Reliable Computing (2007) **13:** 173–194
DOI: 10.1007/s11155-006-9024-3

Interval Finite Elements as a Basis for Generalized Models of Uncertainty in Engineering Mechanics

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(Received: 15 December 2004; accepted: 7 March 2005)

Abstract. Latest scientific and engineering advances have started to recognize the need for defining multiple types of uncertainty. Probabilistic modeling cannot handle situations with incomplete or little information on which to evaluate a probability, or when that information is nonspecific, ambiguous, or conflicting [12], [47], [50]. Many interval-based uncertainty models have been developed to treat such situations.

This paper presents an interval approach for the treatment of parameter uncertainty for linear static structural mechanics problems. Uncertain parameters are introduced in the form of unknown but bounded quantities (intervals). Interval analysis is applied to the Finite Element Method (FEM) to analyze the system response due to uncertain stiffness and loading.

To avoid overestimation, the formulation is based on an element-by-element (EBE) technique. Element matrices are formulated, based on the physics of materials, and the Lagrange multiplier method is applied to impose the necessary constraints for compatibility and equilibrium. Earlier EBE formulation provided sharp bounds only on displacements [32]. Based on the developed formulation, the bounds on the system's displacements and element forces are obtained simultaneously and have the same level of accuracy. Very sharp enclosures for the exact system responses are obtained. A number of numerical examples are introduced, and scalability is illustrated.

1. Introduction

An important issue faced in real life engineering practice is how to deal with variables and parameters of uncertain values. For a proper performance assessment, these uncertainties must be accounted for appropriately. There are various ways in which the types of uncertainty might be classified. One is to distinguish between "aleatory" (or stochastic) uncertainty and "epistemic" uncertainty. The first refers to underlying, intrinsic variabilities of physical quantities, and the latter refers to uncertainty which might be reduced with additional data or information, or better modeling and better parameter estimation [26]. Probability theory is the traditional approach to handling uncertainty. This approach requires sufficient statistical data to justify the assumed statistical distributions. Analysts agree that, given sufficient statistical data, the probability theory describes the stochastic uncertainty well. However, probabilistic modeling cannot handle situations with incomplete or little information on which to evaluate a probability, or when that information is non-specific, ambiguous, or conflicting [12], [47], [50]. Many generalized models of uncertainty have been developed to treat such situations, including fuzzy sets and possibility theory [51], Dempster-Shafer theory of evidence [10], [48], random set [21], probability bounds [6], [12], [13], imprecise probabilities [50], convex model [5], and others.

These generalized models of uncertainty have a variety of mathematical descriptions. However, they are all closely connected with interval analysis [28]. For example, the mathematical analysis associated with fuzzy set theory can be performed as interval analysis on different α levels [23], [30].

Fuzzy arithmetic can be performed as interval arithmetic on α cuts. A Dempster-Shafer structure [10], [48] with interval focal elements can be viewed as a set of intervals with probability mass assignments, where the computation is carried out using the interval focal sets. Probability bounds analysis [6], [12], [13] is a combination of standard interval analysis and probability theory. Uncertain variables are decomposed into a list of pairs of the form (interval, probability). In this sense, interval arithmetic serves as the calculation tool for the generalized models of uncertainty.

Recently, various generalized models of uncertainty have been applied within the context of the finite element method to solve a partial differential equation with uncertain parameters. Regardless of the model adopted, the proper interval solution will represent the first requirement for any further rigorous formulation. The finite element method with interval valued parameters results in the Interval Finite Element Method (IFEM), the numerical solution of which is the focus of this paper. The use of IFEM solution techniques can be broadly classified into two groups, namely the optimization approach and the non-optimization approach. In the optimization approach [1], [22], [27], [42], optimizations are performed to compute the minimal and maximal structural responses when the uncertain parameters are constrained to belong to intervals.

This approach often encounters practical difficulties. Firstly it requires an efficient and robust optimization algorithm. In most structural engineering problems, the interval finite element objective function is nonlinear and complicated, thus often only an approximate solution is achievable. Secondly, this approach is computationally expensive. For each response quantity, two optimization problems must be solved to find the lower and the upper bounds.

More recently, non-optimization approaches for interval finite element analysis have been developed in a number of papers. For linear elastic problems, this approach leads to a system of linear interval equations, then the solution is sought using various methods developed for this purpose. The major difficulty associated with this approach is the “dependency problem” [17], [29], [32], [35]. The dependency in interval arithmetic leads to an overestimation of the solution.

A straightforward replacement of the system parameters with interval ones without taking care of the dependency problem is known as a naïve application of interval arithmetic in the finite element method (naïve IFEM). Usually such a use results in meaninglessly wide and even catastrophic results [32].

In the non-optimization category, a number of developments can be presented. A combinatorial approach (based on an exhaustive combination of the extreme values of the interval parameters) was used in [30], [41]. This approach gives exact solution in linear elastic problems. However, it is computationally tedious and expensive, and is limited to the solutions of small-scale problems. A convex modeling and superposition approach was proposed to analyze load uncertainty in [38], and exact solution was obtained. However, the superposition is only applicable to load uncertainty. The combinatorial approach was used in [15] to treat interval modulus of elasticity. Chen et al. [8] have developed a static displacement bounds analysis using matrix perturbation theory. The first-order perturbation was used, and the second-order term was neglected. The result is approximate and not guaranteed to contain the exact bounds. McWilliam [25] proposed two methods for determining the static displacement bounds of structures with interval parameters. The first method is a modified version of perturbation analysis. The second method is based on the assumption that the displacement surface is monotonic. However, for the general case, the validity of monotonicity is difficult to verify. Dessombz et al. [11] have introduced an interval FEM in which the interval parameters were factored out during the assembly process of the stiffness matrix. Then Rump's iterative algorithm [46] was employed for solving the linear interval equation. In this work, the overestimation control becomes more difficult with the increase in the number of the interval parameters, which does not lead to useful results for practical problems. In the work of Muhanna and Mullen [30], and of Mullen and Muhanna [33], [34], an interval-based fuzzy finite element has been developed for treating uncertain loads in static structural problems. Load dependency was eliminated, and the exact solution was obtained. Also, Muhanna and Mullen [32] have developed an IFEM based on the element-by-element technique and Lagrange multiplier method. Uncertainty in the modulus of elasticity was considered. Most sources of overestimation were eliminated, and a sharp result for displacement was obtained. However, this formulation can only handle uncertain modulus of elasticity, and it can not obtain the sharp enclosures for element internal forces.

This paper extends the results of Muhanna and Mullen by introducing a new formulation for interval finite element analysis for linear static structural problems. Material and load uncertainties are handled simultaneously, and sharp enclosures on the system's displacements as well as the internal forces are obtained efficiently. Two other papers in this special issue also address interval-based structural uncertainty, the lead authors of these papers are Arnold Neumaier and George Corliss. A brief review of interval arithmetic is presented, the formulation is described, and numerical examples are given.

