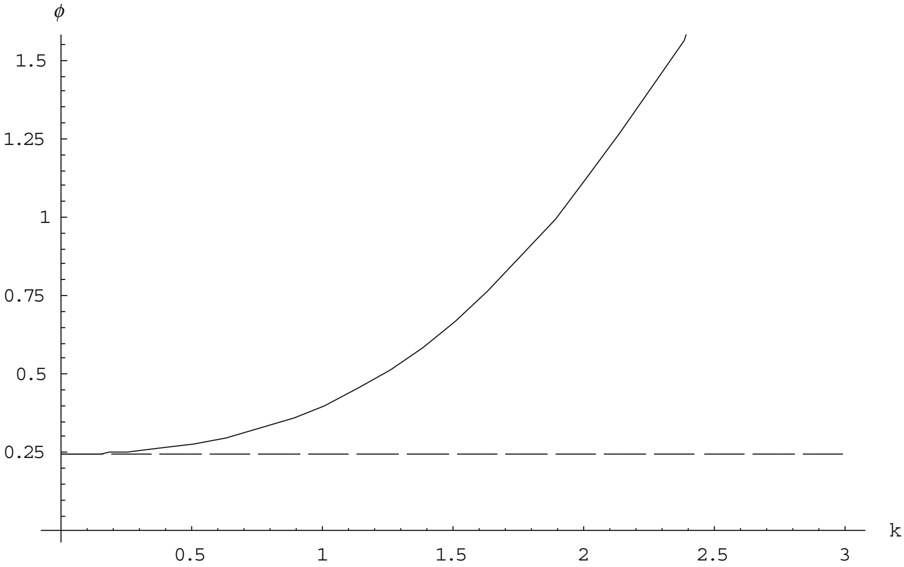

Fig. 2 The equity premium ϕ (in % per year) as a function of the degree of heterogeneity of beliefs ($\gamma = 2$)

between μ_o and μ_p equals four times the standard deviation, the equity premium is increased from 0.25 to 1.12 per year.

Of course, when the degree of heterogeneity in beliefs in the economy, more mutually advantageous exchanges of AD securities between pessimists and optimists emerge. In Figure 3, we have drawn the equilibrium equity

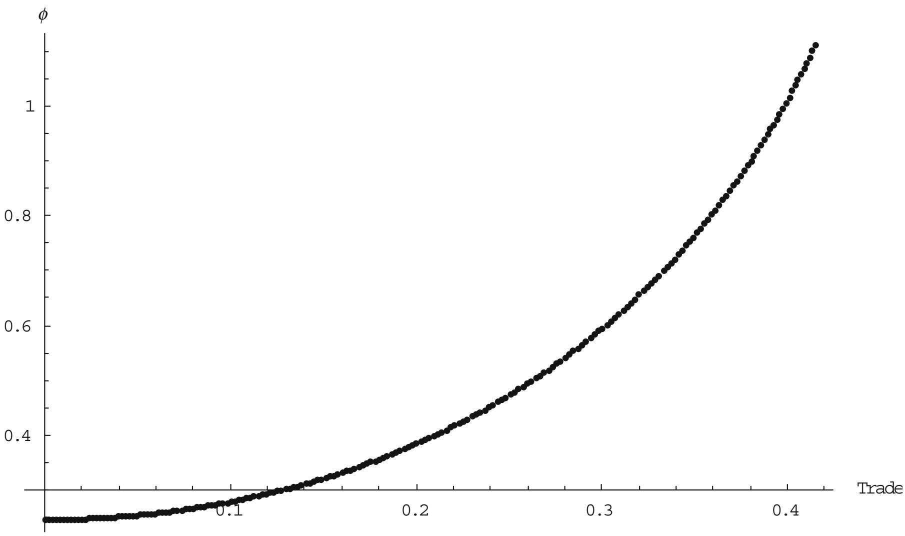

Fig. 3 Relationship between the volume of trade and the equity premium

premium as a function of the volume of trade in the economy, which is defined as $\Sigma_s \pi_s |c_{os} - \omega_s|$. There is an increasing and convex relationship between the volume of trade and the equity premium. The causality is that an increase in the degree of heterogeneity of beliefs affects positively both the volume of trade and the equity premium.

5 Conclusion

Our aim in this paper was to characterize the beliefs that should be used for collective decision making when individuals differ about their expectations. The key property of the aggregation of beliefs is that an increase in the subjective state probability of agent θ should raise the collective probability proportionally to agent θ's degree of absolute risk tolerance. If an agent has a risk tolerance which represents a share x of the collective risk tolerance, he should bear a share x of the collective risk, and a one percent increase in his subjective probability should raise the collective probability by x percents. This result has several important consequences.

