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The Set of Hausdorff Continuous Functions—The Largest Linear Space of Interval Functions

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Abstract. Hausdorff continuous (H-continuous) functions are special interval-valued functions which are commonly used in practice, e.g. histograms are such functions. However, in order to avoid arithmetic operations with intervals, such functions are traditionally treated by means of corresponding semi-continuous functions, which are real-valued functions. One difficulty in using H-continuous functions is that, if we add two H-continuous functions that have interval values at same argument using point-wise interval arithmetic, then we may obtain as a result an interval function which is not H-continuous. In this work we define addition so that the set of H-continuous functions is closed under this operation. Moreover, the set of H-continuous functions is turned into a linear space. It has been also proved that this space is the largest linear space of interval functions. These results make H-continuous functions an attractive tool in real analysis and provides a bridge between real and interval analysis.

1. Introduction

Interval-valued functions of real arguments (briefly: interval functions) are special case of set-valued (multi-valued) functions that are extensively used in rapidly developed mathematical fields like non-smooth and nonlinear analysis [7], [9], [10], [15], [18], differential inclusions [8], [14], [19], [22], convex analysis [15], [25], optimization and (optimal) control theory [15], [22], and other applied areas. These novel mathematical tools are typically based on the classical concept of semi-continuous functions, which are real-valued functions, used to model non-smooth and discontinuous functions considered as boundaries of set-valued mappings. However, in many situations it is more convenient to use simple interval functions.

Three concepts of continuity of interval functions play important roles in relation to the above mentioned theories. In this paper we give a definition to these concepts, briefly referring to them resp. as S-continuity, D-continuity, and H-continuity. We show that these concepts of continuity can be obtained as a natural hybrid between the concept of interval function and the well-known concept of *semi-continuity* of real functions. The lower and upper semi-continuous functions have been well-known at least since the beginning of the 20th century and are credited to Baire [11]. The *normal* upper semi-continuous functions were introduced in 1950 by Dilworth in connection with the order completion of the lattice of continuous functions, see [17]. The concepts of S-continuity and H-continuity are due to Sendov, see [26], and have been motivated by applications to the theory of Hausdorff approximations [27]. These concepts were also studied in [3] in connection with the analysis of interval-valued functions. Pairing a lower semi-continuous function $\underline{f}$ with an upper semi-continuous function $\overline{f}$ in an interval function $[\underline{f}, \overline{f}]$ produces a completely new concept from both algebraic and topological points of view, namely, the concept of S-continuous interval functions. It is shown in [27] that the set of all S-continuous functions on a compact subset of $\mathbb{R}$ is a complete metric space with respect to the Hausdorff distance between their graphs and has the rare and particularly useful property of being completely bounded.

Similarly to the concept of S-continuity, in the recent paper [1] interval functions given by pairs $[\underline{f}, \overline{f}]$ consisting of a normal lower semi-continuous function $\underline{f}$ and a normal upper semi-continuous function $\overline{f}$, such that $\underline{f} \leq \overline{f}$, are introduced and studied. Although Dilworth has not considered such functions as pairs (but only individually), in honor of his contribution in this direction, we call these interval functions Dilworth continuous, or shortly D-continuous.

In many situations, it is more natural to make use of simple interval functions as a basic tool rather than real-valued functions. The recent publications [2] and [5] present such situations. Moreover, it is shown in [4] that the traditional application of real-valued semi-continuous functions to viscosity solutions of partial differential equations can be replaced by H-continuous interval-valued functions, which offer an instructive geometrical meaning to these solutions. In this work we further show the advantages of classes of interval functions, namely, the classes of H-continuous and D-continuous interval functions, with regard to their algebraic and inclusion isotone properties when compared with the real-valued semi-continuous functions. Roughly speaking, classical real analysis becomes more simple and complete through the introduction of H-continuous and D-continuous interval functions.

H-continuous functions can be characterized by the property that they are approximated arbitrary well from above and below by means of uniformly continuous functions [21]. Such a two-sided approximation generates in fact a continuous (in the usual sense) interval function. Due to the minimality condition they satisfy, the H-continuous functions are suitable for constructing interval enclosures of real functions only in particular circumstances. However, we show that the D-continuous functions, which can be obtained as pairs of H-continuous functions, provide interval enclosures to bounded sets of continuous real functions. Furthermore, the algebraic operations in the set of D-continuous functions are inclusion isotone, a property of considerable importance in relation to interval computations. This is one more motivation for these functions to be of special interest for interval analysis.

H-continuous functions [27] may possess interval values for certain arguments and comprise interval functions commonly used in the engineering practice like Heaviside functions (step functions) and histograms. How do we add two H-continuous functions? One possibility is to add them point-wise using the interval arithmetic addition. However, if the functions have interval values for the same value of the the argument(s), then we may obtain as result an interval function, which is not H-continuous. In addition, the value of the resulting function at some points may have nothing in common with its values in neighboring points, violating thus the main idea of continuity.

How should we define addition so that the set of H-continuous functions is closed under this operation? Can we define addition, and also multiplication by scalars, in such a way that the set of H-continuous functions is a linear space? Is this space the largest space of interval functions which is linear? These are some of the questions considered in the present paper.

The traditional definition of the operations between real functions is point-wise on the domain of the functions. However, we show that this approach does not help for keeping the linear structure of interval functions. In this paper we extend the operations of the linear space of real-valued continuous functions to the largest possible set of interval functions in a special way (not point-wise) so that the linear space structure is preserved.

Hausdorff continuous functions are special class of Dilworth continuous interval functions satisfying a minimality condition with respect to the inclusion of graphs. These two concepts generalize the concept of continuity of real functions. It is important to note that both concepts retain the property that a continuous function is completely determined by its values on a dense subset of its domain. The definitions of these concepts and related terminology are given in Section 2 and Section 3. In Section 4 the operations of a linear space of functions are extended to Hausdorff continuous functions. It is shown that, under suitable natural assumptions, the set of Hausdorff continuous functions is the largest set of Dilworth continuous interval functions which is a linear space. It is essential that the proposed addition of Hausdorff continuous functions is such that at a point where both summands have interval values the value of the sum is not determined by the values of the summands only at that point but rather by the values of the summands in a neighborhood of the point. In this sense our addition of interval functions is not point-wise. Some of the results in Section 4 such as Theorems 4.2 and 4.3 are reported in [28] without proofs. A full account of these results is given in Section 4. In Section 5 both operations are extended further to the set of all D-continuous interval functions introducing on it the structure of a quasi-linear space. A natural relationship between the addition of continuous functions and the addition of the interval hulls of sets of continuous functions considered as D-continuous interval functions is also established in this section.

2. General Setting

2.1. The Graph Completion Operator

The real line is denoted by $\mathbb{R}$ and the set of all finite real intervals $[\underline{a}, \overline{a}] = \{x : \underline{a} \leq x \leq \overline{a}\}$ by $\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \leq \overline{a}\}$. Given an interval $a = [\underline{a}, \overline{a}] \in \mathbb{IR}$, $w(a) = \overline{a} - \underline{a}$ is the width of a, while $|a| = \max\{|\underline{a}|, |\overline{a}|\}$ is the modulus of a. An interval a is called proper interval, if $w(a) > 0$ and point interval, if $w(a) = 0$. Identifying $a \in \mathbb{R}$ with the point interval $[a, a] \in \mathbb{IR}$, we consider $\mathbb{R}$ as a subset of $\mathbb{IR}$. We denote by $\mathbb{A}(\Omega)$ the set of all locally bounded interval valued functions defined on the open set $\Omega \subset \mathbb{R}^n$, that is,

$$\mathbb{A}(\Omega) = \{f : \Omega \to \mathbb{IR},\ f\text{-locally bounded}\}.$$