2. Short Review of Interval Arithmetic

In this paper, the notation follows the recommendation of [20]. Interval quantities (interval number, interval vector, interval matrix) are introduced in boldface. Real quantities are introduced in non-bold face.

2.1. Basic Definition

An interval number is a closed set in $\mathbb{R}$ that includes the possible range of an unknown real number, where $\mathbb{R}$ denotes the set of real numbers. Therefore, a real interval is a set of the form

$$\boldsymbol{x} = [\underline{x}, \overline{x}], \tag{2.1}$$

where $\underline{x}$ and $\overline{x}$ are the lower and upper bounds (endpoints) of the interval number $\boldsymbol{x}$ respectively. The set of real intervals is denoted by $\mathbb{IR}$. The midpoint$\check{x}$ of $\boldsymbol{x}$ is

$$\check{x} \equiv \text{mid}(\boldsymbol{x}) := \frac{\underline{x} + \overline{x}}{2}. \tag{2.2}$$

Sometimes it is convenient to write the interval in the midpoint form

$$\boldsymbol{x} = \check{x}(1 + \boldsymbol{\alpha}) \tag{2.3}$$

in which $\boldsymbol{\alpha}$ is a 0-midpoint interval. For example, when we say $\boldsymbol{x}$ has 4% uncertainty, it means $\boldsymbol{\alpha} = [-0.02, 0.02]$, and $\boldsymbol{x} = \check{x}(1 + [-0.02, 0.02])$.

The four elementary operations of real arithmetic, namely addition (+), subtraction (−), multiplication (∗), and division (/) can be extended to intervals. Operations over intervals $\circ \in \{+, -, *, /\}$ are defined by the general rule

$$\boldsymbol{x} \circ \boldsymbol{y} = \{x \circ y \mid x \in \boldsymbol{x},\ y \in \boldsymbol{y}\}. \tag{2.4}$$

It is easy to see that the set of all possible results when applying an operator $\circ$ to $\boldsymbol{x} \circ \boldsymbol{y}$ forms a closed interval (for 0 not in a denominator interval), and the endpoints can be calculated by

$$\boldsymbol{x} \circ \boldsymbol{y} = [\min(x \circ y), \max(x \circ y)] \qquad for \ \circ \in \{+, -, *, /\}. \tag{2.5}$$

Detailed information about interval arithmetic can be found in a series of books and publications such as [2], [18], [28], [35], [45], [49].

2.2. Dependency Problem in Interval Arithmetic

The quality of interval analysis is measured by the width of the interval results, and a sharp enclosure for the exact solution is desirable. However, the width of results may be unnecessarily wide in some occasions due to the dependency effect. For example, if the interval expression $\boldsymbol{x} - \boldsymbol{x}$ is evaluated with $\boldsymbol{x} = [a, b] = [1, 2]$, the interval subtraction rule gives the result: $\boldsymbol{x} - \boldsymbol{x} = [a - b,\ b - a] = [-1, 1]$, which contains the exact solution $[0, 0]$, but is much wider than needed. The interval arithmetic implicitly made the assumption that all intervals are independent, namely it treats $\boldsymbol{x} - \boldsymbol{x}$ as if evaluating the expression $\boldsymbol{x} - \boldsymbol{y}$, and $\boldsymbol{x}, \boldsymbol{y}$ are two independent interval quantities that happen to have the same bounds. This phenomenon is referred to as overestimation due to "dependency" of the variables [17], [29], [32], [35]. Reducing the overestimation is a crucial issue to a successful interval analysis. In general, sharp results are obtained with the proper understanding of the physical nature of the problem and reduction of the dependence. In the above example, the exact solution could be achieved in evaluating $\boldsymbol{x} - \boldsymbol{x}$ as $\boldsymbol{x}(1 - 1) = 0$.

2.3. Interval Vectors and Matrices

An interval matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times k}$ is interpreted as a set of real $n \times k$ matrices by the convention $\boldsymbol{A} = \{A \in \mathbb{R}^{n\times k} \mid A_{ij} \in \boldsymbol{A}_{ij} \text{ for } i = 1, \ldots, n,\ j = 1, \ldots, k\}$. The set of $n \times k$ interval matrices is denoted by $\mathbb{IR}^{n\times k}$. An $n \times 1$ interval matrix is an interval vector, denoted by $\mathbb{IR}^n$. Operations on interval matrices are extended naturally from the corresponding deterministic (point values) matrix operations. Algebraic properties of interval matrix operations are provided in [3], [24], [35].

2.4. Linear Interval Equations

A linear interval equation with coefficient matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and right-hand side $\boldsymbol{b} \in \mathbb{IR}^n$ is defined as the family of linear equations

$$Ax = b \qquad (A \in \boldsymbol{A},\ b \in \boldsymbol{b}). \tag{2.6}$$

Therefore, a linear interval equation represents systems of equations in which the coefficients are unknown numbers ranging in certain intervals. The solution set of (2.6) is given by:

$$\Sigma(\boldsymbol{A}, \boldsymbol{b}) = \{x \in \mathbb{R}^n \mid \exists A \in \boldsymbol{A},\ \exists b \in \boldsymbol{b} : Ax = b\}. \tag{2.7}$$

The solution set $\Sigma(\boldsymbol{A}, \boldsymbol{b})$ usually is not an interval vector, and does not need even to be convex; in general, $\Sigma(\boldsymbol{A}, \boldsymbol{b})$ has a very complicated structure. To guarantee that the solution set $\Sigma(\boldsymbol{A}, \boldsymbol{b})$ is bounded, it is required that the matrix $\boldsymbol{A}$ be regular, i.e. that every matrix $A \in \boldsymbol{A}$ is nonsingular. The interval hull of the solution set $\Sigma(\boldsymbol{A}, \boldsymbol{b})$ is an interval vector which has the narrowest possible interval components that contains the solution set and denoted as

$$\boldsymbol{A}^H\boldsymbol{b} := \lozenge\Sigma(\boldsymbol{A}, \boldsymbol{b}), \tag{2.8}$$

where

$$\boldsymbol{A}^H\boldsymbol{b} = \lozenge\{A^{-1}b \mid A \in \boldsymbol{A},\ b \in \boldsymbol{b}\} \qquad \text{for } \boldsymbol{b} \in \mathbb{IR}^n. \tag{2.9}$$

In fact, computing the hull of the solution set for the general case is an NP-Hard problem [43]. The solution of interest is seeking an enclosure, i.e., an interval vector $\boldsymbol{x}$ containing $\boldsymbol{A}^H\boldsymbol{b}$, while narrow enough to be practically useful:

$$\boldsymbol{A}^H\boldsymbol{b} \subseteq \boldsymbol{x}. \tag{2.10}$$

A number of methods have been developed to find $\boldsymbol{x}$ for the general linear interval equations such as Interval Gauss elimination, Interval Gauss-Seidel iteration, Krawczyk's iteration, and fixed-point iteration [16], [19], [35], [36], [44], [46]. These algorithms usually involve a preconditioning of the coefficient matrix, and then iterations are performed to get the enclosure. The present work uses Brouwer's fixed point theorem and Krawczyk's operator. This method has been discussed in the works of [16], [19], [36], [37], [44], [46].