First, it implies that the socially efficient collective probability distribution depends upon the aggregate wealth level of the group. This is because the aggregate wealth level affects the way risks should be allocated in the group. However, when agents have the same HARA utility function, changes in aggregate wealth have no effect on the allocation of risks. This implies that the collective probability distribution is independent of wealth in that case. We showed that the identically-sloped HARA case is the *only* case in which such separability property between beliefs and utility holds.

Second, we derived various results that are useful to understand the effect of the divergence of opinions on the shape of the collective probability distribution. To do this, we defined the concept of increasing relative disagreement. In short, there is more relative disagreement about state s' than about state s if the individual subjective probabilities are more dispersed in state s' than in state s. We showed that, with such a shift in the distribution of individual probabilities, the rate of increase of the collective probability is larger than the mean rate of increase of individual probabilities if and only if absolute risk aversion is decreasing.

The last step is to link the structure of disagreement at the global level to the global properties of the collective probability distribution. When most disagreements are concentrated in the tails of the distribution, the collective distribution function is dominated by the average individual probability distribution in the sense of second-order stochastic dominance. This tends to raise the equity premium. We showed in a simple numerical example that the heterogeneity of individual beliefs may have a sizeable effect on the equity premium.

The critical assumption of this model is that the group can allocate risk efficiently. This assumption is difficult to test. For example, the inefficient coverage of earthquake coverage in various regions can be interpreted in two ways. The optimistic view is that homeowners are more pessimistic than insurers about the risk, which implies that the low insurance coverage is socially efficient. But alternatively, it could be interpreted as a proof that markets are incomplete. A similar problem arises to explain the insurance crisis after 9/11/01, or about the difficulty to share the risk related to global warming on an international basis. A possible extension of this work would be to consider an economy with incomplete markets.

Appendix: The case of ISHARA preferences

In this appendix, we first prove Proposition 2.

Proof of Proposition 2 Fully differentiating Eq. 8 with respect to z and using property (6) yields that $\partial R/\partial z$ evaluated at (z, P) has the same sign that

$$\frac{\partial T^u}{\partial c}(c(z, P, \theta), \theta) - \sum_{\theta'=1}^{N} \frac{T^u(c(z, P, \theta'), \theta')}{T^v(z, P)} \frac{\partial T^u}{\partial c}(c(z, P, \theta'), \theta'), \tag{21}$$

For ISHARA preferences, $\partial T^u/\partial c$ is a constant, which implies that the above expression is uniformly equal to zero, implying that R is independent of the per capita wealth in the group. Reciprocally, R independent of z implies that

$$\frac{\partial T^u}{\partial c}(c(z, P, \theta), \theta) = \sum_{\theta'=1}^{N} \frac{T^u(c(z, P, \theta'), \theta')}{T^v(z, P)} \frac{\partial T^u}{\partial c}(c(z, P, \theta'), \theta')$$

for all θ and P. This can be possible only if $\partial T^u/\partial c$ is independent of c and θ, which means that the group has ISHARA preferences. □

We now derive an analytical solution in the ISHARA case. It is easy to check that the set of utility functions that satisfies the ISHARA property must be parameterized as follows:

$$u(c, \theta) = \kappa \left(\frac{c - a(\theta)}{\gamma} \right)^{1-\gamma} \tag{22}$$

These utility functions are defined over the consumption domain such that $\gamma^{-1}(c - a(\theta)) > 0$. In this particular case, the first-order condition to state-dependent the Pareto program (3) implies that

$$c(z, P, \theta) - a(\theta) = k \left[\lambda(\theta) p(\theta) \right]^{1/\gamma}.$$

Since $T^u(c, \theta) = (c - a(\theta))/\gamma$, property (8) can be rewritten in the ISHARA case as

$$R(z, P, \theta) = \frac{\left[\lambda(\theta) p(\theta) \right]^{1/\gamma}}{NE\left[\lambda(\widetilde{\theta}) p(\widetilde{\theta}) \right]^{1/\gamma}}, \tag{23}$$

where $Ef(\widetilde{\theta}) = N^{-1}\sum_{\theta=1}^{N} f(\theta)$. The definition of R applied to the ISHARA case implies that

$$R(z, P, \theta) = \frac{p(\theta)p_{\theta}^{v}(P)}{p^{v}(P)}, \tag{24}$$

where $p_{\theta}^{v} = \partial p^{v}/\partial p(\theta)$. Combining Eqs. 23 and 24 yields

