ARTICLE

Mechanized analysis of Anselm's modal ontological argument

John Rushby[1]

Abstract

We use a mechanized verification system, PVS, to examine the argument from Anselm's Proslogion Chapter III, the so-called "Modal Ontological Argument." We consider several published formalizations for the argument and show they are all essentially similar. Furthermore, we show that the argument is trivial once the modal axioms are taken into account. This work is an illustration of Computational Philiosophy and, in addition, shows how these methods can help detect and rectify errors in modal reasoning.

Introduction

The Ontological Argument is a proof of the existence of God presented by St. Anselm of Canterbury in his *Proslogion* (Anselm 1078). It has been studied and debated by philosophers and theologians ever since.

The Ontological Argument has traditionally been identified with Chapter II of the Proslogion, but in the middle years of the last century philosophers noted that Chapter III presents an argument that could be interpreted as an alternative proof for God's existence. This discovery is generally attributed to Malcolm (1960) although Hartshorne had made similar observations earlier. (Hartshorne (1965, Part 1, Section 8) gives some of the history, and also Smith (2014, Introduction).) The discovery was that the argument of Chapter III can be seen as an independent, self-contained proof; previously, and for many commentators still today (Campbell 2018, 2019; Sobel 2003, p. 82), Chapter III is seen as a continuation of the argument of Chapter II.

We reproduce an English translation of Chapter III in Fig. 1. Its argument, which is found in its second paragraph and highlighted in italics, is couched in terms of

✉ John Rushby
Rushby@csl.sri.com

1 Computer Science Laboratory, SRI International, 333 Ravenswood Ave, Menlo Park, CA 94025, USA

God cannot be conceived not to exist. God is that, than which nothing greater can be conceived. That which can be conceived not to exist is not God.

And it assuredly exists so truly, that it cannot be conceived not to exist. *For, it is possible to conceive of a being which cannot be conceived not to exist; and this is greater than one which can be conceived not to exist. Hence, if that, than which nothing greater can be conceived, can be conceived not to exist, it is not that, than which nothing greater can be conceived. But this is an irreconcilable contradiction. There is, then, so truly a being than which nothing greater can be conceived to exist, that it cannot even be conceived not to exist; and this being you are, O Lord, our God.*

So truly, therefore, do you exist, O Lord, my God, that you can not be conceived not to exist; and rightly. For, if a mind could conceive of a being better than you, the creature would rise above the Creator; and this is most absurd. And, indeed, whatever else there is, except you alone, can be conceived not to exist. To you alone, therefore, it belongs to exist more truly than all other beings, and hence in a higher degree than all others. For, whatever else exists does not exist so truly, and hence in a less degree it belongs to it to exist. Why, then, has the fool said in his heart, there is no God (Psalms xiv. 1), since it is so evident, to a rational mind, that you do exist in the highest degree of all? Why, except that he is dull and a fool?

Fig. 1 The complete Chapter III of St. Anselm's Proslogion (1078)

beings whose existence is *necessary* (i.e., "that which cannot be conceived not to exist") and has come to be known as Anselm's *Modal* Argument, since qualifiers such as "necessary" (with dual "possible") indicate *modes* of truth. Philosophers since Aristotle have attempted to construct specialized logics for reasoning about these but modern *modal logics* were developed only in the twentieth century (Cresswell and Hughes 1996).

In this paper, I subject several published formalized variants of Anselm's Modal Argument to mechanized analysis in the PVS Verification system; specifically, I consider versions by Eder and Ramharter (2015, Sect. 4.1), Kane (1984), Malcolm (1960) as reported by Hartshorne (1961), Adams (1971) as reported by Matthews (2005), and Hartshorne (1962, pp. 50–51). This work is an elementary example of *Computational Philosophy* (Grim and Singer 2020), that is, application of mechanized computation and reasoning to philosophical topics.

There are several other applications of automated reasoning to ontological arguments, including first and higher-order treatments of Anselm's traditional argument (Rushby 2020; Oppenheimer and Zalta 2011; Rushby 2013), modal treatments of that argument (Rushby 2019), and Gödel's modal argument (Benzmüller and Woltzenlogel-Paleo 2014). These other applications use moderately advanced logical constructions, such as definite descriptions (Oppenheimer and Zalta 2011; Rushby 2013), higher order logic (Rushby 2020), and first order modal logic (Rushby 2019; Benzmüller and Woltzenlogel-Paleo 2014), whose mechanized support is fairly intricate and prone to errors (Garbacz 2012).