One typical approach to find the solution of a linear system $Ax = b$ is to transform it into a fixed point equation $g(x) = x$, in which

$$g(x) = x - R(Ax - b) = Rb + (I - RA)x, \tag{2.11}$$

and R is a nonsingular matrix. From Brouwer's fixed point theorem, it follows that for some interval vector $\boldsymbol{x} \in \mathbb{IR}^n$

$$Rb + (I - RA)x \in \boldsymbol{x} \qquad \forall x \in \boldsymbol{x} \tag{2.12}$$

implies

$$\exists x \in \boldsymbol{x} : Ax = b. \tag{2.13}$$

One has to verify condition (2.12) is a range determination problem, and can be reduced to interval arithmetic:

$$Rb + (I - RA)\boldsymbol{x} \subseteq \boldsymbol{x}. \tag{2.14}$$

If an interval vector $\boldsymbol{x}$ satisfying (2.14) can be found, then $\boldsymbol{x}$ contains the solution of $Ax = b$. The result can be extended to find the enclosure of the solution set of linear interval equation $\boldsymbol{Ax} = \boldsymbol{b}$ [35], [45]. Based on this result, the following theorem can be presented:

THEOREM 2.1 Rump 2001 [45]. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$, $\boldsymbol{b}, \boldsymbol{x} \in \mathbb{IR}^n$ be given, if*

$$R\boldsymbol{b} + (I - R\boldsymbol{A})\boldsymbol{x} \subseteq \operatorname{int}(\boldsymbol{x}), \tag{2.15}$$

then R and every matrix $A \in \boldsymbol{A}$ is nonsingular, and

$$\Sigma(\boldsymbol{A}, \boldsymbol{b}) = \{x \in \mathbb{R}^n \mid \exists A \in \boldsymbol{A},\ \exists b \in \boldsymbol{b} : Ax = b\} \subseteq \boldsymbol{x}, \tag{2.16}$$

where $\operatorname{int}(\boldsymbol{x})$ *denotes the interior of* $\boldsymbol{x}$.

Expression (2.16) provides a guaranteed enclosure of the solution set of the linear interval equation $\boldsymbol{Ax} = \boldsymbol{b}$. The residual form of (2.15) can be given in the form [35]:

$$R\boldsymbol{b} - R\boldsymbol{A}x_0 + (I - R\boldsymbol{A})\boldsymbol{x}^* \subseteq \text{int}(\boldsymbol{x}^*), \tag{2.17}$$

where $\boldsymbol{x} = x_0 + \boldsymbol{x}^*$ and x_0 is a real vector. In particular, $\check{A}^{-1}$ is a good choice for R [35], and $x_0 = R\check{b}$. Assigning $z = R\boldsymbol{b} - R\boldsymbol{A}x_0$, $\boldsymbol{C} = (I - R\boldsymbol{A})$, iteration could be constructed [46] in the following form

$$\boldsymbol{x}^{*n+1} = z + \boldsymbol{C}(\varepsilon\boldsymbol{x}^{*n}) \qquad (for\ \ n = 0, 1, 2, \ldots), \tag{2.18}$$

and the stopping criteria (2.17) becomes

$$\boldsymbol{x}^{*n+1} \subseteq \text{int}(\boldsymbol{x}^{*n}). \tag{2.19}$$

In (2.18) ε is a constant interval number, serving as an "inflation parameter" to enforce finite termination of the algorithm. If condition (2.19) is satisfied after n iterations, then $\boldsymbol{x}^{*n+1} + x_0$ is an enclosure of the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$. The quality (how sharp the enclosure is) of the enclosure provided in (2.18) depends mainly on the width of the iterative matrix $\boldsymbol{C}$. Crucial for the solution convergence is the condition that the spectral radius $\rho(|\boldsymbol{C}|) < 1$ [44].

The above algorithm is designed for non-parametric linear interval equations; i.e., the coefficients in the system are assumed to vary independently between their bounds. For many engineering problems, the coefficients have complex dependency relations. For example, the stiffness matrix in FEM is symmetric and positive definite. To account for the dependency effect, one approach is to adapt the solver for non-parametric interval equations. This approach usually involves reformulation of the coefficient matrix and the right hand side vector. It has been shown that a sharp or even exact enclosure could be obtained in some cases [11], [31], [32].

3. Interval Finite Element Analysis

3.1. Overestimation in IFEM

A naïve use of interval arithmetic in FEM (naïve IFEM), i.e., replacing real numbers in conventional FEM with interval numbers and solving the system as non-parametric interval equation results in meaningless wide results [11], [32]. Let us consider the two step bar shown in Figure 1. The structure is subjected to a unit load at node 3. The conventional FEM gives the equilibrium equations:

$$Ku = p \tag{3.1}$$

or

$$\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.2}$$

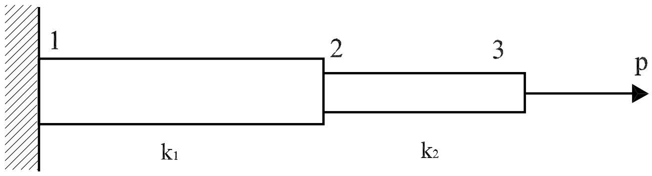

Figure 1. Original two-step bar.

If the stiffness terms k_1 and k_2 are introduced as the interval parameters $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, and the interval numbers of [0.99, 1.01] and [1.98, 2.02] are assigned for $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ respectively, the naïve IFEM takes the following form:

$$\begin{pmatrix} [2.97, 3.03] & [-2.02, -1.98] \\ [-2.02, -1.98] & [1.98, 2.02] \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.3}$$

whose solution using Theorem 2.1 is

$$\begin{aligned} \boldsymbol{u}_1 &= [0.876, 1.123], \\ \boldsymbol{u}_2 &= [1.349, 1.651]. \end{aligned} \tag{3.4}$$

On the other hand, the exact solution can be achieved by solving (3.2) symbolically

$$\begin{aligned} \boldsymbol{u}_1 &= \frac{1}{\boldsymbol{k}_1} = \frac{1}{[0.99, 1.01]} = [0.990, 1.010], \\ \boldsymbol{u}_2 &= \frac{\boldsymbol{k}_1 + \boldsymbol{k}_2}{\boldsymbol{k}_1 \boldsymbol{k}_2} = \frac{1}{\boldsymbol{k}_1} + \frac{1}{\boldsymbol{k}_2} = \frac{1}{[0.99, 1.01]} + \frac{1}{[1.98, 2.02]} = [1.485, 1.515]. \end{aligned} \tag{3.5}$$

The above-presented results for the interval solution of a simple two-step bar problem provide insight about some aspects of the interval finite element formulation and reveal the most important sources of overestimation. The main two factors that lead to overestimation are the element coupling and multiple occurrences of the interval variables. The four parametric coefficients $\boldsymbol{k}_2$ in (3.2) represent the same physical quantity. In the computational process, interval arithmetic treats this physical quantity as four independent interval variables of equal endpoints. Evidently, the same physical quantity cannot have two different values at the same time. The way the sources of overestimation are handled is critical to the formulation of interval finite element analysis.

