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Outer Interval Solution of the Eigenvalue Problem under General Form Parametric Dependencies

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Abstract. The paper addresses the problem of determining an outer interval solution of the parametric eigenvalue problem $A(p)x = \lambda x$, $A(p) \in \mathbb{R}^{n \times n}$ for the general case where the matrix elements $a_{ij}(p)$ are continuous nonlinear functions of the parameter vector p, p belonging to the interval vector $\boldsymbol{p}$. A method for computing an interval enclosure of each eigenpair $(\lambda_\mu, x^{(\mu)})$, $\mu = 1, \dots, n$, is suggested for the case where λ_μ is a simple eigenvalue. It is based on the use of an affine interval approximation of $a_{ij}(p)$ in $\boldsymbol{p}$ and reduces, essentially, to setting up and solving a real system of n or $2n$ incomplete quadratic equations for each real or complex eigenvalue, respectively.

1. Problem Statement

Consider the parametric eigenvalue problem

$$A(p)x = \lambda x, \qquad A(p) \in \mathbb{R}^{n \times n}, \tag{1.1a}$$
$$p \in \boldsymbol{p}, \tag{1.1b}$$

where $p \in (p_1, \dots, p_m) \in \mathbb{R}^m$, is a real parameter vector and $\boldsymbol{p}$ is a given interval vector (throughout the paper ordinary letters will denote real entities while boldface letters will stand for their interval counterparts). Such a "perturbed" eigenvalue problem is related to the so-called robust stability analysis of linear control systems or electric circuits (cf., e.g., [8, Chapter 4]). The elements of $A(p)$

$$a_{ij}(p) = a_{ij}(p_1, \dots, p_m), \tag{1.2a}$$
$$p_l \in \boldsymbol{p}_l, \qquad l = 1, \dots, m \tag{1.2b}$$

are, in the general case, nonlinear functions of the parameters involved. On account of (1.1) each eigenvalue and its eigenvector are functions of p, i.e.

$$\lambda_\mu = \lambda_\mu(p), \qquad x^{(\mu)} = x^{(\mu)}(p) = \left(x_1^{(\mu)}(p), \dots x_n^{(\mu)}(p)\right), \tag{1.3a}$$
$$p \in \boldsymbol{p}. \tag{1.3b}$$

For a fixed μ, $\lambda_\mu(p)$ (and, hence, $x^{(\mu)}(p)$) can be real for some $p \in \boldsymbol{p}$ and complex for some other $p \in \boldsymbol{p}$. In this paper, it is assumed that this does not occur, i.e. $\lambda_\mu(p)$

is either real or complex for all $p \in \boldsymbol{p}$. More specifically, we have the following definitions.

DEFINITION 1.1. An eigenvalue $\lambda_\mu(p)$ is called structurally stable if it remains either real or complex for all $p \in \boldsymbol{p}$.

DEFINITION 1.2. The eigenvalue problem (1.1) is structurally stable if each $\lambda_\mu(p)$, $\mu = 1, \ldots, n$, is structurally stable.

In this paper we make the following assumption.

ASSUMPTION 1.1. The eigenvalue $\lambda_\mu(p)$ considered is structurally stable.

Later (in Sections 2 and 3), a sufficient condition will be given to check the validity of that assumption.

We now proceed to stating the outer interval solution problem related to (1.1). We first consider the case where $\lambda_\mu(p)$ is real. Let $\boldsymbol{\lambda}^*$ denote the range of $\lambda_\mu(p)$ over $\boldsymbol{p}$, i.e.

$$\boldsymbol{\lambda}^* = \{\lambda_\mu(p) : p \in \boldsymbol{p}\}. \tag{1.4}$$

Similarly, let $\boldsymbol{X}_i^*$ be the range of $x_i^{(\mu)}(p)$ over $\boldsymbol{p}$; introduce the vector $\boldsymbol{X}^* = (\boldsymbol{X}_1^*, \ldots, \boldsymbol{X}_n^*)$. An outer interval solution to (1.1) (for the fixed μ) is any interval $\boldsymbol{\lambda}$ and interval vector $\boldsymbol{X}$ such that

$$\lambda^* \subset \boldsymbol{\lambda}, \qquad \boldsymbol{X}^* \subset \boldsymbol{X}. \tag{1.5}$$

In the case of a complex eigenvalue, we consider the real and imaginary part $\mathrm{Re}(\lambda_\mu(p))$ and $\mathrm{Im}(\lambda_\mu(p))$ of the eigenvalue as well as $\mathrm{Re}(x_i^{(\mu)}(p))$ and $\mathrm{Im}(x_i^{(\mu)}(p))$ of the components $x_i^{(\mu)}(p)$ of $x^{(\mu)}(p)$. Thus, we have the ranges

$$\boldsymbol{\lambda}_{\mathrm{Re}}^* = \{\mathrm{Re}(\lambda_\mu(p)) : p \in \boldsymbol{p}\}, \tag{1.6a}$$
$$\boldsymbol{\lambda}_{\mathrm{Im}}^* = \{\mathrm{Im}(\lambda_\mu(p)) : p \in \boldsymbol{p}\}. \tag{1.6b}$$

In a similar way, let $\boldsymbol{X}_{\mathrm{Re}}^*$ denote the interval vector made up of the ranges $\{\mathrm{Re}(x_i^{(\mu)}(p)) : p \in \boldsymbol{p}\}$ while $\boldsymbol{X}_{\mathrm{Im}}^*$ is composed of the ranges $\{\mathrm{Im}(x_i^{(\mu)}(p)) : p \in \boldsymbol{p}\}$. An outer interval solution in this case is any interval vector $(\boldsymbol{\lambda}_{\mathrm{Re}}, \boldsymbol{X}_{\mathrm{Re}}, \boldsymbol{\lambda}_{\mathrm{Im}}, \boldsymbol{X}_{\mathrm{Im}})$ such that

$$\boldsymbol{\lambda}_{\mathrm{Re}}^* \subset \boldsymbol{\lambda}_{\mathrm{Re}}, \qquad \boldsymbol{X}_{\mathrm{Re}}^* \subset \boldsymbol{X}_{\mathrm{Re}}, \qquad \boldsymbol{\lambda}_{\mathrm{Im}}^* \subset \boldsymbol{\lambda}_{\mathrm{Im}}, \qquad \boldsymbol{X}_{\mathrm{Im}}^* \subset \boldsymbol{X}_{\mathrm{Im}}. \tag{1.7}$$

The problem of finding an outer interval solution to (1.1) in the general case of arbitrary nonlinear functions (1.2a) is rather difficult. Most known results treat the special case $\boldsymbol{A}x = \lambda x$ where $\boldsymbol{A}$ is an interval matrix (its elements are independent intervals), e.g. [2], [16], [18]. Techniques for finding bounds on the eigenvalues for the still more particular case where $\boldsymbol{A}$ in a symmetric interval matrix can be found in [3] to [5].