Since $\mathbb{R} \subseteq \mathbb{IR}$ the set $\mathbb{A}(\Omega)$ contains the set

$$\mathcal{A}(\Omega) = \{f : \Omega \to \mathbb{R},\ f\text{-locally bounded}\}$$

of all locally bounded real functions defined on Ω. Let us recall that a real function or an interval-valued function f defined on Ω is called locally bounded if for every $x \in \Omega$ there exist $\delta > 0$ and $M \in \mathbb{R}$ such that

$$|f(y)| < M, \qquad y \in B_\delta(x),$$

where $B_\delta(x)$ denotes the open δ-neighborhood of x in Ω, that is,

$$B_\delta(x) = \{y \in \Omega : ||x - y|| < \delta\}.$$

Let D be a dense subset of Ω. The mappings $I(D, \Omega, \cdot)$, $S(D, \Omega, \cdot) : \mathbb{A}(D) \to \mathcal{A}(\Omega)$ defined for $f \in \mathbb{A}(D)$ and $x \in \Omega$ by

$$\begin{aligned} I(D, \Omega, f)(x) &= \sup_{\delta > 0} \inf\{f(y) : y \in B_\delta(x) \cap D\}, \\ S(D, \Omega, f)(x) &= \inf_{\delta > 0} \sup\{f(y) : y \in B_\delta(x) \cap D\}, \end{aligned}$$

are called lower and upper Baire operators, respectively [11]. The mapping $F : \mathbb{A}(D) \to \mathbb{A}(\Omega)$, called a *graph completion operator*, where

$$F(D, \Omega, f)(x) = [I(D, \Omega, f)(x), S(D, \Omega, f)(x)], \qquad x \in \Omega,\ \ f \in \mathbb{A}(D),$$

is well defined. In the case when $D = \Omega$ the sets D and Ω will usually be omitted from the arguments, that is, we write

$$I(f) = I(\Omega, \Omega, f), \qquad S(f) = S(\Omega, \Omega, f), \qquad F(f) = F(\Omega, \Omega, f).$$

The name of the operator F is derived from the fact that, considering the graphs of f and $F(f)$ as subsets of the topological space $\Omega \times \mathbb{R}$, the graph of $F(f)$ is the minimal closed set, which is a graph of interval-valued function on Ω and contains the graph of f.

Let $f \in \mathbb{A}(\Omega)$. For every $x \in \Omega$ the value of f is an interval $[\underline{f}(x), \overline{f}(x)] \in \mathbb{IR}$. Hence, the function f can be written in the form $f = [\underline{f}, \overline{f}]$ where $\underline{f}, \overline{f} \in \mathcal{A}(\Omega)$ and $\underline{f}(x) \leq \overline{f}(x)$, $x \in \Omega$. The lower and upper Baire operators as well as the graph completion operator of an interval valued function f can be conveniently represented in terms of the functions $\underline{f}$ and $\overline{f}$, namely, for every dense subset D of Ω:

$$\begin{aligned} I(D,\Omega,f) &= I(D,\Omega,\underline{f}), \\ S(D,\Omega,f) &= S(D,\Omega,\overline{f}), \\ F(D,\Omega,f) &= [I(D,\Omega,\underline{f}), S(D,\Omega,\overline{f})]. \end{aligned}$$

2.2. Isotonicity Issues

The graph completion operator is inclusion isotone with respect to the functional argument, that is, if $f, g \in \mathbb{A}(D)$, where D is dense in Ω, then

$$f(x) \subseteq g(x),\ \ x \in D \quad \Longrightarrow \quad F(D,\Omega,f)(x) \subseteq F(D,\Omega,g)(x),\ \ x \in \Omega. \tag{2.1}$$

Furthermore, the graph completion operator is inclusion isotone with respect to the set D in the sense that if D_1 and D_2 are dense subsets of Ω and $f \in \mathbb{A}(D_1 \cup D_2)$ then

$$D_1 \subseteq D_2 \quad \Longrightarrow \quad F(D_1,\Omega,f)(x) \subseteq F(D_2,\Omega,f)(x),\ \ x \in \Omega.$$

This, in particular, means that for any dense subset D of Ω and $f \in \mathbb{A}(\Omega)$ we have

$$F(D,\Omega,f)(x) \subseteq F(f)(x), \quad x \in \Omega. \tag{2.2}$$

A partial order which extends the total order on $\mathbb{R}$ can be defined on $\mathbb{IR}$ in more than one way. However, it was shown [2], [5], [13], [23], that it is particularly useful to consider on $\mathbb{IR}$ the partial order $\leq$ defined by

$$[\underline{a}, \overline{a}] \leq [\underline{b}, \overline{b}] \quad \Longleftrightarrow \quad \underline{a} \leq \underline{b},\ \ \overline{a} \leq \overline{b}. \tag{2.3}$$

The partial order induced in $\mathbb{A}(\Omega)$ by (2.3) in a point-wise way, i.e.,

$$f \leq g \quad \Longleftrightarrow \quad f(x) \leq g(x),\ \ x \in \Omega, \tag{2.4}$$

is an extension of the usual point-wise order in the set of extended real valued functions $\mathcal{A}(\Omega)$.

For the results discussed here it is important that the operators I, S and F are all isotone with respect to the partial order (2.4), that is, for any dense subset D of Ω and any two functions $f, g \in \mathbb{A}(D)$

$$f(x) \leq g(x),\ x \in D \Longrightarrow \begin{cases} I(D,\Omega,f)(x) \leq I(D,\Omega,g)(x), & x \in \Omega, \\ S(D,\Omega,f)(x) \leq S(D,\Omega,g)(x), & x \in \Omega, \\ F(D,\Omega,f)(x) \leq F(D,\Omega,g)(x), & x \in \Omega. \end{cases} \tag{2.5}$$

3. Continuity Concepts

3.1. Definitions

DEFINITION 3.1. A function $f \in \mathbb{A}(\Omega)$ is called S-continuous, if $F(f) = f$.

DEFINITION 3.2. A function $f \in \mathbb{A}(\Omega)$ is called Dilworth continuous or shortly D-continuous if for every dense subset D of Ω we have $F(D,\Omega,f) = f$.

DEFINITION 3.3. A function $f \in \mathbb{A}(\Omega)$ is called Hausdorff continuous, or H-continuous, if for every function $g \in \mathbb{A}(\Omega)$ which satisfies the inclusion $g(x) \subseteq f(x)$, $x \in \Omega$, we have $F(g)(x) = f(x)$, $x \in \Omega$.

The three concepts defined above can be considered as generalizations of the concept of continuity of real functions with H-continuity being the strongest and S-continuity the weakest [1]:

$$\text{H-continuous} \implies \text{D-continuous} \implies \text{S-continuous}. \tag{3.1}$$

3.2. Characterizations

With every interval function f one can associate S-continuous, D-continuous, and H-continuous functions as stated in the next theorem, which combines results from [1] and [27].

THEOREM 3.1. *Let* $f \in \mathbb{A}(\Omega)$. *Then*

(i) for every dense subset D *of* Ω *the function* $F(D,\Omega,f)$ *is S-continuous;*

(ii) the function $G(f) = [I(S(I(f))), S(I(S(f)))]$ *is D-continuous;*

(iii) both functions $F(S(I(f)))$ *and* $F(I(S(f)))$ *are H-continuous.*

This theorem is illustrated by the following example.

EXAMPLE 3.1. Consider the function $f \in \mathbb{A}(\mathbb{R})$ given by

$$f(x) = \begin{cases} [-1,1], & \text{if } x \in \mathbb{Z}, \\ 0, & \text{if } x \in (-\infty,0) \setminus \mathbb{Z}, \\ [0,1], & \text{if } x \in (0,\infty) \setminus \mathbb{Z}, \end{cases}$$

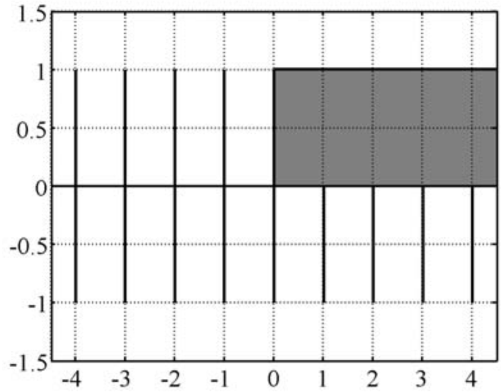

Figure 1. The function f.

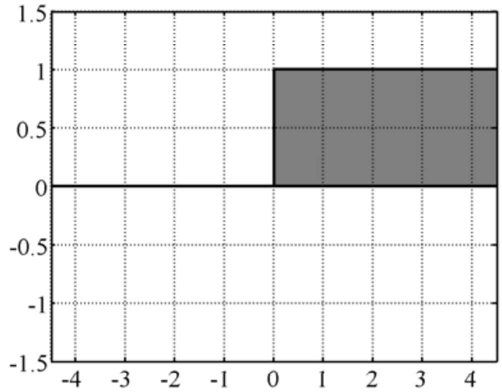

Figure 2. The D-continuous function $G(f)$.

where $\mathbb{Z}$ denotes the set of integers, see Figure 1.

We have $F(f) = f$ meaning that f is S-continuous.