3.2. Present Formulation

To reduce the overestimation in the interval finite element solutions, the issues of coupling and multiple occurrences of interval variables have to be handled properly.

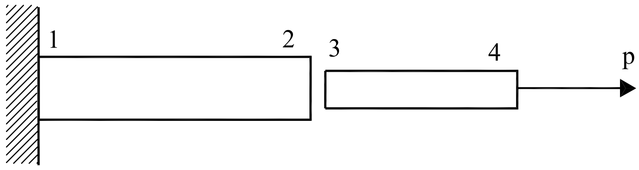

Figure 2. EBE two-step bar model.

In this work, an element-by-element technique (EBE) is used to circumvent the element coupling problem [32]. The EBE technique can be illustrated by the two-step bar problem in Figure 1. The elements are disjointed as shown in Figure 2, thus the system stiffness matrix K takes a block-diagonal structure with dimension of $a \times a$, where a = degrees of freedom per element × number of elements in the structure. The EBE approach adds to the number of degrees of freedom (DOF) in the system but avoids the element coupling. The system stiffness matrix K in the EBE approach is singular, and Lagrange multiplier method is used to ensure the compatibility conditions and eliminate the singularity of K.

In steady-state analysis, the variational formulation for a deterministic case (no uncertainty is involved) of a discrete structural model is given in the following form [4], [14]:

$$\Pi = \frac{1}{2} u^T K u - u^T p \tag{3.6}$$

with the conditions

$$\frac{\partial \Pi}{\partial u_i} = 0 \qquad \text{for all } \; i, \tag{3.7}$$

where Π, K, u, and p are total potential energy, stiffness matrix, displacement vector, and load vector, respectively. Assume that we want to impose onto the solution the m linearly independent discrete constraints $Cu - t = 0$, where C and t contain constants. To impose constraints by Lagrange multipliers, we premultiply $Cu - t$ by a row vector λ that contains as many Lagrange multipliers λ_i as there are constraint equations, and add this to the potential energy (3.6) [9]. Thus

$$\Pi^* = \frac{1}{2} u^T K u - u^T p + \lambda^T (Cu - t). \tag{3.8}$$

Invoking the stationarity of Π^*, i.e., $\partial \Pi^* / \partial u = 0$ and $\partial \Pi^* / \partial \lambda = 0$, we obtain

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} p \\ t \end{pmatrix}. \tag{3.9}$$

Considering the compatibility conditions in the present case takes the form $Cu = 0$ and $t = 0$, (3.9) reduces to

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}. \tag{3.10}$$

Equation (3.10) stands for the deterministic FEM formulation. In the interval case, where the material and the load are considered to be interval numbers, the deterministic linear equation (3.10) becomes the interval linear equation

$$\begin{pmatrix} \boldsymbol{K} & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix}. \tag{3.11}$$

The coefficient matrix in (3.11) represents the combination of two parts: the interval element-by-element stiffness matrix $\boldsymbol{K}$ and the constant deterministic Lagrange multipliers matrix C.

The linear interval equation (3.11) can be solved by Theorem 2.1. However, Theorem 2.1 carries the implicit assumption that the coefficients of $\boldsymbol{A}$ are independent among themselves and that the components of $\boldsymbol{b}$ vary independently. Special treatment has to be applied to reduce the dependency effect.

For an element with interval parameter of modulus of elasticity $\boldsymbol{E}$, the interval parameter could be factored out from the element stiffness matrix. Consider the ith finite element in the structure, assume the uncertainty in the modulus of elasticity is $\boldsymbol{\alpha}_i$, i.e., $\boldsymbol{E}_i = \check{E}_i(1+\boldsymbol{\alpha}_i)$, the element stiffness matrix $\boldsymbol{K}_i$ can be expressed in the form $\boldsymbol{K}_i = \check{K}_i(I + \boldsymbol{d}_i)$. $\check{K}_i$ is the midpoint of $\boldsymbol{K}_i$, I is identity matrix, and $\boldsymbol{d}_i$ is an interval diagonal matrix containing the interval quantity $\boldsymbol{\alpha}_i$. For example, let us take a truss element whose element stiffness matrix can be written as

$$\begin{pmatrix} \dfrac{\check{E}A}{L} & -\dfrac{\check{E}A}{L} \\ -\dfrac{\check{E}A}{L} & \dfrac{\check{E}A}{L} \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\alpha} & 0 \\ 0 & \boldsymbol{\alpha} \end{pmatrix} \right). \tag{3.12}$$

Later in the formulation, care will be taken of the multiple occurrence of $\boldsymbol{\alpha}$ in (3.12).

Following the same procedure for each element, the system stiffness matrix $\boldsymbol{K}$ constructed by EBE model can be expressed as:

$$\boldsymbol{K} = \check{K}(I + \boldsymbol{d}). \tag{3.13}$$

$\check{K}$ is the midpoint of $\boldsymbol{K}$, and $\boldsymbol{d}$ is an interval diagonal matrix; their submatrices are $\check{K}_i$ and $\boldsymbol{d}_i$, respectively, $i = 1, 2, \dots, m$, where m is the number of elements in the structure.