The present paper suggests a method for computing an outer solution $(\boldsymbol{\lambda}, \boldsymbol{X})$ to (1.1) in the case of at least continuous, otherwise arbitrary nonlinear functions (1.2a). It is based on a recently proposed affine approximation of continuous functions in the form [7]

$$\boldsymbol{L}_{ij}(p) = \sum_{k=1}^{m} \alpha_{ijk} p_k + \boldsymbol{g}_{ij}, \qquad p_k \in \boldsymbol{p}_k, \tag{1.8}$$

where α_{ijk} are real constants while $\boldsymbol{g}_{ij}$ is an interval. The method suggested covers both real and complex structurally stable eigenvalues. It reduces, essentially, to setting up and solving a mildly nonlinear system of n real equations in the case of a real eigenvalue or of $2n$ real equations in the case of a complex eigenvalue. It is shown that the method suggested is applicable if the solution of the respective system is positive.

We end up this section with a result that will be useful in the sequel.

LEMMA 1.1. *The eigenvalue $\lambda_\mu(p)$ is structurally stable, if it is a simple eigenvalue over $\boldsymbol{p}$.*

Proof. *a*) Consider first the case of an eigenvalue $\lambda_\mu(p)$ that is a complex number for some $p = p^0 \in \boldsymbol{p}$. Assume that there exists another $p^1 \in \boldsymbol{p}$ such that $\mathrm{Im}(\lambda_\mu(p^1)) = 0$. It is easily seen that $\lambda_\mu(p^1)$ is not simple. Indeed, suppose that $p^1 \in \mathrm{int}(\boldsymbol{p})$. In that case, there is a pair of complex conjugate eigenvalues $\lambda_\mu(p')$ and $\lambda_\mu(p'')$ in the vicinity of p^1. If now $p' \to p^1$ and $p'' \to p^1$, then $\lambda_\mu(p') \to \lambda_\mu(p^1)$ and $\lambda_\mu(p'') \to \lambda_\mu(p^1)$, showing that $\lambda_\mu(p^1)$ has (algebraic) multiplicity $m_a = 2$. The same argument remains valid if p^1 is on the boundary of $\boldsymbol{p}$. The latter case can be reduced to the previous one by inflating $\boldsymbol{p}$ to $\breve{\boldsymbol{p}} = \boldsymbol{p} + [-\varepsilon, \varepsilon]$ and then letting $\varepsilon \to 0$. Thus, the complex eigenvalue $\lambda_\mu(p)$ remains structurally stable only if it remains simple over $\boldsymbol{p}$.

b) Now consider the case of an eigenvalue $\lambda_\mu(p)$ that is a real number for some $p = p^0 \in \boldsymbol{p}$. The same argument as in case (*a*) shows that $\lambda_\mu(p)$ can give rise to a pair of complex conjugate eingenvalues $\lambda_\mu(p')$ and $\lambda_\mu(p'')$ only if $\lambda_\mu(p)$ becomes double real eigenvalue $\lambda_\mu(p)$ for some $p = p^1$. Once again, the real eigenvalue $\lambda_\mu(p)$ remains structurally stable only if it remains simple over $\boldsymbol{p}$. □

2. Real Eigenvalue

In this section, we are interested in finding an outer solution $(\boldsymbol{\lambda}, \boldsymbol{X})$, i.e. an interval vector with the property (1.5). First, we shall, however, slightly change the problem formulation.

Any eigenvector $x(p)$ (related to an eigenvalue $\lambda_\mu(p)$) for a fixed $p \in \boldsymbol{p}$ can be normalized so that

$$\|x(p)\| = \left(\sum_{i=1}^{n} x_i^2(p) \right)^{1/2} = 1. \tag{2.1}$$

This is the traditional approach. Following [13], we adopt an alternative approach. We choose an index s (the choice of s will be made clear later on) and make the following assumption.

ASSUMPTION 2.1. The component

$$x_s(p) \neq 0, \qquad \forall p \in \boldsymbol{p} \tag{2.2}$$

so the eigenvector is normalized (by dividing $x(p)$ by $x_s(p)$) to have

$$x_s(p) = 1, \qquad \forall p \in \boldsymbol{p}. \tag{2.3}$$

Using the new normalization rule, the problem to solve is as follows.

PROBLEM 2.1. Find an outer solution $(\boldsymbol{\lambda}, \boldsymbol{X})$ to the parametric problem (1.1) when the normalization rule for the eigenvector $x(p)$ is (2.3).

Now we show that Problem 2.1 can be reformulated as follows. Let $J = \{1, \dots, n\}$ and $J' = \{1, \dots, s-1, s+1, \dots, n\}$. We first introduce the n-dimensional real vector

$$y = (y_1, \dots, y_n)^T \tag{2.4a}$$

with

$$y_i = x_i(p), \qquad i \in J', \quad y_s = \lambda(p). \tag{2.4b}$$

Using (2.3) and (2.4), (1.1) is rewritten as

$$\begin{aligned} \sum_{j \in J'} a_{ij}(p) y_j - y_s y_i + a_{is}(p) &= 0, \qquad i \in J', \\ \sum_{j \in J'} a_{sj}(p) y_j - y_s + a_{ss}(p) &= 0, \end{aligned} \tag{2.5a}$$

$$p \in \boldsymbol{p}. \tag{2.5b}$$

Let $\boldsymbol{y}_i^*$ denote the range of the ith component $y_i(p)$, $p \in \boldsymbol{p}$, $i \in J$ of the solution y to (2.5). Let $\boldsymbol{y}^*$ be the vector made up of $\boldsymbol{y}_i^*$. Consider the following problem.

PROBLEM 2.2. Find an outer interval solution $\boldsymbol{y}$ to (2.5), i.e. an interval vector enclosing the range vector $\boldsymbol{y}^*$:

$$\boldsymbol{y}^* \subset \boldsymbol{y}. \tag{2.6}$$

Obviously, the solution $\boldsymbol{y}$ to Problem 2.2 is a solution to the original Problem 2.1.