The D-continuous function $G(f)$ is given by, see Figure 2:

$$G(f)(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0), \\ [0,1], & \text{if } x \in [0, \infty). \end{cases}$$

Finally we have the H-continuous functions, Figure 3:

$$F\Big(S\big(I(f)\big)\Big)(x) = 0, \qquad x \in \mathbb{R},$$

$$F\Big(I\big(S(f)\big)\Big)(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0), \\ [0,1], & \text{if } x = 0, \\ 1, & \text{if } x \in (0, \infty). \end{cases}$$

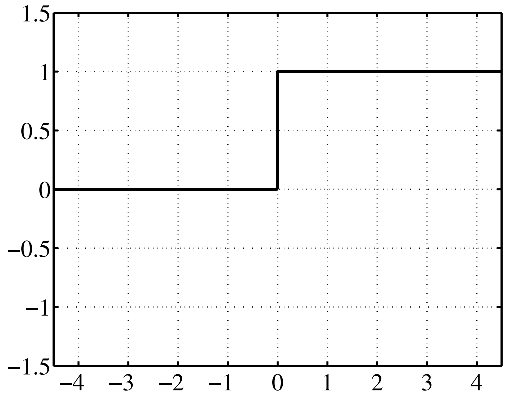

Figure 3. The H-continuous function $F(I(S(f)))$.

The concepts of continuity given in Definitions 3.1, 3.2, and 3.3 are strongly connected to the concepts of semi-continuity of real functions. We have the following characterization of the fixed points of the lower and upper Baire operators [11]:

$$\begin{aligned} I(f) &= f \iff f \text{ is lower semi-continuous on } \Omega, \\ S(f) &= f \iff f \text{ is upper semi-continuous on } \Omega. \end{aligned} \tag{3.2}$$

Hence for an interval function $f = [\underline{f}, \overline{f}] \in \mathcal{A}(\Omega)$

$$f \text{ is S-continuous} \iff \begin{cases} \overline{f} \text{ is upper semi-continuous,} \\ \underline{f} \text{ is lower semi-continuous.} \end{cases} \tag{3.3}$$

The D-continuous interval functions admit a similar characterization through the normal upper semi-continuous and normal lower semi-continuous functions, see [17]. We have

$$f \text{ is D-continuous} \iff \begin{cases} \overline{f} \text{ is normal upper semi-continuous,} \\ \underline{f} \text{ is normal lower semi-continuous.} \end{cases} \tag{3.4}$$

Let us recall that a function φ is called normal upper semi-continuous, see [17], if $S(I(\varphi)) = \varphi$ and it is called normal lower semi-continuous if $I(S(\varphi)) = \varphi$. In view of Theorem 3.1 we also have the following characterization:

- If φ is upper semi-continuous then

$$\varphi \text{ is normal upper semi-continuous } \iff F(\varphi) \text{ is H-continuous.} \tag{3.5}$$

- If φ is lower semi-continuous then

$$\varphi \text{ is normal lower semi-continuous } \iff F(\varphi) \text{ is H-continuous.} \tag{3.6}$$

The minimality condition used in the definition of Hausdorff continuous functions can also be formulated in terms of semi-continuous functions, namely, if $f = [\underline{f}, \overline{f}]$ is S-continuous, then f is H-continuous if and only if

$$\{\phi \in \mathcal{A}(\Omega) : \phi \text{ is semi-continuous}, \underline{f} \leq \phi \leq \overline{f}\} = \{\underline{f}, \overline{f}\}.$$

3.3. Continuity and H-continuity

The H-continuous functions, representing the strongest type of continuity among the three types considered above, are also similar to the usual continuous real functions in that they assume point values on a dense subset of the domain Ω. This is obtained from a Baire category argument. It was shown in [2] that for every $f \in \mathbb{H}(\Omega)$ the set

$$W_f = \{x \in \Omega : w(f(x)) > 0\} \tag{3.7}$$

is of first Baire category. Since $\Omega \subseteq \mathbb{R}^n$ is open this implies that for every $f \in \mathbb{H}(\Omega)$ the set

$$D_f = \{x \in \Omega : w(f(x)) = 0\} = \Omega \setminus W_f \tag{3.8}$$

is dense in Ω. Since a finite or countable union of sets of first Baire category is also a set of first Baire category we have that for every finite or countable set $\mathcal{F}$ of Hausdorff continuous functions the set

$$D_{\mathcal{F}} = \{x \in \Omega : w(f(x)) = 0,\ f \in \mathcal{F}\} = \Omega \setminus \bigcup_{f \in \mathcal{F}} W_f \tag{3.9}$$

is dense in Ω.

The property that (3.8) is dense in Ω can also be used to characterize H-continuous functions as follows.

THEOREM 3.2. *If the interval function f is D-continuous and assumes point values on a dense subset D of Ω, that is, $w(f(x)) = 0$, $x \in D$, then f is H-continuous.*

Proof. Let $g \in \mathbb{A}(\Omega)$ satisfy the inclusion $g(x) \subseteq f(x)$, $x \in \Omega$. Obviously then $g(x) = f(x)$, $x \in D$. Using this identity together with the inclusion isotonicity of the operator F given in (2.1), (2.2), and the Definition 3.2 we have for $x \in \Omega$

$$F(g)(x) \subseteq F(f)(x) = f(x) = F(D, \Omega, f)(x) = F(D, \Omega, g)(x) \subseteq F(g)(x).$$

Therefore $F(g) = f$ which implies that f is H-continuous, see Definition 3.3. □

In the sequel we shall use the following notations:

- $\mathbb{F}(\Omega)$—the set of all S-continuous functions defined on Ω;
- $\mathbb{G}(\Omega)$—the set of all finite D-continuous functions defined on Ω;

- $\mathbb{H}(\Omega)$—the set of all finite H-continuous functions defined on Ω.

The implications (3.1) indicate the inclusions $\mathbb{H}(\Omega) \subseteq \mathbb{G}(\Omega) \subseteq \mathbb{F}(\Omega)$.

3.4. The Set of Hausdorff Continuous Functions in Real Analysis and the Analysis of Differential Equations

Although not well known enough, in many respects the set of Hausdorff continuous functions $\mathbb{H}(\Omega)$ is a natural and particularly useful extension of the set of continuous functions $C(\Omega)$. For example, $\mathbb{H}(\Omega)$ is the Dedekind order completion of $C(\Omega)$ with respect to the point-wise defined partial order, see [2]. This result solves a long time open problem which asked what is the Dedekind order completion of $C(\Omega)$. The best previous result was obtained by Dilworth back in 1950, [17], and fell far short of the complete answer. A further topological connection between $C(\Omega)$ and $\mathbb{H}(\Omega)$ is established in [6]. It is shown that if the vector space $C(\Omega)$ is equipped with the order convergence structure using the usual point-wise defined order, its completion as a convergence vector space is $\mathbb{H}(\Omega)$. Even though, the theory of Hausdorff continuous functions is new and presently still under development, their applications show that they do play an important role in Real Analysis. In particular one may note that one of the main motivations behind the development of the various spaces in Real Analysis as well as in Functional Analysis is the partial differential equations with the need to assimilate the various types of "weak" solutions. A very general result concerning the application of Hausdorff continuous functions to the analysis of PDEs is given in [5]. It is shown that the solutions of large classes of systems of nonlinear partial differential equations can be assimilated with Hausdorff continuous functions. This class is of the general form

$$g\big(x, u(x), \ldots, D_x^p u(x), \ldots\big) = f(x), \qquad x \in \Omega, \;\; p \in \mathbb{N}^n, \;\; |p| \leq m,$$

where g is only supposed to be jointly continuous in all its arguments and $f \in \mathbb{H}(\Omega)$. Hence the set $\mathbb{H}(\Omega)$ might be a viable alternative to some of the presently used functional spaces (e.g. $L^p(\Omega)$, Sobolev spaces) with the advantage of being both more regular and universal, as well as significantly simpler.

Hausdorff continuous functions are applicable in more particular cases as well. As an illustration we will consider in detail the viscosity solutions of the Hamilton-Jacobi equation

$$\Phi\big(x, u(x), \nabla u(x)\big) = 0, \qquad x \in \Omega, \tag{3.10}$$

where $u : \Omega \to \mathbb{R}$ is the unknown function, ∇u is the gradient of u and the given function $\Phi : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is jointly continuous in all its arguments. The theory of viscosity solutions rests on two fundamental concepts, namely, of subsolution and of supersolution. Let $USC(\Omega)$ denote the set of all real upper semi-continuous functions on Ω and let $LSC(\Omega)$ denote the set of all real lower semi-continuous functions on Ω.