Applying this factorization, the system equation (3.11) can be written as

$$\left(\begin{pmatrix} \check{K} & C^T \\ C & 0 \end{pmatrix} + \begin{pmatrix} \check{K}\boldsymbol{d} & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix}. \tag{3.14}$$

To use the Theorem 2.1 in the present formulation, (3.14) is introduced as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{3.15}$$

with

$$\boldsymbol{A} = \left(\begin{pmatrix} \check{K} & C^T \\ C & 0 \end{pmatrix} + \begin{pmatrix} \check{K}\boldsymbol{d} & 0 \\ 0 & 0 \end{pmatrix} \right), \quad \boldsymbol{x} = \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix}. \tag{3.16}$$

$\boldsymbol{A}$ can be decomposed further

$$\begin{aligned} \boldsymbol{A} &= \begin{pmatrix} \check{K} & C^T \\ C & 0 \end{pmatrix} + \begin{pmatrix} \check{K} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{d} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \check{A} + S\boldsymbol{D}. \end{aligned} \tag{3.17}$$

Using the residual form (2.17) to construct fixed point iteration (2.18)

$$\boldsymbol{x}^{*n+1} = z + \boldsymbol{C}(\boldsymbol{\varepsilon x}^{*n}) \qquad for \;\; n = 0, 1, 2, \ldots \tag{3.18}$$

in which $z = R\boldsymbol{b} - R\boldsymbol{A}x_0$, $\boldsymbol{C} = (I - R\boldsymbol{A})$, $R = \check{A}^{-1}$, and $x_0 = R\check{b}$. By substituting z and $\boldsymbol{C}$, the iteration (3.18) becomes

$$\begin{aligned} \boldsymbol{x}^{*l+1} &= \big(R\boldsymbol{b} - R(\check{A} + S\boldsymbol{D})x_0\big) + \big(I - R(\check{A} + S\boldsymbol{D})\big)(\boldsymbol{\varepsilon x}^{*l}) \\ &= R\boldsymbol{b} - x_0 - RS\boldsymbol{D}x_0 - RS\boldsymbol{D}(\boldsymbol{\varepsilon x}^{*l}) \\ &= R\boldsymbol{b} - x_0 - RS\boldsymbol{D}(x_0 + \boldsymbol{\varepsilon x}^{*l}) \\ &= R\boldsymbol{b} - x_0 - RS\boldsymbol{M}^l\boldsymbol{\delta}. \end{aligned} \tag{3.19}$$

In the problems with deterministic right hand side, we have $\boldsymbol{b} = \check{b}$, and (3.19) reduces to a simpler form

$$\boldsymbol{x}^{*n+1} = -RS\boldsymbol{M}^n\boldsymbol{\delta}. \tag{3.20}$$

A key point in the formulation (3.19) is that $\boldsymbol{D}(x_0 + \boldsymbol{\varepsilon x}^{*n})$ has been introduced as $\boldsymbol{M}^n\boldsymbol{\delta}$ using the $\boldsymbol{M}$ matrix concept [32], [33] to handle the dependency problem in $\boldsymbol{D}(x_0 + \boldsymbol{\varepsilon x}^{*n})$. $\boldsymbol{M}$ is an interval matrix with the dimensions $(n \times m)$, and n = dimensions of the system. It contains the components from $(x_0 + \boldsymbol{\varepsilon x}^{*n})$, and it will be updated with each iteration. $\boldsymbol{\delta}$ is a constant interval vector with the dimensions of m, and the components are the uncertainties $\boldsymbol{\alpha}_i$ of the modulus of elasticity of each element, $i = 1, \ldots, m$. Every interval parameter $\boldsymbol{\alpha}_i$ associated with element i occurs only once in $\boldsymbol{\delta}$. The following example shows how generally $\boldsymbol{Dx}$ can be rewritten as $\boldsymbol{M\delta}$. Suppose there are two interval parameters $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$

$$\begin{pmatrix} \boldsymbol{\alpha}_1 & 0 & 0 & 0 \\ 0 & \boldsymbol{\alpha}_1 & 0 & 0 \\ 0 & 0 & \boldsymbol{\alpha}_2 & 0 \\ 0 & 0 & 0 & \boldsymbol{\alpha}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_1 & 0 \\ \boldsymbol{x}_2 & 0 \\ 0 & \boldsymbol{x}_3 \\ 0 & \boldsymbol{x}_4 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{pmatrix}. \tag{3.21}$$

This treatment eliminates the multiple occurrences of $\boldsymbol{\alpha}_i$ in $\boldsymbol{D}$, thus reduces the overestimation due to dependency problem. If condition (2.19) is satisfied after n iterations, the enclosure $\boldsymbol{x}$ is given by

$$\boldsymbol{x} = \boldsymbol{x}^{*n+1} + x_0. \tag{3.22}$$

The obtained interval vector $\boldsymbol{x}$ contains two parts: $\boldsymbol{x} = [\boldsymbol{u} \quad \boldsymbol{\lambda}]$. The first part, $\boldsymbol{u}$, is the enclosure for the system's nodal displacements. The second part $\boldsymbol{\lambda}$ can be used to find the enclosure for the element internal forces, as shown in below.

In conventional deterministic FEM, element internal forces in global coordinates can be calculated by

$$F_i = K_i u_i \tag{3.23}$$

in which K_i is element stiffness matrix, and u_i is element nodal displacement in global coordinates. The element internal forces in local coordinates can be obtained by premultiplying by a transformation matrix T_i. In the IFEM, following the same procedure to calculate element forces will bring in overestimation, making the bounds of the element forces unnecessarily wide. The reason is that both $\boldsymbol{K}_i$ and $\boldsymbol{u}_i$ are functions of the same interval parameter $\boldsymbol{\alpha}_i$, this multiple occurrences of $\boldsymbol{\alpha}_i$ should be eliminated. In the present IFEM formulation, element forces are calculated from Lagrange multipliers. From (3.11), it follows

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{p} - C^T\boldsymbol{\lambda}. \tag{3.24}$$

Because of its element-by-element structure, (3.24) produces the element forces directly (in global coordinates). Instead of calculating the left hand side of (3.24), we calculate its right hand side to handle dependence problem. Suppose the enclosure $\boldsymbol{x}$ has been achieved after n iterations. Then $\boldsymbol{\lambda}$ can be obtained from $\boldsymbol{x}$ by the element connectivity matrix L, i.e.,

$$\boldsymbol{\lambda} = L\boldsymbol{x}. \tag{3.25}$$

The interval load $\boldsymbol{p}$ can be rewritten as

$$\boldsymbol{p} = N\boldsymbol{b} \tag{3.26}$$

in which N is a constant matrix populated with zeros and ones. Substituting (3.19), (3.22), (3.25), and (3.26) into $\boldsymbol{p} - C^T\boldsymbol{\lambda}$ yields

$$\begin{aligned} \boldsymbol{p} - C^T\boldsymbol{\lambda} &= \boldsymbol{p} - C^T L(\boldsymbol{x}^{*n+1} + x_0) \\ &= N\boldsymbol{b} - C^T L(R\boldsymbol{b} - RS\boldsymbol{M}^n\boldsymbol{\delta}) \\ &= (N - C^T LR)\boldsymbol{b} + C^T LRS\boldsymbol{M}^n\boldsymbol{\delta}. \end{aligned} \tag{3.27}$$

Equation (3.27) may be premultiplied by a coordinate transformation matrix T to get the element forces in local coordinates [9], i.e.,

$$\boldsymbol{f} = T(\boldsymbol{p} - C^T\boldsymbol{\lambda}) = T(N - C^T LR)\boldsymbol{b} + TC^T LRS\boldsymbol{M}^n\boldsymbol{\delta}. \tag{3.28}$$

In (3.28), the multiple occurrences of the interval load $\boldsymbol{b}$ and interval material parameter $\boldsymbol{\delta}$ have been minimized, and a very sharp results for element force response are obtained.