We now proceed to solving Problem 2.2. The approach adopted is based on ideas suggested recently in [11], [12] for the special case where the elements a_{ij} of A are independent parameters $a_{ij} \in \boldsymbol{a}_{ij}$. First, to simplify the presentation, we introduce the vector nonlinear function $h^0(y)$ with elements

$$h_i^0(y) = \begin{cases} y_s y_i, & i \in J', \\ y_s, & i = s. \end{cases} \tag{2.7}$$

System (2.5) can now be written in a more compact form

$$\sum_{j \in J'} a_{ij}(p) y_j - h_i^0(y) + a_{is}(p) = 0, \qquad i \in J, \tag{2.8a}$$

$$p \in \boldsymbol{p}. \tag{2.8b}$$

At this point, we linearize the nonlinear functions $a_{ij}(p)$, $p \in \boldsymbol{p}$ using an approach suggested in [7]. It is shown there that if $a_{ij}(p)$ are at least continuous functions in $\boldsymbol{p}$, then they can be approximated by the following affine interval form

$$\boldsymbol{L}_{ij}(p) = \sum_{k=1}^{m} \alpha_{ijk} p_k + \boldsymbol{g}_{ij}, \qquad p_k \in \boldsymbol{p}_k, \tag{2.9}$$

where α_{ijk} are real coefficients while $\boldsymbol{g}_{ij}$ is an interval. An algorithm for automatic computation of α_{ijk} and $\boldsymbol{g}_{ij}$ is given in [7]. The form (2.9) has the following inclusion property

$$a_{ij}(p) \in \boldsymbol{L}_{ij}(p), \qquad p \in \boldsymbol{p}. \tag{2.10}$$

For simplicity of notation, we introduce the real column vector

$$\omega_{ij} = (\alpha_{ij1}, \ldots, \alpha_{ijm})^T \tag{2.11}$$

and the inner product

$$b_{ij} = \omega_{ij}^T p \tag{2.12}$$

so

$$L_{ij}(p, g_{ij}) = \omega_{ij}^T p + g_{ij}, \qquad p \in \boldsymbol{p}, \;\; g_{ij} \in \boldsymbol{g}_{ij}. \tag{2.13}$$

Replacing $a_{ij}(p)$ in (2.8a) with (2.13), we get a “relaxed” system

$$\sum_{j \in J'} (\omega_{ij}^T p + g_{ij}) y_j - h_i^0(y) + \omega_{is}^T p + g_{is} = 0, \qquad i \in J, \tag{2.14a}$$

$$p \in \boldsymbol{p}, \qquad g_{ij} \in \boldsymbol{g}_{ij}. \tag{2.14b}$$

It is important to stress that the solution set of system (2.14) is larger than the solution set of system (2.8) because of the inclusion property (2.10). Therefore, the solution to Problem 2.2 can be found by computing an outer interval solution to

(2.14). To do so, following an idea suggested in [13], we put all the known variables in (2.14b) in the centered form

$$p_k = p_k^0 + u_k, \quad u_k \in \boldsymbol{u}_k, \qquad g_{ij} = g_{ij}^0 + t_{ij}, \quad t_{ij} \in \boldsymbol{t}_{ij}, \qquad i,j \in J, \tag{2.15a}$$

where p_k^0 and g_{ij}^0 are the centers of the respective intervals $\boldsymbol{p}_k$ and $\boldsymbol{g}_{ij}$ while $\boldsymbol{u}_k$ and $\boldsymbol{t}_{ij}$ are their symmetric intervals of radii r_{p_k} and $r_{g_{ij}}$, respectively. By analogy with (2.15a) the unknown y_j are represented as

$$y_j = y_j^0 + v_j, \qquad v_j \in \boldsymbol{v}_j, \tag{2.15b}$$

where, in contrast to (2.15a), the center y_j^0 and the radius r_{y_j} of the interval $\boldsymbol{v}_j$ are not known and are to be determined. With this in mind, we first substitute (2.15) into (2.14) to get the following system

$$\sum_{j\in J'}(\omega_{ij}^T p^0 y_j^0 + \omega_{ij} p^0 v_j + \omega_{ij}^T u y_j^0 + \omega_{ij}^T u v_j) + \sum_{j\in J'}^{n}(g_{ij}^0 y_j^0 + g_{ij}^0 v_j + y_j^0 t_{ij} + t_{ij} v_j)$$

$$- h_i(v) + \omega_{is}^T p^0 + \omega_{is}^T u + g_{is}^0 + t_{is} = 0, \qquad i \in J, \tag{2.16a}$$

$$u \in \boldsymbol{u}, \qquad t_{ij} \in \boldsymbol{t}_{ij}, \tag{2.16b}$$

where

$$h_j(v) = \begin{cases} y_s^0 y_i^0 + y_s^0 v_i + y_i^0 v_s + v_i v_s, & i \in J', \\ y_s^0 + v_s, & i = s. \end{cases} \tag{2.16c}$$

Now we put all the increments u, v_j, t_{ij} in (2.16) equal to zero to get

$$\sum_{i\in J'} a_{ij}^0 y_j^0 - y_i^0 y_s^0 + a_{is}^0 \;=\; 0, \qquad i \in J', \tag{2.17a}$$

$$\sum_{i\in J'} a_{sj}^0 y_j^0 - y_s^0 + a_{ss}^0 \;=\; 0 \tag{2.17b}$$

with

$$a_{ij}^0 = \omega_{ij}^T p^0 + g_{ij}^0, \qquad i,j \in J. \tag{2.17c}$$

Note that in general $a_{ij}^0 \neq a_{ij}(p^0)$. The center y^0 can now be found as the solution to the nonlinear system (2.17).

There is, however, an easier way to compute y^0. Indeed, it is seen that the system (2.17) has the same structure as system (2.5). Therefore, on account of the equivalence between Problem 2.1 and Problem 2.2, y^0 can be found by first solving the eigenvalue problem

$$A^0 x = \lambda x, \tag{2.18}$$

where the elements of A^0 are given in (2.17c). Let (λ^0, x^0) denote the pair (λ_μ^0, x_μ^0) of eigenvalue and eigenvector corresponding to the real eigenvalue considered (computed by the normalization rule (2.1)). Finally, the center y^0 sought will be found through dividing x^0 by x_s^0 and replacing its sth component (equal to 1) with λ^0.

Now we can specify how to choose the index s: We require that $|x_s|$ be the maximum among all the magnitudes $|x_i|$, $i \in J$. This choice seems to be the best in order to ensure Assumption 2.1.

Next, we have to compute the radius r_v of the outer interval solution sought. On account of (2.16) and (2.17), consider the system

$$\begin{aligned} &\sum_{j\in J'} a_{ij}^0 v_j - y_s^0 v_i - y_i^0 v_s = b_i, \qquad i \in J', \\ &\sum_{j\in J'} a_{sj}^0 v_j - v_s = b_s, \end{aligned} \tag{2.19a}$$

$$\begin{aligned} b_i \;=\; &-\left(\sum_{j\in J'} y_j^0 \omega_{ij}^T + \omega_{is}^T\right) u - t_{is} - \sum_{j\in J'} y_j^0 t_{ij} \\ &- \sum_{j\in J'} t_{ij} v_j - \sum_{j\in J'} \omega_{ij}^T u v_j + h_i'(v), \qquad i \in J, \end{aligned} \tag{2.19b}$$

$$h_i'(v) = \begin{cases} v_s v_i, & i \in J', \\ 0, & i \in s, \end{cases} \tag{2.19c}$$

$$u \in \boldsymbol{u}, \qquad t_{ij} \in \boldsymbol{t}_{ij}. \tag{2.19d}$$

Let $T = \{t_{ij}\}$ and $\boldsymbol{T} = \{\boldsymbol{t}_{ij}\}$; system (2.19) can be written in matrix form

$$\tilde{A}^0 v = b(u, T, v), \qquad u \in \boldsymbol{u}, \;\; T \in \boldsymbol{T}, \tag{2.20}$$

where the elements of $\tilde{A}^0$ are the corresponding coefficients before the unknowns in the left-hand side of (2.19a) while the elements b_i of $b(u, T, v)$ are given in (2.19b).