DEFINITION 3.4. A function $u \in USC(\Omega)$ is called a viscosity subsolution of the equation (3.10) if for any $\varphi \in C^1(\Omega)$ we have $\Phi(x_0, u(x_0), \nabla\varphi(x_0)) \leq 0$ at any local maximum point x_0 of $u-\varphi$. Similarly, $u \in LSC(\Omega)$ is called a viscosity supersolution of the equation (3.10) if for any $\varphi \in C^1(\Omega)$ we have $\Phi(x_0, u(x_0), \nabla\varphi(x_0)) \geq 0$ at any local minimum point x_0 of $u - \varphi$.

Naturally, a solution should be required somehow to incorporate the properties of both a subsolution and a supersolution. In the classical viscosity solutions theory, see [16], a viscosity solution is a function u which is both a subsolution and a supersolution. Since $USC(\Omega) \cap LSC(\Omega) = C(\Omega)$, this clearly implies that the viscosity solutions defined in this way are all continuous functions.

The concept of viscosity solution for functions u which are not necessarily continuous is introduced by using the lower and upper semi-continuous envelopes $I(u)$ and $S(u)$, see [20].

DEFINITION 3.5. A function $u : \Omega \to \mathbb{R}$ is called a viscosity solution of (3.10) if $S(u)$ is a viscosity subsolution of (3.10) and $I(u)$ is a viscosity supersolution of (3.10).

The first important point to note about the interval approach is as follows. Interval valued functions appear naturally in the context of noncontinuous viscosity solutions. Namely, they appear as *graph completions*. Indeed, the above definition places requirements not on the function u itself but on its lower and upper semi-continuous envelopes or, in other words, on its graph completion $F(u)(x) = [I(u)(x), S(u)(x)]$, $x \in \Omega$. Clearly, Definition 3.5 treats functions which have the same upper and lower semi-continuous envelopes, that is, have the same graph completion, as identical functions. On the other hand, since different functions can have the same graph completion, a function can not in general be identified from its graph completion, that is, functions with the same graph completion are indistinguishable. Therefore, no generality will be lost if only interval valued functions representing graph completions are considered. We define the concept of viscosity solution for the interval valued functions in $\mathbb{F}(\Omega)$.

DEFINITION 3.6. A function $u = [\underline{u}, \overline{u}] \in \mathbb{F}(\Omega)$ is called a viscosity solution of (3.10) if $\underline{u}$ is a supersolution of (3.10) and $\overline{u}$ is a subsolution of (3.10).

A second advantage of the interval approach is as follows. A function $u \in \mathcal{A}(\Omega)$ is a viscosity solution of (3.10) in the sense of Definition 3.5 if and only if the interval valued function $F(u)$ is a viscosity solution of (3.10) in the sense of Definition 3.6. In this way the level of the regularity of a solution u is manifested through the width of the interval valued function $F(u)$. The distance between $I(u)$ and $S(u)$ is an essential measure of the regularity of any solution u, irrespective of whether it is given as a point valued function or as an interval valued function. If no restriction is placed on the distance between $I(u)$ and $S(u)$ we will have some quite meaningless solutions as shown in [12]. On the other hand, a strong restriction like $I(u) = S(u)$ gives only solutions which are continuous. This issue was addressed in [4] where the Hausdorff continuous viscosity solutions are introduced. If a Hausdorff continuous function u is a viscosity solution according to Definition 3.6 then, in terms of the Hausdorff distance (the distance between the closures of the graphs of the functions, [27]), the functions $I(u)$ and $S(u)$ are as close as they can be, namely, the Hausdorff distance is zero. Traditionally, discontinuous viscosity solutions are assimilated through the concept of envelope viscosity solution, see [12]. It is shown in [4] that all envelope viscosity solutions can be associated with Hausdorff continuous viscosity solutions through the graph completion operator. Furthermore, it is demonstrated that the main ideas within the classical theory of continuous viscosity solutions can be extended to the wider set of Hausdorff continuous functions.

3.5. The Interval Operations

We recall the interval-arithmetic operations for addition and multiplication by scalars. For $\gamma \in \mathbb{R}$, $[\underline{a}, \overline{a}], [\underline{b}, \overline{b}] \in \mathbb{IR}$ we have

$$[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b},\ \overline{a} + \overline{b}], \tag{3.11}$$

$$\gamma * [\underline{a}, \overline{a}] = [\min\{\gamma * \underline{a},\ \gamma * \overline{a}\},\ \max\{\gamma * \underline{a},\ \gamma * \overline{a}\}]. \tag{3.12}$$

One can define corresponding operations for S-continuous functions point-wise using the interval arithmetic operations (3.11), (3.12).

We use the notation $f + g$ for the point-wise sum of the functions $f = [\underline{f}, \overline{f}]$ and $g = [\underline{g}, \overline{g}]$ given by

$$(f + g)(x) = [\underline{f}(x) + \underline{g}(x),\ \overline{f}(x) + \overline{g}(x)], \qquad x \in \Omega. \tag{3.13}$$

For the scalar multiplication we shall use the interval arithmetic operation (3.12) point-wise. For every $f = [\underline{f}, \overline{f}] \in \mathbb{F}(\Omega)$ and $x \in \Omega$ we have

$$\begin{aligned}(\alpha * f)(x) &= \alpha * \big(f(x)\big) \\ &= [\min\{\alpha * \underline{f}(x),\ \alpha * \overline{f}(x)\},\ \max\{\alpha * \underline{f}(x),\ \alpha * \overline{f}(x)\}].\end{aligned} \tag{3.14}$$

We conclude this section with an example showing that the point-wise definition of addition for interval functions using the familiar addition of intervals does not preserve the continuity properties of the summands.

EXAMPLE 3.2. Consider the functions $f, g \in \mathbb{H}(\mathbb{R})$ given by

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ [0, 1], & \text{if } x = 0, \\ 1, & \text{if } x > 0; \end{cases}$$

$$g(x) = \begin{cases} 0, & \text{if } x < 0, \\ [-1, 0], & \text{if } x = 0, \\ -1, & \text{if } x > 0. \end{cases}$$

Using (3.11) we have

$$f(x)+g(x)=\begin{cases}0, & \text{if } x<0,\\ [-1,1], & \text{if } x=0,\\ 0, & \text{if } x>0.\end{cases}$$

One should expect that the sum $f(x)+g(x)$ is zero for all x since the two functions closely resemble continuous functions and from (3.12) we have

$$g(x)=(-1)*f(x), \qquad x\in\mathbb{R}.$$

However, we observe that the summands are H-continuous functions while the sum is not, that is, the set $\mathbb{H}(\Omega)$ is not closed under the operation addition (3.13) based on point-wise interval arithmetic.

4. The Set of Hausdorff Continuous Functions as a Linear Space

4.1. Linear Operations: Definition

We shall extend the arithmetic operations of real continuous functions to D-continuous interval functions in such a way that the subset $\mathcal{P}$ of $\mathbb{G}(\Omega)$ on which the linear space structure is preserved is as large as possible. For the scalar multiplication we shall use the point-wise interval arithmetic operation (3.14). (At the end of this section, see Example 4.2 we shall clarify why we consider D-contionuous functions and not S-continuous ones.) It should be noted that this operation preserves the continuity properties (in terms of Definitions 3.1, 3.2, and 3.3) of the interval functions.

With regard to addition, we define the sum of $f,g\in\mathbb{G}(\Omega)$ in a point-wise way at all points of the domain Ω at which the value of at least one of the operands is a point interval. At the remaining points of the domain (where both operands are intervals with nonzero width) the value of the sum will not be defined in a point-wise way and remains undetermined for the moment. In order to avoid possible confusion with the point-wise sum (3.13) we shall denote this new addition in $\mathcal{P}$ by "⊕". Symbolically, for $f=[\underline{f},\overline{f}]\in\mathbb{G}(\Omega)$ and $g=[\underline{g},\overline{g}]\in\mathbb{G}(\Omega)$ we have

$$(f\oplus g)(x)=\begin{cases}[f(x)+\underline{g}(x),\ f(x)+\overline{g}(x)], & \text{if } w\big(f(x)\big)=0,\\ [\underline{f}(x)+g(x),\ \overline{f}(x)+g(x)], & \text{if } w\big(g(x)\big)=0.\end{cases} \tag{4.1}$$

Naturally,

$$(f\oplus g)(x)=f(x)+g(x), \qquad x\in D_{fg}, \tag{4.2}$$

where

$$D_{fg}=\{x\in\Omega : w\big(f(x)\big)=w\big(g(x)\big)=0\}. \tag{4.3}$$

Now we shall define the operation addition on $\mathbb{H}(\Omega)$ in such a way that $\mathbb{H}(\Omega)$ is a linear space. Example 3.2 shows that this operation can not be defined point-wise using interval addition.