In contrast, Anselm's modal argument requires only propositional modal logic. However, this is mechanized by embedding it in the classical logic of a verification system, which requires some skill. Nonetheless, these are among the simplest examples of Computational Philosophy and can serve as an introduction to the field. I will show, however, that despite its simplicity, this exercise delivers useful and novel results of both a positive and negative kind—the latter being to expose a lamentable error in a paper published in this very journal.

The structure of the paper is as follows. Section 2 provides a short introduction to the embedding of propositional modal logic in PVS, which is the system that provides the mechanization used here; the five variants of the Argument are presented and analyzed in Sect. 3; Sect. 4 discusses the value of this exercise and some related analyses, and Sect. 5 provides conclusions.

Propositional modal logic in PVS

Modal logics allow reasoning about various *modes* of truth: for example, what it means for something to be *necessarily* true, or to *know* that something is true as opposed to merely *believing* it.

The modal *qualifier* $\Box$ and its dual $\Diamond$ (defined as $\neg\Box\neg$)[1] are used to indicate expressions that should be interpreted modally. All modal logics share the same basic structure but they employ different sets of axioms and make other adjustments according to the mode attributed to the qualifiers. For example, in an *Alethic* modal logic, where $\Box$ is interpreted as necessity, we will expect the formula $\Box P \supset P$ to hold: if something is necessarily true, then it should be true. But we would not expect this formula to hold in a *Doxastic* logic, where $\Box$ is interpreted as belief. Instead, we might expect $\Box P \supset \Diamond P$ to hold: if I *believe* P is true, then I cannot also believe it to be false (reading $\Diamond P$ as $\neg\Box\neg P$). There is a collection of formulas such as these that have standard names (the two above are called T and D, respectively) and that are used in various combinations to axiomatize different modal logics. Some of the common ones are listed below.

K $\Box(P \supset Q) \supset (\Box P \supset \Box Q)$ this formula is true in all modal logics,

T $\Box P \supset P$,

D $\Box P \supset \Diamond P$,

B $P \supset \Box\Diamond P$,

4 $\Box P \supset \Box\Box P$,

5 $\Diamond P \supset \Box\Diamond P$.

The semantics of modal logics are interpreted relative to a set of *possible worlds*, so that $\Box p$ means true in *all* worlds and $\Diamond p$ means true in *some* world. To represent different interpretations for the modalities, we impose an *accessibility* relation on possible worlds and refine the statements above to true in *all accessible* and *some accessible* worlds, respectively.

There is a relationship between the standard modal axioms and properties of the accessibility relation. For example, T and D, mentioned above, correspond to accessibility relations that are reflexive and serial, respectively. The modal ontological argument is expressed in Alethic modal logic, and such logics have accessibility relations that are equivalence relations. This corresponds to the combination of modal axioms T (reflexive) and either 5 (Euclidean) or 4 (transitive) and B (symmetric).[2]

---

[1] We use $\neg$ for negation, $\wedge$ for conjunction, $\vee$ for disjunction and $\supset$ for material implication.

[2] The modal logic with no accessibility relation has the same theorems as one whose accessibility relation is an equivalence.

We will undertake our examination of the modal ontological argument using the PVS Verification System. Verification systems are tools from computer science that are generally used for exploration and verification of software or hardware designs and algorithms; they comprise a specification language, which is essentially a rich logic, and a collection of powerful deductive engines (e.g., satisfiability solvers for combinations of theories, model checkers, and automated and interactive theorem provers). In particular, PVS has a specification language based on higher-order logic and its proof automation is guided interactively. It is generally applied to analysis and verification of computational systems and has more than 4000 citations. We do not describe PVS in detail here but we do try to provide enough information to make this presentation self contained; see PVS (constantly updated) for the PVS system description and Rushby (2018, 2020, 2019) for previous applications to the (traditional, Proslogion Chapter II) Ontological Argument.

There is a standard translation from modal logic to classical first- or higher-order logic (Wikipedia 2019) and we use this to provide a *shallow embedding* into the dependently-typed higher-order classical logic of PVS in a way that preserves much of its proof automation. The embedding is described in tutorial detail elsewhere (Rushby 2017). The core of the embedding is shown in Fig. 2; the basic idea is that modal expressions are "lifted" so that they become functions on worlds. The function `val(x)(w)` provides the valuation for propositional constant `x` in world `w`, and `~`, `&`, , and `=>` provide the "lifted" versions of negation, conjunction, disjunction, and material implication, respectively. We then define the modalities as described above, and define modal validity as truth in all worlds.

When we write a modal formula such as `<>P => P` in PVS, we really intend its classical embedding, `valid(<>P => P)`, and this is taken care of automatically using the PVS `CONVERSION`s (see Rushby 2017) specified at the bottom of the theory, and in some of the theories that appear later.