It is easily seen that $\tilde{A}^0$ can be constructed in the following manner. Form the matrix

$$A_a = A_c - y_s^0 E, \tag{2.21}$$

where E stands for the identity matrix. The auxiliary matrix A_a is then transformed by replacing its sth column A_a^s with the normalized (according to (2.3)) eigenvector $(-x^0)$ to get the new matrix $\tilde{A}^0$. We shall now show that $\tilde{A}^0$ is almost always invertible. Indeed, on account of (2.21) $\det(A_a) = 0$ since y_s^0 is an eigenvalue of A_c. On the other hand, A_a is transformed as indicated above by replacing its column A_a^s with the eigenvector $(-x^0)$. As in general $A_c^s \neq -x^0$, almost always $\det(\tilde{A}^0) \neq 0$.

Let $C = (\tilde{A}^0)^{-1}$; multiplying (2.20) by C, we get

$$v = Cb(u, T, v) =: b'(u, T, v), \qquad u \in \boldsymbol{u}, \ \ T \in \boldsymbol{T}. \tag{2.22}$$

At this point, we replace (2.22) with

$$\boldsymbol{v}' = b'(\boldsymbol{u}, \boldsymbol{T}, \boldsymbol{v}'), \tag{2.23a}$$

where $\boldsymbol{v}'$ is the fixed-point solution to

$$\boldsymbol{v} = b'(\boldsymbol{u}, \boldsymbol{T}, \boldsymbol{v}). \tag{2.23b}$$

Due to a well-known result in [1] the solution set of (2.22) is contained in $\boldsymbol{v}'$. Thus, the radius r_v sought can be found as the radius r' of $\boldsymbol{v}'$. To find r', we have first to find the radius of both sides of (2.23b). On account of (2.23), (2.22), (2.20), and (2.19) we have

$$\begin{aligned} r_i \;=\; & \sum_{k\in J} \left| c_{ik} \left(\sum_{j\in J'} y_j^0 \omega_{kj} + \omega_{ks} \right)^T \right| r_p + \sum_{k\in J} |c_{ik}| r_{g_{ks}} + \sum_{k\in J} \sum_{j\in J'} |c_{ik} y_j^0| r_{g_{kj}} \\ & + \sum_{k\in J} \sum_{j\in J'} |c_{ik} r_{g_{kj}}| r_j + \sum_{k\in J} \sum_{j\in J'} |c_{ik} \omega_{kj}^T| r_p r_j + \sum_{k\in J} |c_{ik}| h'_k(r), \quad i \in J. \end{aligned} \tag{2.24}$$

We now introduce the auxiliary column-vector d and the matrix D with elements:

$$\begin{aligned} d_i \;=\; & \sum_{k\in J} \left| c_{ik} \left(\sum_{j\in J'} y_j^0 \omega_{kj} + \omega_{ks} \right)^T \right| r_p \\ & + \sum_{k\in J} |c_{ik}| r_{g_{ks}} + \sum_{k\in J} \sum_{j\in J'} |c_{ik} y_j^0| r_{g_{kj}}, \qquad i \in J; \end{aligned} \tag{2.25a}$$

$$D_{ij} \;=\; \sum_{k\in J'} |c_{ik} \omega_{kj}|^T r_p + \sum_{k\in J'} |c_{ik}| r_{g_{ks}}, \qquad i,j \in J. \tag{2.25b}$$

Thus, (2.24) can be written in vector form as

$$r = d + Dr + |C| h'(r). \tag{2.26a}$$

Writing out (2.26a) in componentwise form

$$r_i \;=\; d_i + \sum_{j\in J} d_{ij} r_j + \sum_{j\in J} |c_{ij}| h'_j(r), \qquad i \in J; \tag{2.26b}$$

$$h'_j(r) \;=\; \begin{cases} r_s r_j, & i \in J', \\ 0, & j = s, \end{cases} \tag{2.26c}$$

it is seen that (2.26) is a system of incomplete quadratic form. The problem of how to solve that system will be postponed for Section 4.

If system (2.26) has a positive solution $r' > 0$ ($r'_i > 0$), then we can introduce the intervals

$$\boldsymbol{y}_i = y_i^0 + [-r'_i, r'_i], \qquad i \in J, \tag{2.27}$$

which solves the equivalent Problem 2.2 and, hence, the original Problem 2.1. Indeed, let $\boldsymbol{X}^*$ denote the eigenvector (associated with the μth eigenvalue $\lambda_\mu(p)$) if it is obtained using the normalization rule (2.3). More precisely, we have the following theorem.

THEOREM 2.1. *If the nonlinear system (2.26) has a positive solution $r' > 0$, then:*

(i) the interval

$$\boldsymbol{y}_s = y_s^0 + [-r'_s, r'_s] \tag{2.28a}$$

is an outer interval estimate of the range $\boldsymbol{\lambda}^*$ of the μth eigenvalue considered; the interval vector $\boldsymbol{X}$, whose μth component $\boldsymbol{X}_s = 1$ while the remaining components are

$$\boldsymbol{X}_i = \boldsymbol{y}_i, \qquad i \in J', \tag{2.28b}$$

is an outer interval enclosure of the normalized range interval vector $\boldsymbol{X}^*$ related to the μth real eigenvalue considered;

(ii) Assumption 2.1 is valid;

(iii) Assumption 1.1 is valid.

Proof. (i) Since r' is a radius, we are interested in a positive solution to (2.26). In view of the equivalence between Problem 2.1 and Problem 2.2 it suffices to prove that $\boldsymbol{y}$ defined by (2.27) is an enclosure of $\boldsymbol{y}^*$. The latter assertion follows from the following considerations. The transformation of (2.8) into (2.14) is a relaxation, i.e. the solution set S_1 of (2.8) is contained in the solution set S_2 of (2.14). This is due to the specific way (2.8) is replaced with (2.14) and the inclusion property (2.10). Further, the solution set S_2 can be put in the form $S_2 = y^0 + S_3$ where S_3 is the solution set of (2.21). On the other hand, it follows from [1] that the fixed-point solution $\boldsymbol{v}'$ to (2.23b) contains S_3. Thus, it has been proved that $S_1 \subset y^0 + \boldsymbol{v}'$. On the other hand, by the construction of the radius vector r', it is clear that r' is, in fact, the radius of $\boldsymbol{v}'$. Hence, the vector y with components (2.27) has the inclusion property (2.6). Finally, the assertion (i) of the theorem follows from the equivalence between Problem 2.1 and Problem 2.2.