$$\frac{p_{\theta}^{v}(P)}{p^{v}(P)} = \frac{\lambda(\theta)^{1/\gamma}p(\theta)^{-1+1/\gamma}}{NE\left[\lambda(\widetilde{\theta})p(\widetilde{\theta})\right]^{1/\gamma}} \tag{25}$$

for $\theta = 1, ..., N$. The solution to this system of partial differential equations has the following form:

$$p^{v}(P) = C \quad \left[E\left[\lambda(\widetilde{\theta})p(\widetilde{\theta})\right]^{1/\gamma}\right]^{\gamma}, \tag{26}$$

where C is a constant. In order for p^{v} to be a probability distribution, we need to select the particular solution with

$$p^{v}(P(s)) = \frac{\left[E_{\widetilde{\theta}}\left[\lambda(\widetilde{\theta})p(s,\widetilde{\theta})\right]^{1/\gamma}\right]^{\gamma}}{\sum_{t=1}^{S}\left[E_{\widetilde{\theta}}\left[\lambda(\widetilde{\theta})p(t,\widetilde{\theta})\right]^{1/\gamma}\right]^{\gamma}}. \tag{27}$$

Calvet, Grandmont and Lemaire (2001) and Jouini and Napp (2007) obtained the same solution. Jouini and Napp (2007) and Chapman and Polkovnichenko (2006) derived this result in the special case of CRRA ($a = 0$).

Three special cases are worthy to examine.

- Consider first the case with γ tending to zero. This corresponds to risk-neutral preferences above a minimum level of subsistence. Under this specification, condition (27) is rewritten as

$$p^{v}(P(s)) = p^{n}(P(s))$$
$$=_{def} \frac{\max_{\theta\in\Theta}\lambda(\theta)p(s,\theta)}{\sum_{t=1}^{S}\max_{\theta\in\Theta}\lambda(\theta)p(t,\theta)} \quad \text{for all } s. \text{ (risk-neutral case)} \tag{28}$$

  With risk-neutral preferences, the efficient allocation produces a flip-flop strategy where the cake in state s is entirely consumed by the agent with the largest Pareto-weighted probability associated to that state. It implies that the group will use a state probability p^{n} proportional to it to determine its attitude toward risk ex ante.
- In the case of logarithmic preferences ($a = 0,\ \gamma = 1$), the denominator in Eq. 27 equals $E\lambda(\widetilde{\theta})$ since

$$\sum_{t=1}^{S} E_{\widetilde{\theta}}\lambda(\widetilde{\theta})p(t,\widetilde{\theta}) = E_{\widetilde{\theta}}\left[\lambda(\widetilde{\theta})\sum_{t=1}^{S}p(t,\widetilde{\theta})\right] = E\lambda(\widetilde{\theta}) = 1.$$

It implies that

$$p^v(P(s)) = p^{\ln}(P(s)) =_{def} E\lambda(\widetilde{\theta})p(s,\widetilde{\theta}) \text{ for all } s. \text{ (logarithmic case)} \quad (29)$$

With these Bernoullian preferences, the efficient probability that should be associated to any state s is just the weighted mean $p^{\ln}(s)$ of the individual subjective probabilities of that state s. This is the limit case $T_c^u \equiv 1$ of the result presented in Proposition 5.

- In the CARA case, we assume that $u(c,t,\theta) = -\exp(-c/t(\theta))$, which is equivalent to γ tending to $+\infty$, and $a(\theta)/\gamma$ tending to $-t(\theta)$. Equation 8 implies in that case that

$$p^v(P) = p^{CARA}(P) = K\prod_{\theta=1}^{N} p(\theta)^{\frac{t(\theta)}{\sum_{\theta'=1}^{N} t(\theta')}}, \quad (30)$$

where K is a normalizing constant. Aggregation rule (30) and (29) are due to Rubinstein (1974). This aggregation rule is particularly easy to use when all individual beliefs are normally distributed. Suppose that agent θ, $\theta = 1, ..., N$, believes that states are normally distributed with mean $\mu(\theta)$ and variance $\sigma^2(\theta)$. An easy consequence of Eq. 30, first observed by Lintner (1969), is that the collective beliefs p^v are also normally distributed with mean

$$\mu^v = \frac{\sum_{\theta=1}^{N} \frac{t(\theta)\mu(\theta)}{\sigma^2(\theta)}}{\sum_{\theta=1}^{N} \frac{t(\theta)}{\sigma^2(\theta)}}, \quad (31)$$

and variance

$$\sigma^v = \left[\frac{\sum_{\theta=1}^{N} \frac{t(\theta)}{\sigma^2(\theta)}}{\sum_{\theta=1}^{N} t(\theta)}\right]^{-0.5}. \quad (32)$$

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