Assume first that the operation addition "$\oplus$" can be defined on $\mathbb{H}(\Omega)$ in such a way that $\mathbb{H}(\Omega)$ is a linear space and see what properties should be satisfied. For any $f, g \in \mathbb{H}(\Omega)$ the set D_{fg} defined through (4.3) is dense in Ω, see (3.9). Using (4.2) we have for $x \in \Omega$:

$$F(D_{fg}, \Omega, f + g)(x) = F(D_{fg}, \Omega, f \oplus g)(x) \subseteq F(f \oplus g)(x) = (f \oplus g)(x).$$

The minimality property of the H-continuous functions, see Definition 3.3, implies that the only possible way to define the sum of $f, g \in \mathbb{H}(\Omega)$ is given in the following definition.

DEFINITION 4.1. For every $f, g \in \mathbb{H}(\Omega)$ the interval function

$$(f \oplus g)(x) = F(D_{fg}, \Omega, f + g)(x), \qquad x \in \Omega, \tag{4.4}$$

is the sum of f and g.

Clearly, the point-wise sum $f + g$ given by (3.13) is S-continuous but not necessarily H-continuous. It is easy to see that for every $f, g \in \mathbb{H}(\Omega)$ we have the inclusion

$$(f \oplus g)(x) \subseteq (f + g)(x), \qquad x \in \Omega. \tag{4.5}$$

Indeed, since the function $f + g$ is S-continuous and

$$(f \oplus g)(x) = f(x) + g(x), \qquad x \in D_{fg},$$

then

$$(f \oplus g)(x) = F(D_{fg}, \Omega, f + g)(x) \subseteq F(f + g)(x) = (f + g)(x), \qquad x \in \Omega.$$

4.2. The Linear Operations: Properties

THEOREM 4.1. *The set* $\mathbb{H}(\Omega)$ *is closed under the operation "$\oplus$", that is,* $f \oplus g \in \mathbb{H}(\Omega)$ *whenever* $f, g \in \mathbb{H}(\Omega)$.

Proof. We shall use Definition 3.3 to show that $f \oplus g$ is H-continuous. The function $f \oplus g$ defined in (4.4) is S-continuous, because it is given by a graph completion operator, see Theorem 3.1(*i*). Assume that φ is an interval function satisfying the inclusion $\varphi(x) \subseteq (f \oplus g)(x)$, $x \in \Omega$. Then

$$\varphi(x) \subseteq (f \oplus g)(x) \subseteq f(x) + g(x), \qquad x \in \Omega,$$

implies that

$$\varphi(x) = (f \oplus g)(x) = f(x) + g(x), \qquad x \in D_{fg}.$$

Therefore

$$\begin{aligned}(f \oplus g)(x) &= F(D_{fg}, \Omega, f+g)(x) = F(D_{fg}, \Omega, \varphi)(x) \\ &\subseteq F(\varphi)(x) \subseteq F(f \oplus g)(x) = (f \oplus g)(x), \qquad x \in \Omega,\end{aligned}$$

which implies $F(\varphi) = f \oplus g$. Hence $f \oplus g \in \mathbb{H}(\Omega)$. □

It is important to note that the values of the sum $f \oplus g$ at the points where both operands assume interval values can not be determined point-wise, i.e., from the values of f and g at these points. This is illustrated by the following example.

EXAMPLE 4.1. Consider the functions $f, g \in \mathbb{H}(\mathbb{R})$ given by

$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0, \\ [-1, 1], & \text{if } x = 0; \end{cases}$$

$$g(x) = \begin{cases} \cos(1/x), & \text{if } x \neq 0, \\ [-1, 1], & \text{if } x = 0. \end{cases}$$

We have

$$(f \oplus g)(x) = \begin{cases} \sqrt{2}\cos(1/x + \pi/4), & \text{if } x \neq 0, \\ [-\sqrt{2}, \sqrt{2}], & \text{if } x = 0. \end{cases}$$

Clearly $(f \oplus g)(0)$ can not be obtained just from the values $f(0)$ and $g(0)$.

4.3. The Linear Space $\mathbb{H}(\Omega)$

THEOREM 4.2. *The set* $\mathbb{H}(\Omega)$ *with the operations addition defined by (4.4) and multiplication by a real number given in (3.14) is a linear space.*

Proof. The proofs of the axioms of a linear space use similar techniques. As an illustration we show below the second distributive law, namely, that for $\alpha, \beta \in \mathbb{R}$ and $f \in \mathbb{H}(\Omega)$ we have

$$(\alpha + \beta) * f = (\alpha * f) \oplus (\beta * f).$$

This law is typically violated in interval spaces [24]. For example, in the space of real intervals $(\mathbb{IR}, +, \mathbb{R}, *)$, if $A \in \mathbb{IR}$ is origin symmetric ($A = (-1) * A$), then we have for $\alpha = 1$, $\beta = -1$, $0 = 0 * A \neq A + (-1) * A = A + A$, which is true only for $A = 0$. In general only the inclusion

$$(\alpha + \beta) * A \subseteq (\alpha * A) + (\beta * A), \qquad \alpha, \beta \in \mathbb{R}, \; A \in \mathbb{IR}, \tag{4.6}$$

holds, which is strict whenever $\alpha\beta < 0$ and $w(A) > 0$. Using inclusion (4.6) and (3.14) we obtain

$$\big((\alpha + \beta) * f\big)(x) = (\alpha + \beta) * f(x) \subseteq \alpha * f(x) + \beta * f(x), \qquad x \in \Omega. \tag{4.7}$$

We also have, see (4.5),

$$\begin{aligned}\big((\alpha * f) \oplus (\beta * f)\big)(x) &\subseteq \big((\alpha * f) + (\beta * f)\big)(x) \\ &= \alpha * f(x) + \beta * f(x), \qquad x \in \Omega. \end{aligned} \tag{4.8}$$

From inclusions (4.7) and (4.8) it follows that

$$(\alpha + \beta) * f(x) = \big((\alpha * f) \oplus (\beta * f)\big)(x) = \alpha * f(x) + \beta * f(x), \qquad x \in D_f,$$

where D_f is the dense subset of Ω on which f assumes point values, see (3.8). The functions $(\alpha + \beta) * f$ and $(\alpha * f) \oplus (\beta * f)$ are both H-continuous and therefore D-continuous as well, see (3.1). Then, using Definition 3.2 the above identity implies $(\alpha + \beta) * f = (\alpha * f) \oplus (\beta * f))$. □

THEOREM 4.3. *Assume that the set $\mathcal{P} \subseteq \mathbb{G}(\Omega)$ is closed under inclusion in the sense that*

$$\left.\begin{array}{l} f \in \mathcal{P}, \ \ g \in \mathbb{G}(\Omega), \\ g(x) \subseteq f(x), \ \ x \in \Omega \end{array}\right\} \Longrightarrow g \in \mathcal{P}. \tag{4.9}$$

If $\mathcal{P} \subseteq \mathbb{G}(\Omega)$ is a linear space with respect to an operation addition satisfying (4.1) and the multiplication by scalar (3.14), then $\mathcal{P} \subseteq \mathbb{H}(\Omega)$.

The proof of Theorem 4.3 is rather technical and is given in the Appendix. Theorem 4.3 indicates that a linear space consisting of D-continuous functions can not be larger than $\mathbb{H}(\Omega)$. It was shown already that $\mathbb{H}(\Omega)$ is a linear space, see Theorem 4.2. Furthermore, using Definition 3.3 it can be easily seen that $\mathbb{H}(\Omega)$ has the property (4.9). Therefore, $\mathbb{H}(\Omega)$ is the maximal linear space, which satisfies the assumptions of Theorem 4.3.

It is important to note that the condition $\mathcal{P} \subseteq \mathbb{G}(\Omega)$ is essential for the validity of Theorem 4.3. It is shown in the example below that one can construct linear spaces of interval functions which are S-continuous but not D-continuous. However, in our view, the functions involved in this example show that the inclusion $\mathcal{P} \subseteq \mathbb{G}(\Omega)$ is a natural condition.