(ii) Assume that there exists at least one vector $\breve{p}$ such that $x_s(\breve{p}) = 0$. We can always construct a sequence $p^{(\rho)}$ such that the distance between $\breve{p}$ and $p^{(\rho)}$ decreases monotonically with ρ, i.e. $\lim_{\rho \to \infty} p^{(\rho)} = \breve{p}$ and $x_s(p^{(\rho)}) \neq 0$ for $p^{(\rho)} \neq \breve{p}$. For small ρ, the normalization through dividing by $x_s(p^{(\rho)})$ results in normalized vectors $y(p^{(\rho)})$ of bounded width. However, as $\rho \to \infty$, $y(p^{(\rho)})$ becomes unbounded, which is in contradiction with the premises of the theorem that (2.26) has a bounded solution r'.

(*iii*) Let system (2.8) be written as

$$f(y,p) = 0, \qquad p \in \boldsymbol{p}. \tag{2.29}$$

Since $a_{ij}(p)$ are continuous functions in $\boldsymbol{p}$, so is $f(y,p)$ in $B = S_1 \times \boldsymbol{p}$. Hence, for each $z, y \in S_1$ and each $p \in \boldsymbol{p}$

$$f(z,p) - f(y,p) = S(z-y), \tag{2.30}$$

where S is the slope matrix; algorithms for determining S are given in [9], [14], [19]. Since the solution set S_1 is bounded, the difference $z - y$ is also bounded, which entails that each S is invertible to have from (2.30)

$$z - y = S^{-1}\big(f(z,p) - f(y,p)\big). \tag{2.31}$$

If both z and y satisfy (2.29), then (2.31) implies $z = y$. Hence, for all $p \in \boldsymbol{p}$, the equation (2.29) has a unique solution y. Therefore, due to the equivalence between Problem 2.1 and Problem 2.2, $\lambda_\mu(p)$ is a simple eigenvalue. Finally, on account of Lemma 1.1, the eigenvalue considered is structurally stable. □

It is seen from the foregoing that the present method for solving the original Problem 2.1 comprises, essentially the following computations. First, one solves the complete eigenvalue problem (2.18), (2.17) to find the n' pairs $(\lambda_\mu^0, \boldsymbol{x}_\mu^0)$, corresponding to real eigenvalues λ_μ^0, $\mu \in J$, $n' \leq n$. For each fixed μ, the index s is determined as explained above. Then the nonlinear (incomplete quadratic) system (2.26) is set up and solved. If its solution r' is positive, then the outer interval solution of the original eigenvalue Problem 2.1 is given by (2.27), (2.28) where y_i^0 is given by the corresponding "nominal" eigenvalue λ_μ^0.

3. Complex Eigenvalue

In this section, the method suggested in the previous section will be extended to the case of complex eigenvalues

$$\lambda_\mu(p) = \mathrm{Re}\big(\lambda_\mu(p)\big) + j\mathrm{Im}\big(\lambda_\mu(p)\big). \tag{3.1}$$

The original problem is now stated.

PROBLEM 3.1. Find (for a fixed μ) an outer interval solution $(\boldsymbol{\lambda}_{\mathrm{Re}}, \boldsymbol{X}_{\mathrm{Re}}, \boldsymbol{\lambda}_{\mathrm{Im}}, \boldsymbol{X}_{\mathrm{Im}})$ having the inclusion property (1.7).

Let the "nominal" eigenvalue and the respective eigenvector be

$$\lambda^0 = \lambda_{\mathrm{Re}}^0 + j\lambda_{\mathrm{Im}}^0, \qquad x_i^0 = x_{i,\mathrm{Re}}^0 + jx_{i,\mathrm{Im}}^0, \qquad i \in J. \tag{3.2}$$

Let the index s have already been chosen (the actual choice of s and the determination of the "nominal" pair (λ^0, x^0) will be given later on). Also, let α be the quotient of

$x^0_{s,\mathrm{Im}}/x^0_{s,\mathrm{Re}}$. The eigenvector x^0 is now normalized through dividing all components of x^0 by $x^0_{s,\mathrm{Re}}$; thus

$$x^0_{s,\mathrm{Re}} = 1, \tag{3.3a}$$
$$x^0_{s,\mathrm{Im}} = \alpha. \tag{3.3b}$$

This choice is natural, if

$$|x^0_{s,\mathrm{Re}}| \geq |x^0_{s,\mathrm{Im}}|; \tag{3.4}$$

otherwise it is preferable to normalize x^0 through dividing it by $x^0_{s,\mathrm{Im}}$. Assuming (3.3), we are led to make the following assumption.

ASSUMPTION 3.1. The component

$$x_{s,\mathrm{Re}}(p) \neq 0, \qquad \forall p \in \boldsymbol{p}, \tag{3.5}$$

so the eigenvector considered is normalized to have

$$x_{s,\mathrm{Re}}(p) = 1, \qquad \forall p \in \boldsymbol{p}, \tag{3.6a}$$
$$x_{s,\mathrm{Im}}(p) = \alpha, \qquad \forall p \in \boldsymbol{p}. \tag{3.6b}$$

We now introduce the $2n$-dimensional real vector y with components

$$\begin{aligned} y_i &= x_{i,\mathrm{Re}}(p), & i \in J', \qquad y_s &= \lambda_{\mathrm{Re}}(p), \\ y_{i+n} &= x_{i,\mathrm{Im}}(p), & i \in J', \qquad y_{s+n} &= \lambda_{\mathrm{Im}}(p). \end{aligned} \tag{3.7}$$

On substitution of (3.2) into (1.1) and on account of (3.6) and (3.7), the following system is obtained

$$\begin{aligned} \sum_{j\in J'} a_{ij}(p)y_j - y_s y_i + a_{is}(p) + y_{s+n}y_{i+n} &= 0, \qquad i \in J', \\ \sum_{j\in J'} a_{sj}(p)y_j - y_s + a_{ss}(p) + \alpha y_{s+n} &= 0, \\ \sum_{j\in J'} a_{ij}(p)y_{j+n} - y_s y_{i+n} + \alpha a_{is}(p) - y_{s+n}y_i &= 0, \qquad i \in J', \\ \sum_{j\in J'} a_{sj}(p)y_{j+n} - \alpha y_s + \alpha a_{ss}(p) - y_{s+n} &= 0, \end{aligned} \tag{3.8a}$$

$$p \in \boldsymbol{p}. \tag{3.8b}$$

We consider the following problem related to (3.8).