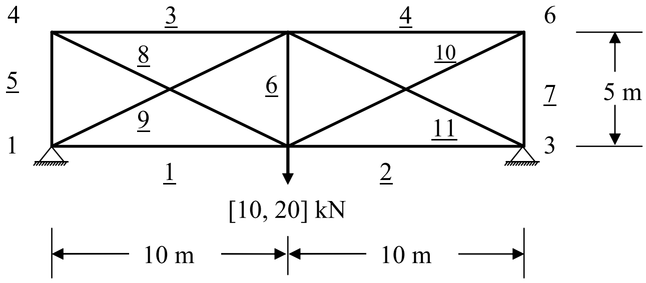

Figure 3. Two-bay truss.

Table 1. Solutions for selected vertical and horizontal displacements of two-bay truss.

Nodal displacements	$\underline{v}_2$(m)	$\overline{v}_2$(m)	$\underline{u}_4$(m)	$\overline{u}_4$(m)	$\underline{v}_4$(m)	$\overline{v}_4$(m)
Combinatorial approach × 10^{-5}	−20.0326	−9.9166	1.9489	4.0041	−0.9984	−0.4886
Present IFEM × 10^{-5}	−20.0690	−9.8296	1.9104	4.0158	−1.0005	−0.4811
Naïve IFEM × 10^{-5}	−21.6608	−8.2378	1.4747	4.4516	−1.1059	−0.3757
Present IFEM error	0.18%	0.87%	1.98%	0.29%	0.21%	1.53%
Naïve IFEM error	8.13%	16.93%	24.33%	11.17%	10.76%	23.10%

Table 2. Solutions for axial forces of two-bay truss [compression(−)].

Axial forces	$\underline{N}_3$(kN)	$\overline{N}_3$(kN)	$\underline{N}_9$(kN)	$\overline{N}_9$(kN)
Combinatorial approach	−7.9496	−3.9270	−13.5797	−6.7364
Present IFEM	−7.9663	−3.8863	−13.5985	−6.6909
Naïve IFEM	−9.2159	−2.6664	−15.1364	−5.2001
Present IFEM error	0.21%	1.04%	0.14%	0.67%
Naïve IFEM error	15.93%	32.10%	11.46%	22.80%

4. Examples

The present formulation for IFEM is illustrated by numerical solutions for three problems with stiffness and load uncertainty.

The first example is a two-bay truss as shown in Figure 3. The truss is subjected to a concentrated load, applied at the middle lower joint. The load is uncertain and described by an interval [10, 20] kN.

Each element has a cross-sectional area $A_i = 0.01$ m^2, and an uncertain modulus of elasticity $\boldsymbol{E}_i = [199, 201]$ GPa, $i = 1, \ldots, 11$. The modulus of elasticity of each element are assumed to be varied independently.

The results for selected displacements and element forces are given in Table 1 and Table 2, respectively. The solutions of IFEM are compared with those of all

Table 3. Interval properties for the members of the two-bay two-floor frame.

Member	Shape	***A*** (cm^2)	***I***(cm^4)	***E***(GPa)
C_1	W12×19	[35.76, 36.12]	[5383.95, 5438.06]	[199, 201]
C_2	W14×132	[249.07, 251.57]	[63364.99, 64001.83]	[199, 201]
C_3	W14×109	[205.42, 207.48]	[51354.63, 51870.76]	[199, 201]
C_4	W10×12	[22.72, 22.95]	[2228.13, 2250.52]	[199, 201]
C_5	W14×109	[205.42, 207.48]	[51354.63, 51870.76]	[199, 201]
C_6	W14×109	[205.42, 207.48]	[51354.63, 51870.76]	[199, 201]
B_1	W27×84	[159.20, 160.80]	[118032.83, 119219.09]	[199, 201]
B_2	W36×135	[254.85, 257.41]	[323037.21, 326283.81]	[199, 201]
B_3	W18×40	[75.75, 76.51]	[25346.00, 25600.73]	[199, 201]
B_4	W27×94	[177.82, 179.60]	[135427.14, 136788.21]	[199, 201]

possible combinations (exact in the case of linear truss systems) and the Näve IFEM ones. In the truss case, the element stiffness matrix is a rank one matrix, and the determinant of a rank one matrix multiplied by a parameter is linear with respect to the parameter. Therefore, the exact bounds on the interval solutions of truss systems are in full agreement with the bounds obtained from solutions to all possible combinations of extreme parameter values.

The present approach captured the bounds of the system response with errors within a range of 0.18% to 1.98%. However, the näve IFEM overestimated the bounds of displacements by a range of 8.13% to 24.33%, and the errors escalated to as big as 32% in the element force calculation.

The second example is a two-bay two-story planar frame shown in Figure 4. The problem was studied in [52]. The example illustrates the handling of the case of stiffness and load uncertainty. The frame is adopted from the work of [7].

In the figure, the column is denoted as "C" and the beam as "B." Subscripts indicate member number. The frame is subjected to uniform loads acting on the member B_1, B_2, B_3, and B_4. The geometric and material properties of each member are summarized in Table 3.

The four uncertain uniform loads are $\mathbf{w}_i$ ($i = 1, 2, 3, 4$) and described by the following interval variables:

$$\begin{aligned} \mathbf{w}_1 &= [105.8,\ 113.1]\ \text{kN/m}, & \mathbf{w}_2 &= [105.8,\ 113.1]\ \text{kN/m}, \\ \mathbf{w}_3 &= [49.255,\ 52.905]\ \text{kN/m}, & \mathbf{w}_4 &= [49.255,\ 52.905]\ \text{kN/m}. \end{aligned} \tag{4.1}$$

All other parameters (the cross-sectional area, moment of inertia, and modulus of elasticity of each member) are considered uncertain as well, and the variations are 1% of their nominal values. The intervals used are summarized in Table 3. In both cases, it is assumed that all interval variables vary independently within their bounds.