EXAMPLE 4.2. The set

$$\left\{ f \in \mathbb{A}(\mathbb{R}) : \exists \varphi_f \in C(\mathbb{R}), \exists a_f \in \mathbb{R} : f(x) = \left\{ \begin{array}{ll} \varphi_f(x), & x \neq 0, \\ \varphi_f(0) + \operatorname{sgn}(a_f)[0, |a_f|], & x = 0 \end{array} \right. \right\}$$

with addition

$$(f \boxplus g)(x) = \left\{ \begin{array}{ll} \varphi_f(x) + \varphi_g(x), & \text{if } x \neq 0, \\ \varphi_f(0) + \varphi_g(0) + \operatorname{sgn}(a_f + a_g)[0, |a_f + a_g|], & \text{if } x = 0, \end{array} \right.$$

and scalar multiplication given in (3.14) is a linear space. One can easily see that the functions in this space are S-continuous but not D-continuous.

5. The Set of D-continuous Functions as a Quasi-Linear Space

5.1. Extension of Addition

In this section we show that the operation addition derived in the preceding section for Hausdorff continuous functions can be extended to D-continuous functions introducing in $\mathbb{G}(\Omega)$ the structure of a quasi-linear space. To this end we use an equivalent formulation of Definition 4.1 using the lower and upper Baire operators as well as the operator G given in Theorem 3.1(*ii*).

THEOREM 5.1. *For every $f, g \in \mathbb{H}(\Omega)$ we have*

$$f \oplus g = [I\big(S(\underline{f} + \underline{g})\big), S\big(I(\overline{f} + \overline{g})\big)] = G(f + g). \tag{5.1}$$

Proof. Using the monotonicity of the operators I and S, see (2.5), and the fact the $\underline{f}, \underline{g}$ are lower semi-continuous while $\overline{f}, \overline{g}$ are upper semi-continuous we have

$$\begin{aligned} I\big(S(\underline{f} + \underline{g})\big) &\geq I(\underline{f} + \underline{g}) = \underline{f} + \underline{g}, \\ S\big(I(\overline{f} + \overline{g})\big) &\leq S(\overline{f} + \overline{g}) = \overline{f} + \overline{g}, \end{aligned}$$

which implies the inclusion

$$G(f + g)(x) \subset (f + g)(x), \qquad x \in \Omega. \tag{5.2}$$

The above inclusion (5.2) together with inclusion (4.5) imply

$$(f \oplus g)(x) = G(f + g)(x) = (f + g)(x), \qquad x \in D_{fg},$$

where the set D_{fg} is defined through (4.3). The functions $G(f + g)$ and $f \oplus g$ are both D-continuous, see Theorem 3.1(*ii*) and Theorem 4.1. Since D_{fg} is dense in Ω, see (3.9), it follows from Definition 3.2 that these functions are equal on Ω. □

The representation (5.1) of the operation addition "⊕" can be applied to D-continuous functions as well.

DEFINITION 5.1. For every $f, g \in \mathbb{G}(\Omega)$ the interval function

$$f \oplus g = G(f + g)$$

is the sum of f and g.

Note that due to Theorem 3.1 for every $f, g \in \mathbb{G}(\Omega)$ the function $G(f + g)$ is D-continuous. Therefore, $\mathbb{G}(\Omega)$ is closed under the operation addition "⊕" defined above.

Due to the involvement of the operator G in the definition of addition on $\mathbb{G}(\Omega)$ the following lemma will be useful.

LEMMA 5.1. *The operator G is inclusion isotone on $\mathbb{A}(\Omega)$, that is, for every two functions $f, g \in \mathbb{A}(\Omega)$ we have*

$$f(x) \subseteq g(x), \ \ x \in \Omega \implies G(f)(x) \subseteq G(g)(x), \ \ x \in \Omega.$$

Proof. Using the monotonicity (2.5) of the operators I and S from the inequalities

$$\underline{f} \geq \underline{g}, \qquad \overline{f} \leq \overline{g}$$

it follows that

$$I(S(\underline{f})) \geq I(S(\underline{g})), \qquad S(I(\overline{f})) \leq S(I(\overline{g})).$$

The above inequalities imply the required inclusion. □

5.2. Relation to Interval Hulls of Continuous Functions

Definition 5.1 naturally relates to the addition of continuous functions. More precisely, let $\mathcal{F}$ be a bounded set of continuous real functions on Ω and let $\hat{\mathcal{F}}$ be the set of all continuous interval enclosures of $\mathcal{F}$, that is,

$$\hat{\mathcal{F}} = \{\psi = [\underline{\psi}, \overline{\psi}] \in \mathbb{A}(\Omega) : \underline{\psi}, \overline{\psi} \in C(\Omega) : \phi(x) \in \psi(x),\ x \in \Omega,\ \phi \in \mathcal{F}\}. \quad (5.3)$$

The interval hull of the set $\mathcal{F}$ given by

$$\text{hull}(\mathcal{F})(x) = \bigcap_{\psi \in \hat{\mathcal{F}}} \psi(x), \qquad x \in \Omega, \quad (5.4)$$

is not always a continuous function. The next theorem shows that the interval hull is a D-continuous function and characterizes its lower and upper functions.

THEOREM 5.2. *Let $\mathcal{F}$ be a bounded subset of $C(\Omega)$ and let $f = [\underline{f}, \overline{f}] = \text{hull}(\mathcal{F})$. Then*

a) f is D-continuous;

b) $\underline{f} = I(\underline{\theta})$ and $\overline{f} = S(\overline{\theta})$ where

$$\underline{\theta}(x) = \inf_{\phi \in \mathcal{F}} \phi(x), \qquad \overline{\theta}(x) = \sup_{\phi \in \mathcal{F}} \phi(x), \qquad x \in \Omega.$$

Proof. It follows from (5.4) that

$$\overline{f}(x) = \inf_{\varphi \in \mathcal{F}_1} \varphi(x), \qquad x \in \Omega,$$

where

$$\mathcal{F}_1 = \{\varphi \in C(\Omega) : \varphi \geq \phi,\ \phi \in \mathcal{F}\}.$$

Using that the set $\mathcal{F}_1$ is a Dedekind cut in $C(\Omega)$ the function $\overline{f}$ is normal upper semi-continuous, see [17]. In a similar way the function $\underline{f}$ is normal lower semi-continuous. Then the D-continuity of f follows from (3.4). Furthermore, considering (3.5) and (3.6) the functions $F(\overline{f})$ and $F(\underline{f})$ are both H-continuous. Using that the set of Hausdorff continuous functions $\mathbb{H}(\Omega)$ is the Dedekind order completion of

$C(\Omega)$, see [2], it follows that $F(\overline{f})$ is exactly the supremum of $\mathcal{F}$ in $\mathbb{H}(\Omega)$. It was also shown in [1,the proof of Theorem 5] that this supremum can be represented through the point-wise supremum $\overline{\theta}$ in the form $F(\overline{f}) = F(S(\overline{\theta}))$. Hence $\overline{f} = S(\overline{f}) = S(\overline{\theta})$. The respective statement for $\underline{f}$ is proved in a similar way. □

5.3. Relation to the Addition of Interval Hulls

The following theorem shows that the addition of D-continuous functions in Definition 5.1 can be considered as an extension of the usual addition of continuous real functions to interval hulls of bounded sets of continuous functions.

THEOREM 5.3. *Let the D-continuous functions f and g be interval hulls, see (5.4), of the bounded sets of continuous functions $\mathcal{F}$ and $\mathcal{G}$, respectively, that is*

$$f = \mathrm{hull}(\mathcal{F}), \qquad g = \mathrm{hull}(\mathcal{G}).$$

Then

$$f \oplus g = \mathrm{hull}(\{\phi + \varphi : \phi \in \mathcal{F},\ \varphi \in \mathcal{G}\}).$$

Proof. Let

$$h = \mathrm{hull}(\{\phi + \varphi : \phi \in \mathcal{F},\ \varphi \in \mathcal{G}\}).$$

Denote

$$\begin{aligned} \overline{\theta}_1(x) &= \sup_{\phi\in\mathcal{F}} \phi(x), & x \in \Omega,\\ \overline{\theta}_2(x) &= \sup_{\varphi\in\mathcal{G}} \varphi(x), & x \in \Omega,\\ \overline{\theta}_3(x) &= \sup_{\substack{\phi\in\mathcal{F}\\ \varphi\in\mathcal{G}}} \big(\phi(x) + \varphi(x)\big), & x \in \Omega. \end{aligned}$$

Since in the definition of $\theta_3(x)$ the functions ϕ and φ vary independently in the sets $\mathcal{F}$ and $\mathcal{G}$ respectively we have

$$\overline{\theta}_3(x) = \overline{\theta}_1(x) + \overline{\theta}_2(x), \qquad x \in \Omega.$$

It follows from Theorem 5.2 that

$$\overline{f} = S(\overline{\theta}_1), \qquad \overline{g} = S(\overline{\theta}_2), \qquad \overline{h} = S(\overline{\theta}_3).$$