PROBLEM 3.2. Find an outer solution $\boldsymbol{y}$ to (3.8), i.e. a $2n$-dimensional interval vector $\boldsymbol{y}$ enclosing the range vector $\boldsymbol{y}^*$:

$$\boldsymbol{y}^* \subset \boldsymbol{y}. \tag{3.9}$$

Obviously, the solution $\boldsymbol{y}$ to Problem 3.2 is a solution to the original Problem 3.1.

To solve Problem 3.2, we appeal to the same approach as that adopted in solving Problem 2.1. Again, we start by introducing the nonlinear vector function $h^0(y)$ with components

$$h_i^0(y) = \begin{cases} -y_s y_i + y_{s+n} y_{i+n}, & i \in J', \\ -y_s + \alpha y_{s+n}, & i = s, \\ -y_s y_{i+n} - y_{s+n} y_i, & i \in J', \\ -\alpha y_s - y_{s+n}, & i = s+n. \end{cases} \tag{3.10}$$

System (3.8) is then written compactly as

$$\begin{aligned} \sum_{j \in J'} a_{ij}(p) y_j + h_i^0(y) + a_{is}(p) &= 0, \qquad i \in J, \\ \sum_{j \in J'} a_{ij}(p) y_{j+n} + h_{i+n}^0(y) + \alpha a_{is}(p) &= 0, \qquad i \in J, \end{aligned} \tag{3.11a}$$

$$p \in \boldsymbol{p}. \tag{3.11b}$$

It is seen that system (3.11) has the same structure as system (2.8). Next, as in Section 2, the functions $a_{ij}(p)$ in (3.11) are linearized by (2.13). Substituting (2.13) into (3.11a), we get a relaxed system of the type (2.14) but of double the size. Using the centered form (2.15) of the known variables, we first determine the "center" y^0 by finding the complex solutions of the eigenvalue problem

$$A^0 x = \lambda x, \tag{3.12}$$

where A^0 is defined as in Section 2.

Following exactly the same approach as in the previous section, we next set up a $2n$-dimensional system of the type (2.24). Its vector form will be (just as (2.26))

$$r = d + Dr + |C| h(r), \tag{3.13}$$

where r is now a $2n$-dimensional vector while d, D, C, and $h(r)$ are *mutandis mutandi*, obtained in the same manner as in Section 2. Finally, if the solution r' to (3.13) is positive, we construct the intervals

$$\boldsymbol{y}_i = y_i^0 + [-r_i', r_i'], \qquad i = 1, \ldots, 2n. \tag{3.14a}$$

Unlike the real eigenvalue case, however, we now have to impose the additional requirement

$$r_{n+s}' \leq |y_{n+s}^0|. \tag{3.14b}$$

This condition ensures that the transformation of (1.1) to (3.8) remains valid for all $p \in \boldsymbol{p}$. Indeed, if (3.14b) is violated, $y_{n+s}(p)$ may change its sign for some $p \in \boldsymbol{p}$, which is inadmissible because of the normalization rule (3.3b) and (3.6b). Thus we have the following theorem.

THEOREM 3.1. *If the nonlinear system (3.13) has a positive solution $r' > 0$ and (3.14b) is fulfilled, then* *(i) the interval*

$$\boldsymbol{y}_s = y_s^0 + [-r'_s, r'_s] \tag{3.15a}$$

is an enclosure for the range $\boldsymbol{\lambda}^*_{\mathrm{Re}}$ *of the real part of the complex eigenvalue* $\lambda_\mu(p)$ *considered and the interval*

$$\boldsymbol{y}_{s+n} = y_{s+n}^0 + [-r'_{s+n}, r'_{s+n}] \tag{3.15b}$$

is an enclosures for the range $\boldsymbol{\lambda}^*_{\mathrm{Im}}$ *of the imaginary part of* $\lambda_\mu(p)$*; the* $2n$*-dimensional interval vector* $\boldsymbol{X}$*, whose sth component* $X_s = 1$ *and* $(s+n)$*-th component* $X_{s+n} = \alpha$ *while the remaining components are*

$$\boldsymbol{X}_i = \boldsymbol{y}_i, \qquad i \in J', \tag{3.15c}$$

$$\boldsymbol{X}_{i+n} = \boldsymbol{y}_{i+n}, \qquad i \in J', \tag{3.15d}$$

is an outer enclosure for the real and imaginary parts of the normalized interval $\boldsymbol{X}^*$ *related to the* μ*th complex eigenvalue considered;*

(ii) Assumption 3.1 is valid;

(iii) Assumption 1.1 is valid.

The proof of assertions (*i*) and (*ii*) in Theorem 3.1 is similar to that of Theorem 2.1. The proof of (*iii*) is a corollary to (*ii*). Indeed, the validity of (3.14b) entails that $\boldsymbol{\lambda}^*_{\mu,\mathrm{Im}}$ is either a positive or negative interval. Hence, the eigenvalue $\lambda_\mu(p)$ considered remains a complex number for all $p \in \boldsymbol{p}$.

4. Solving the Incomplete Quadratic System

In this section, we are concerned with the solution of system (2.26). Since this is a system of polynomial type, all real solutions of (2.26) could be found by a continuation method [15] or using a special interval method [6]. An alternative method for solving (2.26) will be suggested here that is much simpler for implementation and seems to be more effective than either method mentioned above. The new method exploits to the full the specific structure of system (2.26). The only requirement for its applicability in the invertibility of matrix $|C|$.

Indeed, if $|C|$ is regular, then (2.26) can be written in the form

$$Ar = b + h'(r). \tag{4.1}$$

For simplicity of notation, we assume that $s = n$; also, we will need the index sets $I = \{1, \ldots, n\}$, $I' = \{1, \ldots, n-1\}$ and $J = \{2, \ldots, n\}$, $J' = \{2, \ldots, n-1\}$. The system (4.1) is then written as follows

$$\sum_{j \in I} a_{ij} r_j = b_i + r_i r_n, \qquad i \in I', \tag{4.2a}$$

$$\sum_{j \in I} a_{nj} r_{j_i} + a_{nn} r_n = b_n. \tag{4.2b}$$

The present method for solving (4.2) comprised the following three stages.

Stage 1 (Forward sweep).

It turns out that system (4.2) can be transformed into the following form

$$Ur = q(r_n). \tag{4.3}$$

Here U is an upper triangular matrix while each component $q_i(r_n)$ of the column vector $q(r_n)$ is a polynomial in r_n. For $i = 1$ to $i = n-2$, $q_i(r_n)$ is a polynomial of degree $i+1$; for $i = n-1$ and $i = n$, $q_i(r_n)$ is a polynomial of degree n. The transformation of (4.2) into (4.3) is carried out in $n-1$ steps.