There are thirty-four interval variables involved in this case. The combinatorial method requires 2^{34} deterministic FEA, which is computationally infeasible and

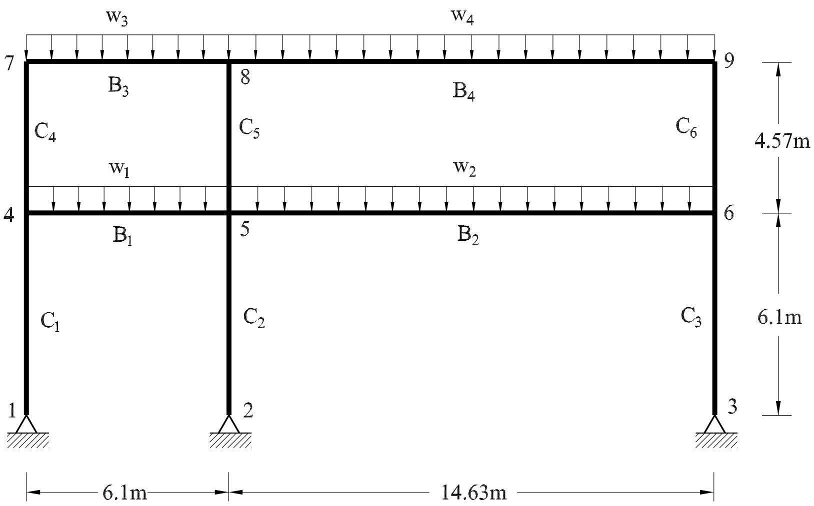

Figure 4. Two-bay two-floor frame.

Table 4. Bounds of selected nodal displacement for the two-bay two-floor frame with stiffness uncertainty and load uncertainty.

	Monte Carlo Sampling		Present IFEA	
Displ.	LB	UB	LB	UB
$\boldsymbol{u}_5$ (cm)	−0.76882	−0.65149	−0.78265	−0.62739
$\boldsymbol{v}_5$ (cm)	−0.24745	−0.22655	−0.24778	−0.22610
$\boldsymbol{\theta}_5$ (rad)	−0.00414	−0.00351	−0.00417	−0.00348
$\boldsymbol{u}_9$ (cm)	−1.52209	−1.34186	−1.56420	−1.29056
$\boldsymbol{v}_9$ (cm)	−0.20962	−0.19119	−0.21009	−0.19064
$\boldsymbol{\theta}_9$ (rad)	0.00519	0.00603	0.00515	0.00607

Table 5. Bounds of selected member nodal forces for the two-bay two-floor frame with stiffness uncertainty and load uncertainty.

		Monte Carlo Sampling		Present IFEA	
Member (node)	Nodal force	LB	UB	LB	UB
B_2	Axial (kN)	218.23	240.98	216.35	242.67
(left node)	Shear (kN)	833.34	892.24	832.96	892.47
	Moment (kN-m)	1842.86	1979.32	1839.01	1982.63
C_5	Axial (kN)	−618.63	−573.34	−619.00	−573.29
(bottom node)	Shear (kN)	−288.69	−261.16	−289.84	−259.59
	Moment (kN-m)	−683.94	−619.79	−688.02	−614.90

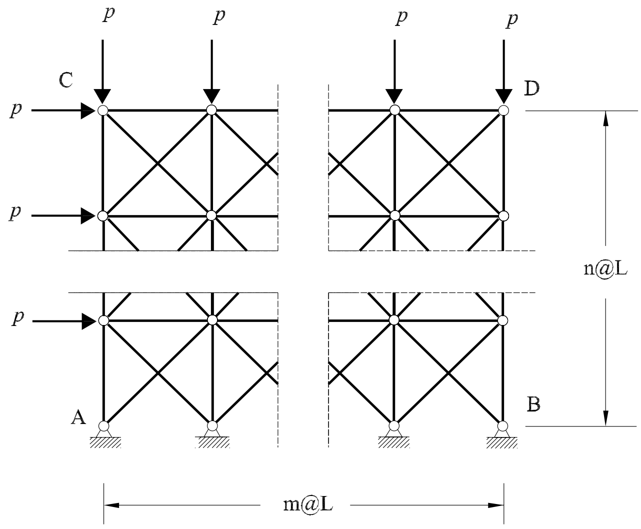

Figure 5. Large scale truss

does not provide the exact solution in this case. Monte Carlo sampling method is used instead to evaluate the quality of the results obtained by the present interval FEA. One million samples are made.

The displacement at node 5 and node 9 and the member nodal force of member B_2 (left node) and C_5 (bottom node) are summarized in Tables 4 and 5, respectively.

As seen from the tables, the solution obtained by the present method tightly encloses the one from Monte Carlo sampling method. This suggests that the overestimation of the bounds obtained by the present interval FEA is small, and tight bounds are obtained.

To investigate the ability of the developed interval FEA to handle problems with a large number of interval variables, its scalability and computational efficiency are analyzed. A series of truss problems with increasing number of members are considered [52]. The trusses are adopted from the work of Pownuk [39], [40]. The configuration of the truss structure is shown in Figure 5, where the truss consists of m bays and n stories.

Concentrated nodal loads are applied in the horizontal direction at the left edge nodes and in the vertical direction at the top edge nodes, as illustrated in Figure 5. The loads are deterministic with value of P for each concentrated nodal load. Each element is assigned two interval variables for its cross-sectional area and modulus of elasticity, respectively. Hence, the total number of interval variables is twice the number of elements in the structure. It is assumed that the midpoint of the cross-sectional area of all elements is A, and the midpoint of the modulus of elasticity of all elements is E. For all interval variables, the introduced uncertainty is 1% of their midpoint values. Therefore, for ith element, the cross-sectional area is

Table 6. Truss structures analyzed.

Truss (story × bay)	Num. elements	Num. interval variables
3 × 10	123	246
4 × 12	196	392
4 × 20	324	648
5 × 22	445	890
5 × 30	605	1210
6 × 30	726	1452
6 × 35	846	1692
6 × 40	966	1932
7 × 40	1127	2254
8 × 40	1288	2576

$\boldsymbol{A}_i = [0.995, 1.005]A$, and the modulus of elasticity is $\boldsymbol{E}_i = [0.995, 1.005]E$. All interval variables are assumed to vary independently within their bounds.

In most prior studies of the FEA dealing with interval variables, the number of interval variables considered is rather small ($\ll 100$). In this example, a total of ten trusses are analyzed with the number of interval variables ranging from 246 to 2576. In this sense, the problems considered here is "large" scale.

Table 6 lists the combinations of story (n) and bay (m) for each truss and the corresponding number of elements and interval variables. The scalability of the present method is examined through the study of how its performance varies with increasing number of interval variables.

Due to the large number of interval variables in this example, the combinatorial method is infeasible, and the naïve interval FEA does not converge. The sensitivity analysis method [39], [40] is used for the evaluation of the present method. The sensitivity analysis method is based on the monotonicity assumption of the response quantities, and it can provide a good inner bound when the parameter uncertainty is small.

The results for a typical displacement, namely the vertical displacement at the right upper corner (node D) of the trusses are summarized in Table 7. The displacement $\boldsymbol{v}_D$ has the form

$$\boldsymbol{v}_D = \boldsymbol{a}\frac{PL}{EA}.$$

Only the dimensionless part $\boldsymbol{a}$ is presented in the table. Table 7 compares the solutions obtained by the present method with those from the sensitivity analysis. The midpoint solution d_0 is also listed, i.e., the deterministic solution obtained when the parameters take their midpoint (nominal) values.