Furthermore, the functions $\overline{\theta}_1$, $\overline{\theta}_2$, and $\overline{\theta}_3$, being suprema of sets of continuous functions, are all lower semi-continuous. The functions $\overline{f}$, $\overline{g}$, and $\overline{h}$, being produced by the operator S, are upper semi-continuous. Therefore, each one of the functions $\overline{\theta}_1$, $\overline{\theta}_2$, $\overline{\theta}_3$, $\overline{f}$, $\overline{g}$, and $\overline{h}$ is discontinuous on a set of first Baire category. Hence the set W_1 on which at least one of the six function is discontinuous is also a set of first

Baire category. Since the functions $\overline{\theta}_1, \overline{\theta}_2$, and $\overline{\theta}_3$ are continuous at every $x \in \Omega \setminus W_1$ we have

$$S(\overline{\theta}_i)(x) = \overline{\theta}_i(x), \qquad i = 1, 2, 3, \;\; x \in \Omega \setminus W_1,$$

which implies

$$\overline{h}(x) = \overline{\theta}(x) = \overline{\theta}_1(x) + \overline{\theta}_2(x) = \overline{f}(x) + \overline{g}(x), \qquad x \in \Omega \setminus W_1.$$

Using that $\overline{f} + \overline{g}$ is continuous at every $x \in \Omega \setminus W_1$ we further obtain

$$\overline{h}(x) = (\overline{f} + \overline{g})(x) = S\big(I(\overline{f} + \overline{g})\big)(x), \qquad x \in \Omega \setminus W_1.$$

In a similar way we find a set of first Baire category W_2 such that

$$\underline{h}(x) = I\big(S(\underline{f} + \underline{g})\big)(x), \qquad x \in \Omega \setminus W_2.$$

Then for $x \in D = \Omega \setminus (W_1 \cap W_2)$

$$h(x) = [I\big(S(\underline{f} + \underline{g})\big)(x), S\big(I(\overline{f}(x) + \overline{g})\big)](x) = G(f + g)(x) = (f \oplus g)(x).$$

Using that D is dense in Ω and that the functions h and $f \oplus g$ are both D-continuous we obtain

$$h = F(D, \Omega, h) = F(D, \Omega, f \oplus g) = f \oplus g$$

which completes the proof. □

The above theorem is illustrated by the following example:

EXAMPLE 5.1. Let us consider the following subsets of $C(\mathbb{R})$

$$\mathcal{F} = \{\phi_\lambda : \lambda > 0\},$$

where

$$\phi_\lambda(x) = \begin{cases} \dfrac{1 - e^{-\lambda x}}{1 + e^{-\lambda x}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

and

$$\mathcal{G} = \{\psi_\mu : \mu > 0\},$$

where

$$\psi_\mu(t) = \begin{cases} 0, & \text{if } t > 0, \\ \dfrac{1 - e^{\mu t}}{1 + e^{\mu t}}, & \text{if } t \leq 0. \end{cases}$$

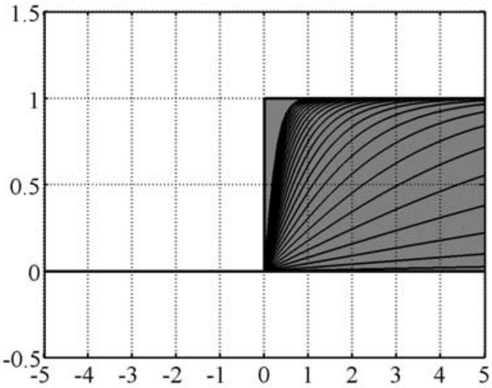

Figure 4. The set $\mathcal{F}$ and its interval hull f.

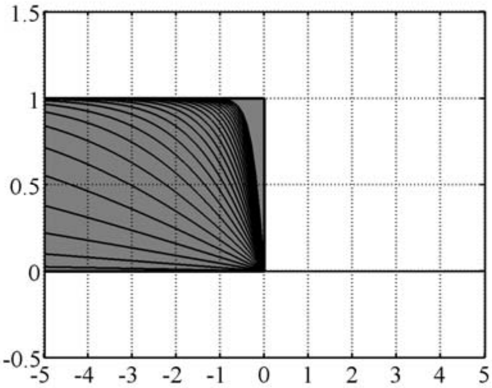

Figure 5. The set $\mathcal{G}$ and its interval hull g.

The interval hulls of these sets are, see Figures 4 and 5:

$$f(x) = \text{hull}(\mathcal{F})(x) = \begin{cases} [0,1], & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

$$g(x) = \text{hull}(\mathcal{G})(x) = \begin{cases} 0, & \text{if } x > 0, \\ [0,1], & \text{if } x \leq 0. \end{cases}$$

Following the usual approach to operations with sets we denote

$$\mathcal{F} + \mathcal{G} = \{\phi_\lambda + \psi_\mu : \lambda, \mu > 0\}.$$

We have, see Figure 6:

$$\text{hull}(\mathcal{F} + \mathcal{G})(x) = [0,1] = (f \oplus g)(x), \qquad x \in \mathbb{R}.$$

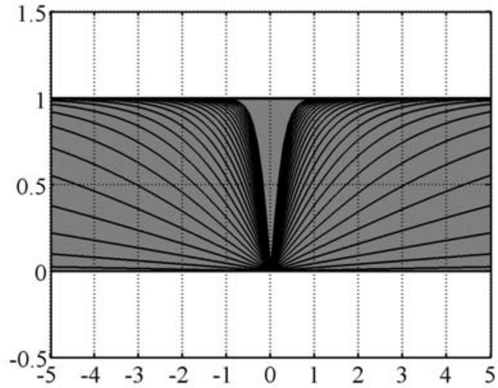

Figure 6. The set $\mathcal{F} + \mathcal{G}$ and its interval hull.

Note that the point-wise sum of f and g

$$(f+g)(x) = \begin{cases} [0,1], & \text{if } x \neq 0, \\ [0,2], & \text{if } x = 0 \end{cases}$$

does not give a sharp bound for the interval hull of the set $\mathcal{F} + \mathcal{G}$ at $x = 0$.

5.4. The Quasi-linear Space $\mathbb{G}(\Omega)$

THEOREM 5.4. *The set $\mathbb{G}(\Omega)$ with the operation addition given by Definition 5.1 and multiplication by a real number defined through (3.14) is a quasi-linear space.*

Proof. The axioms of quasi-linear space [24] are verified similarly to the way the axioms for linear space are verified for H-continuous functions. The difference is in the second distributive law involving addition of scalars, that is,

$$(\alpha+\beta) * f = (\alpha * f) \oplus (\beta * f),$$

which holds for $\alpha\beta \geq 0$. For α, β satisfying $\alpha\beta \geq 0$ we have

$$\big((\alpha+\beta) * f\big)(x) = \alpha * f(x) + \beta * f(x) = (\alpha * f + \beta * f)(x), \qquad x \in \Omega.$$

The function $\alpha * f + \beta * f$ is D-continuous. Therefore

$$(\alpha * f) \oplus (\beta * f) = G(\alpha * f + \beta * f) = \alpha * f + \beta * f = (\alpha+\beta) * f. \qquad \square$$

5.5. Inclusion Isotone Properties

A natural connection between the D-continuous interval functions and the interval hulls of bounded sets of continuous functions functions was established in Sections 5.2 and 5.3. The applicability of the D-continuous functions to constructing interval enclosures is further demonstrated in the next theorem, which establishes the inclusion isotonicity of the operation addition $\oplus$ defined in Section 5.1.

THEOREM 5.5. *For every* $f, g, u, v \in \mathbb{G}(\Omega)$ *we have*

$$f(x) \subseteq u(x),\ g(x) \subseteq v(x),\ x \in \Omega \implies (f \oplus g)(x) \subseteq (u \oplus v)(x),\ x \in \Omega. \quad (5.5)$$

Proof. It is well known that the addition of intervals is inclusion isotone. Hence

$$(f + g)(x) \subseteq (u + v)(x), \qquad x \in \Omega.$$

Then, using the inclusion isotonicity of the operator G, see Lemma 5.1, we have

$$(f \oplus g)(x) = G(f + g)(x) \subseteq G(u + v)(x) = (u \oplus v)(x), \qquad x \in \Omega. \qquad \square$$

6. Conclusion

The operations addition and scalar multiplication of point-valued functions are extended on the set of Hausdorff continuous interval functions in such a way that the latter becomes a linear space. Furthermore, we prove that this space is the largest linear space within the set of D-continuous interval functions with operations addition and multiplication by scalars extending the respective operations of point-valued functions. The operations are further extended to the set of D-continuous interval functions introducing in it the structure of a quasi-linear space. Both H- and D-continuous functions are inclusion isotone.