Step 1. First, multiply each ith equation in (4.2) by a_{ni} to have

$$\sum_{j\in I} \alpha_{1j} r_j = \beta_1 + a_{n1} r_1 r_n, \tag{4.4a}$$

$$\sum_{j\in I} \alpha_{ij} r_j = \beta_i + a_{ni} r_i r_n, \qquad i = 2, \ldots, n-1. \tag{4.4b}$$

Next, add each equation in (4.4b) to the equation in (4.4a) to obtain

$$\sum_{j\in I} \alpha'_{1j} r_j = \beta'_1 + \left(\sum_{j\in I'} a_{nj} r_j \right) r_n. \tag{4.5}$$

From (4.2b)

$$\sum_{j\in I'} a_{nj} r_j = b_n - a_{nn} r_n \tag{4.6}$$

and on substituting (4.6) into (4.5) we get

$$\sum_{j\in I} \alpha'_{1j} r_j = \beta'_1 + b_n r_n - a_{nn} r_n^2 = \beta_{10} + \beta_{11} r_n + \beta_{12} r_n^2. \tag{4.7}$$

Thus, we have managed to modify the first equation in (4.2a) involving the term $r_1 r_n$ in such a way that the right-hand side of equation (4.7) contains only r_n and r_n^2.

Now consider the system

$$\sum_{j\in I} a'_{1j} r_j = \beta_{10} + \beta_{12} r_n^2, \tag{4.8a}$$

$$\sum_{j\in I} a_{ij} r_j = b_i + r_i r_n, \qquad i = 2, \ldots, n-1, \tag{4.8b}$$

$$\sum_{j\in I} a_{nj} r_j = b_n, \tag{4.8c}$$

(where $a'_{1j} = \alpha'_{1j}$ for $j \in I'$ and $a'_{1n} = \alpha'_{1j} - \beta_{11}$). Our next goal is to eliminate the first column in (4.8b), (4.8c). To this end, we apply a Gaussian elimination (assuming

$a'_{11} \neq 0$) to obtain

$$\sum_{j \in I} a'_{1j} r_j = \beta_{10} + \beta_{12} r_n^2, \tag{4.9a}$$

$$\sum_{j \in J} a'_{1j} r_j = b'_1 + r_i r_n + \beta_{i2} r_n^2, \qquad i = 2, \dots, n-1, \tag{4.9b}$$

$$\sum_{j \in J} a'_{nj} r_j = b'_n + \beta_{n2} r_n^2. \tag{4.9c}$$

This completes the first step of Stage 1.

Step 2. Now we consider system (4.9b), (4.9c). As in Step 1, we first modify its first equation multiplying each equation in (4.9b) by the corresponding coefficient a'_{ni} and summing up all resulting equations, we obtain

$$\sum_{j \in J} \alpha'_{2j} r_j = \beta'_2 + \left(\sum_{j \in J'} a'_{nj} r_j \right) r_n + \beta'_{22} r_n^2. \tag{4.10}$$

Replacing the sum in (4.10) with the corresponding expression in (4.9c) and shifting the emerging r_n term from the right hand side to the left hand side, we get

$$\sum_{j \in I'} a''_{2j} r_j = \beta_{20} + \beta_{22} r_n^2 + \beta_{23} r_n^3. \tag{4.11a}$$

We see that now the right-hand side of (4.11a) is a polynomial in r_n of third degree. Adding the remaining equations

$$\sum_{j \in J} a'_{ij} r_j = b'_i + r_i r_n + \beta_{i2} r_n^2, \qquad i = 3, \dots, n-1, \tag{4.11b}$$

$$\sum_{j \in J} a'_{nj} r_j = b'_n + \beta_{n2} r_n^2, \tag{4.11c}$$

we form system (4.11). Now assuming $a''_{22} \neq 0$ we apply a Gaussian elimination to (4.11) to eliminate the first column in (4.11b), (4.11c). At the end of the second step, we get the system

$$\sum_{j \in J} a''_{2j} r_j = \beta_{20} + \beta_{22} r_n^2 + \beta_{23} r_n^3, \tag{4.12a}$$

$$\sum_{j \in I'} a''_{ij} r_j = b''_i + r_i r_n + \beta_{i2} r_n^2 + \beta_{i3} r_n^3, \qquad i = 3, \dots, n-1, \tag{4.12b}$$

$$\sum_{j \in J'} a''_{nj} r_j = b''_n + \beta_{n2} r_n^2 + \beta_{n3} r_n^3. \tag{4.12c}$$

The following steps are similar to Step 2. We shall consider only the last step.

Step $n-1$. At the end of the $(n-2)$-th step, we have a reduced system of the form

$$a_{n-1,n-1}^{(n-2)} r_{n-1} + a_{n-1,n}^{(n-2)} r_n = b_{n-1}^{(n-2)} + r_{n-1} r_n + q'_{n-1}(r_n), \tag{4.13a}$$

$$a_{n,n-1}^{(n-2)} r_{n-1} + a_{n,n}^{(n-2)} r_n = b_n^{(n-2)} + q'_n(r_n). \tag{4.13b}$$

We first multiply (4.13a) by $a_{n,n-1}^{(n-2)}$. Then we replace $a_{n,n-1}^{(n-2)} r_{n-1}$ in (4.13a) with $b_n^{(n-2)} - a_{n,n}^{(n-2)} r_n + q'_n(r_n)$ to obtain the system

$$a_{n-1,n-1}^{(n-2)} r_{n-1} + (a_{n-1,n}^{(n-2)} - b_n^{(n-2)}) r_n = b_{n-1}^{(n-2)} + q''_{n-1}(r_n), \tag{4.14a}$$

$$a_{n,n-1}^{(n-2)} r_{n-1} + a_{n,n}^{(n-2)} r_n = b_n^{(n-2)} + q'_n(r_n), \tag{4.14b}$$

in which the right-hand side contains only r_n: the polynomials $q''_{n-1}(r_n)$ of nth degree and $q'_n(r_n)$ of $(n-1)$-th degree.

Now we apply a Gaussian elimination to (4.14) and get

$$a_{n-1,n-1}^{(n-1)} r_{n-1} + a_{n-1,n}^{(n-1)} r_n = q_{n-1}(r_n), \tag{4.15a}$$

$$a_{n,n}^{(n-1)} r_n = q_n(r_n). \tag{4.15b}$$

It is seen that the last equation (4.15b) contains only the variable r_n. It can be written in the form

$$q(r_n) = q_n(r_n) - a_{n,n}^{(n-1)} r_n = 0, \tag{4.16}$$

where $q(r_n)$ is a polynomial of nth degree.

This completes the first stage of the method.

Remark 4.1. To insure better accuracy, the forward sweep of the present method must be implemented with pivoting at each step as this is done in the actual implementation of the Gaussian elimination scheme. The only difference is that now the pivot element can be chosen only among the elements of the first row of the matrix $A^{(k)}$ except for the last element of the row at each iteration k since another choice would destroy the specific structure of the equations involved.