From Table 7, it is observed that the solutions obtained from the present method are slightly wider than those from the sensitive analysis method in all problems. In the three-story ten-bay truss involving 246 interval variables, the relative difference

Table 7. Bounds for vertical displacement at node D of the trusses, with 1% uncertainty in cross-sectional area and modulus of elasticity.

Truss	Midpoint	Sensitivity Anal.		Present IFEA				
story×bay	solution d_0	LB*	UB*	LB	UB	δ_{LB}	δ_{UB}	wid/d_0
3×10	2.5447	2.5143	2.5756	2.5112	2.5782	0.12%	0.10%	2.64%
4×12	3.4193	3.3782	3.4612	3.3723	3.4664	0.18%	0.15%	2.75%
4×20	3.3001	3.2592	3.3418	3.2532	3.3471	0.18%	0.16%	2.84%
5×22	4.1309	4.0791	4.1837	4.0690	4.1928	0.25%	0.22%	3.00%
5×30	4.1005	4.0486	4.1532	4.0386	4.1624	0.25%	0.22%	3.02%
6×30	4.9246	4.8617	4.9886	4.8462	5.0030	0.32%	0.29%	3.18%
6×35	4.9111	4.8482	4.9751	4.8326	4.9895	0.32%	0.29%	3.19%
6×40	4.9054	4.8425	4.9694	4.8270	4.9838	0.32%	0.29%	3.20%
7×40	5.7201	5.6461	5.7954	5.6236	5.8166	0.40%	0.37%	3.37%
8×40	6.5422	6.4570	6.6289	6.4259	6.6586	0.48%	0.45%	3.56%

$\delta_{LB} = |LB - LB^*| / LB^*$. $\delta_{UB} = |UB - UB^*| / UB^*$. $wid / d_0 = (UB - LB) / d_0$.

between these two solutions is 0.12% and 0.1% for the lower bound and upper bound, respectively. As the problem scale increases to eight-story forty-bay with 2576 interval variables, the relative difference between these two solutions is 0.48% and 0.45% for the lower bound and upper bound, respectively. This comparison indicates that the present method yields sharp results for large scale problems, and the accuracy remains at the same level with the increase of problem size.

Additional useful information listed in Table 7 is the ratio of the present IFEA solution to the midpoint solution. This ratio gives an estimation of the uncertainty in the response resulting from the uncertainty in the parameter. The results show reasonable values of the displacement variations in all problems, ranging from 2.64% to 3.56%.

Next, we investigate the computational efficiency of the present interval FEA. Table 8 summarizes the problem scale, iteration number, and total computational time of each problem solved by the present method. The table also contains iteration time and matrix inversion time, as well as their ratios to the total computational time. The reported time is the CPU time.

The computations were carried out on a PC with Intel Pentium4 2.4 GHz CPU with 1GB RAM under Windows XP. According to the data in Table 8, the total computational time, the iteration time and the matrix inversion time are plotted as functions of the number of interval variables, shown in Figure 6.

Table 8 suggests that the number of iterations needed to achieve convergence are comparable for all the problems. As compared to the matrix inversion, the iterations take much less time, ranging from 8.7% to 19.5% of total computational time.

The results in Table 8 also indicate that the percentage of the CPU time spent calculating the matrix inverse ranges from 78.4% to 90.7%. For most problems, the percentage is around 90%.

Table 8. CPU time for the truss analyses with the present interval FEA (unit: seconds).

Truss (story × bay)	n_v	Iterations	t_i	t_r	t	t_i / t	t_r / t
3 × 10	246	4	0.14	0.56	0.72	19.5%	78.4%
4 × 12	392	5	0.45	2.06	2.56	17.7%	80.5%
4 × 20	648	5	1.27	8.80	10.17	12.4%	86.5%
5 × 22	890	5	2.66	21.48	24.38	10.9%	88.1%
5 × 30	1210	6	6.09	53.17	59.70	10.2%	89.1%
6 × 30	1452	6	11.08	89.06	100.77	11.0%	88.4%
6 × 35	1692	6	15.11	140.23	156.27	9.7%	89.7%
6 × 40	1932	6	20.11	208.64	230.05	8.7%	90.7%
7 × 40	2254	6	32.53	323.14	358.76	9.1%	90.1%
8 × 40	2576	7	48.454	475.72	528.45	9.2%	90.0%

n_v: Number of interval variables. t_i: Total CPU time for iterations.
t_r: CPU time for matrix inverse calculation. t: Total computational CPU time.

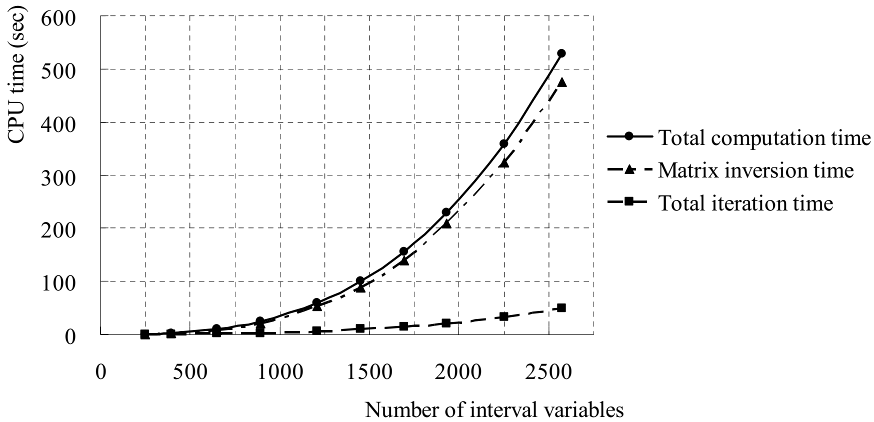

Figure 6. Computation time vs problem scale.

It is observed that the computational time increases approximately cubically with the number of interval variables for the present interval FEA. A good fit to the data in Table 8 can be found with

$$t = 8.5 \times 10^{-8} n_v^{2.8707},$$

where t is the total computational CPU time, and n_v is the number of interval variables. Hence, the present interval FEA has an approximately cubic computational complexity.

5. Conclusion

In this paper a new formulation for interval FEM is presented. Uncertain loads and stiffness are introduced as interval numbers. The major difficulty associated with the IFEM is the overestimation due to the dependency effect: the computed range of the response is much wider than the actual range. For engineering application, the physical nature of the problem must be considered to control the overestimation. In the present approach, an element-by-element technique is used and compatibility conditions are ensured by the Lagrange multiplier method. The resulting linear interval equation is solved using the Brouwer's fixed point theory with Krawczyk's operator and a newly developed overestimation control. We eliminate most sources of overestimation, and a very sharp enclosure for the system's displacement and forces are obtained simultaneously and have the same level of accuracy. The numerical examples also illustrated the present formulation's scalability.

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