Topics for future work. The algebraic properties of the set of D-continuous functions are to be investigated. It is hoped that the study of these practically important objects will contribute for the development of the general abstract theory of partially ordered quasivector spaces (quasivector lattices). Applications of the algebraic properties of D-continuous functions to the formulation of new dynamic problems and their solutions will be studied. Construction of the space of D-continuous functions as interval space over the vector lattice of H-continuous functions is intended.

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Appendix

Proof of Theorem 4.3. Let $f = [\underline{f}, \overline{f}] \in \mathcal{P}$. We shall show that f is H-continuous by proving that it assumes point values on a dense subset of the domain, see Theorem 3.2. This will be established by showing that for every $\varepsilon > 0$ the set

$$W_{f,\varepsilon} = \{x \in \Omega : w(f(x)) \geq \varepsilon\}$$

is closed and nowhere dense. Since f is S-continuous, the functions $\underline{f}$ and $\overline{f}$ are lower and upper semi-continuous functions, respectively, see (3.3). Therefore, the width function $w(f(x)) = \overline{f}(x) - \underline{f}(x)$ is an upper semi-continuous function which implies that $W_{f,\varepsilon}$ is closed.

We shall show that $W_{f,\varepsilon}$ is nowhere dense by assuming the opposite, that is, there exists $\varepsilon > 0$ such that $W_{f,\varepsilon}$ is not nowhere dense in Ω. This means that there exists an open set $V \subseteq \Omega$ such that $W_{f,\varepsilon} \cap V$ is dense in V. Since the function $w(f)$ is upper semi-continuous it is a fixed point of the upper Baire operator, that is, $S(w(f)) = w(f)$, see (3.2). Let $x \in V$. Using that $W_{f,\varepsilon} \cap V$ is dense in V, for any $\delta > 0$ the set $B_\delta(x) \cap W_{f,\varepsilon}$ is not empty. Then, we have

$$\begin{aligned} w\big(f(x)\big) = S\big(w(f)\big)(x) &= \inf_{\delta>0} \sup\{w\big(f(y)\big) : y \in B_\delta(x)\} \\ &\geq \inf_{\delta>0} \sup\{w\big(f(y)\big) : y \in B_\delta(x) \cap W_{f,\varepsilon}\} \geq \varepsilon. \end{aligned}$$

Therefore,

$$w\big(f(x)\big) \geq \varepsilon, \qquad x \in V. \tag{A.1}$$

Let us consider the functions

$$\begin{aligned} \varphi(x) &= \frac{\underline{f}(x) + \overline{f}(x)}{2}, \qquad x \in \Omega, \\ g &= F\Big(S\big(I(\varphi)\big)\Big). \end{aligned}$$

We have $\varphi(x) \in f(x)$, $x \in \Omega$. From the S-continuity of f it follows immediately that $g(x) \subseteq f(x)$, $x \in \Omega$. Since g is H-continuous, see Theorem 3.1, and therefore D-continuous as well, see (3.1), it follows from the assumption (4.9) that $g \in \mathcal{P}$.

Using that the set V is open there exists $a \in V$ and $\delta > 0$ such that

$$\overline{B_\delta(a)} \subset V, \tag{A.2}$$

where $\overline{B_\delta(a)}$ denotes the closure of $B_\delta(a)$. Now consider the function

$$h(x) = \begin{cases} g(x), & \text{if } x \in \Omega \setminus \overline{B_\delta(a)}, \\ \left[\underline{g}(x) - \dfrac{\varepsilon}{2}, \overline{g}(x) + \dfrac{\varepsilon}{2}\right], & \text{if } x \in \overline{B_\delta(a)}. \end{cases}$$

It is easy to see that h is D-continuous. We shall show that

$$h(x) \subseteq f(x), \qquad x \in \Omega. \tag{A.3}$$

Using inequality (A.1) we obtain

$$\overline{f}(x) \geq \underline{f}(x) + \varepsilon\mu(x), \qquad x \in \Omega, \tag{A.4}$$

where

$$\mu(x) = \begin{cases} 1, & \text{if } x \in V, \\ 0, & \text{if } x \in \Omega \setminus V. \end{cases}$$

Therefore for every $x \in \Omega$ we have

$$\varphi(x) = \frac{\underline{f}(x) + \overline{f}(x)}{2} \geq \frac{\underline{f}(x) + \underline{f}(x) + \varepsilon\mu(x)}{2} \geq \underline{f}(x) + \frac{\varepsilon}{2}\mu(x), \tag{A.5}$$

$$\varphi(x) = \frac{\underline{f}(x) + \overline{f}(x)}{2} \leq \frac{\overline{f}(x) - \varepsilon\mu(x) + \overline{f}(x)}{2} \leq \overline{f}(x) - \frac{\varepsilon}{2}\mu(x). \tag{A.6}$$

Inclusion (A.2) and inequalities (A.5) and (A.6) imply

$$\begin{aligned} \underline{h} &\geq \underline{g} - \frac{\varepsilon}{2}\mu = I\Big(S\big(I(\varphi)\big)\Big) - \frac{\varepsilon}{2}\mu \geq I(\varphi) - \frac{\varepsilon}{2}\mu \\ &\geq I\left(\underline{f} + \frac{\varepsilon}{2}\mu\right) - \frac{\varepsilon}{2}\mu \geq I(\underline{f}) + \frac{\varepsilon}{2}I(\mu) - \frac{\varepsilon}{2}\mu = \underline{f}, \\ \overline{h} &= \overline{g} + \frac{\varepsilon}{2}\mu = S\big(I(\varphi)\big) + \frac{\varepsilon}{2}\mu \leq S(\varphi) + \frac{\varepsilon}{2}\mu \\ &\leq S\left(\overline{f} - \frac{\varepsilon}{2}\mu\right) + \frac{\varepsilon}{2}\mu \leq S(\overline{f}) - \frac{\varepsilon}{2}I(\mu) + \frac{\varepsilon}{2}\mu = \overline{f}, \end{aligned}$$

which completes the proof of the inclusion (A.3). Due to assumption (4.9) we have $h \in \mathcal{P}$.

Since $\mathcal{P}$ is a linear space the function

$$\psi = h + (-1) * g$$

is also in $\mathcal{P}$. Using (3.14) and (4.1) for $x \in D_g = \{x \in \Omega : w(g(x)) = 0\}$ we have

$$\begin{aligned} \psi(x) &= [\underline{h}(x) + (-1) * g(x), \overline{h}(x) + (-1) * g(x)] \\ &= \begin{cases} 0, & \text{if } x \in D_g \setminus \overline{B_\delta(a)}, \\ [-\varepsilon, \varepsilon], & \text{if } x \in D_g \cap \overline{B_\delta(a)}. \end{cases} \end{aligned}$$

Since g is H-continuous it follows from (3.8) that the set D_g is dense in Ω. Therefore the function

$$\theta = F(D, \Omega, \psi) \in \mathbb{A}(\Omega)$$

is well defined. It is easy to see that

$$\theta(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus \overline{B_\delta(a)}, \\ [-\varepsilon, \varepsilon], & \text{if } x \in \overline{B_\delta(a)}. \end{cases}$$

Clearly θ is D-continuous. Using also (2.2) we have

$$\theta = F(D, \Omega, \psi) \subseteq F(\psi) = \psi,$$

which implies that $\theta \in \mathcal{P}$, see (4.9). For the function θ, according to (3.14) we have

$$\big((-1) * \theta\big)(x) = \begin{cases} 0, & \text{if } x \in D \setminus \overline{B_\delta(a)}, \\ [-\varepsilon, \varepsilon], & \text{if } x \in D \cap \overline{B_\delta(a)}. \end{cases}$$

Hence

$$(-1) * \theta = \theta. \tag{A.7}$$

Using that $\mathcal{P}$ is a linear space, (A.7) implies $\theta = 0$, which is false. The obtained contradiction shows that for every $\varepsilon > 0$ the set $W_{f,\varepsilon}$ is closed and nowhere dense. The set W_f defined in (3.7) can be represented in the form

$$W_f = \bigcup_{n=1}^{\infty} W_{f,\frac{1}{n}},$$

which implies that it is a set of first Baire category. Hence f assumes point values on a dense subset $D_f = \Omega \setminus W_f$ of the domain Ω. Thus the H-continuity of f follows from Theorem 3.2. □

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