Stage 2. We solve (4.16) for all real solutions $r_n^{(m)}$, $m = 1, \ldots, M$, $M \le n$. This can be done in a reliable way using an appropriate interval method, e.g. the method from [6, Section 3]. Let M^0 denote the number of positive solutions. If $M^0 = 0$, terminate in outcome A: The system (2.26) has no positive solution; otherwise order the positive solutions in increasing value (i.e. $r_n^{(1)} \le r_n^{(2)} \le \cdots \le r_n^{(M^0)}$) and go to the next stage.

Stage 3 (Backward sweep). For $m = 1$, we compute $q_{n-1}(r_n^{(m)})$ and the corresponding $r_{n-1}^{(m)}$, $q_{n-2}(r_n^{(m)})$ and the corresponding $r_{n-2}^{(m)}$ and so on until we compute $r_1^{(m)}$ from (4.9a). If all components $r_i^{(1)} > 0$, terminate in outcome B: A positive solution to system (2.26) has been found. Otherwise, if $r_i^{(1)} < 0$ for some i, resume the backward sweep with $m = 2$, $m = 3$ and so on until either outcome A or outcome B occurs.

The present approach can by extended to tackle the system (3.13) related to the case of complex eigenvalues.

5. Numerical Example

We consider the following matrix $A(p)$ (arising in the field of electric circuits [8, Chapter 4]) with elements:

$$\begin{aligned} a_{11}(p) &= -\frac{1}{p_4}\left(\frac{p_2p_3}{p_2+p_3}+p_1\right), & a_{12}(p) &= -\frac{1}{p_4}\left(\frac{p_2}{p_2+p_3}-1\right), \\ a_{21}(p) &= \frac{p_3}{p_5(p_2+p_3)}, & a_{22}(p) &= \frac{1}{p_5(p_2+p_3)}. \end{aligned} \tag{5.1a}$$

The intervals of the parameters involved are:

$$\begin{aligned} \boldsymbol{p}_1 &= \boldsymbol{p}_3 = [99, 101], & \boldsymbol{p}_2 &= [198, 202], \\ \boldsymbol{p}_4 &= [0.49, 0.51], & \boldsymbol{p}_5 &= [240, 260]\cdot 10^{-6}. \end{aligned} \tag{5.1b}$$

The affine interval approximation of the elements $a_{ij}(p)$ in (5.1) was obtained using the algorithm of [7]: Following t

$$\begin{aligned} \boldsymbol{L}_{11}(p) &= -0.6700p_1 + 0.0001633p_2 - 1.334p_3 + 400.2p_4 - 2001 \\ &\quad + [-0.268, 0.268], \\ \boldsymbol{L}_{12}(p) &= 0.002223p_2 - 0.004446p_3 + 1.3339p_4 - 1.3338 \\ &\quad + 10^{-4}[-6.78, 6.78], \\ \boldsymbol{L}_{21}(p) &= -4.448p_2 + 8.896p_3 - 5.34\cdot 10^6 p_5 + 2669 \\ &\quad + [-3.161, 3.161], \\ \boldsymbol{L}_{22}(p) &= 0.0445p_2 + 0.0445p_3 + 5.347\cdot 10^4 p_5 - 40.085 \\ &\quad + [-0.02234, 0.02234]. \end{aligned} \tag{5.2}$$

Using (2.17), (5.1b), and (5.2), the matrix A^0 in (2.18) can now be computed. The eigenvalues λ_1^0 and λ_2^0 of A^0 for this example are both real:

$$\lambda_1^0 = -195.47, \qquad \lambda_2^0 = -18.245. \tag{5.3}$$

We consider the problem of finding an outer interval enclosure of $\lambda_2(p)$ and of the corresponding eigenvector $x^{(2)}(p)$, $p \in \boldsymbol{p}$, when using the normalization rule (2.3), i.e. Problem 2.1. We solve the outer solution problem considered applying the method from Section 2.

The eigenvector $x^{(0)}$, corresponding to $\lambda^0 = \lambda_2^0$, is

$$x^0 = (-0.0037, -1.0000)^T. \tag{5.4}$$

Thus, the index s (the index of the maximum magnitude component of $x^{(0)}$) is $s = 2$. Hence, from (5.3) and (5.4)

$$y^0 = (-0.0037, -18.245)^T. \tag{5.5}$$

Following the computational scheme of the method, we set up the system for determining the radius r' of the outer solution, which now is

$$\begin{aligned} -0.988r_1 - 0.0000207r_2 - 0.00565r_1r_2 + 0.0000644 &= 0, \\ 45.050r_1 - 2.0277r_2 - 7.5434r_1r_2 + 1.0033 &= 0. \end{aligned} \tag{5.6}$$

The solution r' to (5.6) is

$$r' = (0.0003, 1.2354)^T. \tag{5.7}$$

First, it was obtained by the simple iteration method with starting point $r^{(0)} = (0, 0)$. The solution (5.7) was also computed using the method from Section 4. Since $r' > 0$, the outer solution $\boldsymbol{y}$, according to (2.27), (5.5), and (5.7) is

$$\boldsymbol{y} = ([-0.0034, -0.0040], [-19.4804, -17.0096])^T. \tag{5.8}$$

Since $s = 2$, we finally have

$$\boldsymbol{\lambda}_2 = [-19.4804, -17.0096], \tag{5.9}$$
$$\boldsymbol{X}^{(2)} = ([-0.0034, -0.0040], 1)^T \tag{5.10}$$

for the outer bounds on the eigenvalue $\lambda_2(p)$ and $x^2(p)$, respectively, when $p \in \boldsymbol{p}$.

The new method has been implemented in MATLAB 5.6 environment. The final numerical results (5.9), (5.10) are only reported to four decimal places since no directed roundings have been used.

6. Conclusion

The problem of determining an outer interval solution to the parametric eigenvalue problem (1.1) formulated in the general form of continuous (not necessarily smooth) parameter dependencies (1.2) has been considered.

A method for solving the problem considered in the case of real eigenvalues has been suggested in Section 2. A generalization has been presented in Section 3 for the case of complex eigenvalues. The method suggested reduces, essentially, to solving a system of n nonlinear equations (2.26) or $2n$ nonlinear equations (3.13) for each real or complex eigenvalue, respectively. The systems (2.26) and (3.13) have the same structure of incomplete quadratics forms. The method is applicable if the respective system has a positive solution.

A simple three-stage procedure for solving the nonlinear system (2.26) has been exposed in Section 4.

A numerical example (5.1) has been solved to illustrate the applicability of the method suggested. In its present form, the computer program developed, however, does not account for round-off errors in the computations involved. To insure reliable results, a self-validating version of the new method is to be implemented. This can be done using the toolbox Intlab [17] of MATLAB.

This paper's method can be extended to provide an outer interval solution to the generalized eigenvalue problem.

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