Reconstructions of the modal ontological argument

We will examine five reconstructions of Anselm's modal argument. As we will discover, they are all variants of the same rather trivial deduction.

The version by Eder and Ramharter

I will begin with the version of Eder and Ramharter (2015, Sect. 4.1). They attribute this to Hartshorne (1962, pp. 49–51) but my reading of Hartshorne's version is somewhat different and I will examine that later.[3] The argument is presented in an *Alethic* modal logic: that is, one where the modalities express *necessity* ($\Box$) and *possibility* ($\Diamond$), respectively.

---

[3] Hartshorne uses the arrow $\rightarrow$ to mean *strict* implication, so his version of Hartshorne (1962, pp. 49–51) is that examined in Sect. 3.5. However, elsewhere, Hartshorne (1965, p. 97) does state "Anselm's Principle" in the form used by Eder and Ramharter.

```
shallow_pml: THEORY          ┌──────────────┐
BEGIN                        │     PVS      │
                             └──────────────┘

  worlds: TYPE+
  access: pred [[worlds, worlds]]
  pmlformulas: TYPE = [worlds -> bool]
  pvars: TYPE+

  v, w: VAR worlds
  x, y: VAR pvars
  P, Q: VAR pmlformulas

  val(x)(w): bool

  ~(P)(w): bool = NOT P(w) ;
  &(P, Q)(w): bool = P(w) AND Q(w) ;
  \/(P, Q)(w): bool = P(w) OR Q(w) ;
  =>(P, Q)(w): bool = P(w) IMPLIES Q(w) ;

  □(P)(w): bool = FORALL v: access(w, v) IMPLIES P(v) ;
  <>(P)(w): bool = EXISTS v: access(w, v) AND P(v) ;
  dexpand: LEMMA <> P = ˜□˜ P ;

  |=(w, P): bool = P(w)
  valid(P): bool = FORALL w: w |= P

  validval(x: pvars): bool = valid(val(x))
  CONVERSION valid, val

END shallow_pml
```

Fig. 2 Shallow Embedding of Propositional Modal Logic in PVS

Definition: *g* is an abbreviation for "there exists a being than which there is none greater" (i.e., God).

Premise ER1: $g \supset \Box g$ (i.e., if God exists, then he exists necessarily).

Eder and Ramharter refer to ER1 as *Anselm's Principle* and state that a more precise reading is "if God exists, then his nonexistence is inconceivable" (i.e., reading $\Box g$ as $\neg \Diamond \neg g$).

Premise ER2: $\Diamond g$ (i.e., God's existence is possible).

Conclusion ERC: *g* (i.e., God exists in a classical sense).

Before we proceed to examine the logic of this formulation, we note that it is abstracted quite significantly from Anselm's natural language description as given (in English translation) in the second paragraph of Fig. 1. Anselm speaks of beings that can be "conceived" to exist, and also has a relation of "greater" among beings. Eder and Ramharter's paper is largely concerned with what it means for a logical formulation to be a good reconstruction of an informal argument and they note that this formulation fails several of their criteria. We will return to some of these points in Sect. 4.

Eder and Ramharter's formulas ER1, ER2, and ERC are all rendered straightforwardly into PVS as shown in Fig. 3. The imported theory simple_pml provides the shallow embedding of propositional modal logic shown in the previous section, plus some additional theories including the standard modal axioms.

```
Eder_Ramharter: THEORY                                          PVS
BEGIN

IMPORTING simple_pml

  g: pvars
  P: VAR pmlformulas

  ER1: AXIOM g => □ g

  ER2: AXIOM <> g

  ER_triv: LEMMA symmetric?(access) IMPLIES (P => □P) IMPLIES (<>P => P)

  CONVERSION+ validval

  ERC_tmp: THEOREM g

  ERC: THEOREM symmetric?(access) => g

END Eder_Ramharter
```

Fig. 3 Eder and Ramharter's Version in PVS

Automated proof

The formula `ERC_tmp` states the basic conclusion, which we attempt to prove using the following PVS proof commands.

```
(lemma "ER1") (lemma "ER2")                         PVS proof steps
(grind :if-match nil)
(inst?)
(grind :polarity? t)
```

These install the two premises, then invoke the standard automated proof command `grind` with instantiation of variables disabled (this just unwinds the embedding of modal logic); next we perform heuristic instantiation on one of the premises, and finally use `grind` again, with variable instantiation instructed to pay attention to the polarity of variables.

This delivers the following proof sequent.

```
{-1}   access(w!1, v!1)                              PVS sequent
{-2}   val(g)(v!1)
   |-------
{1}    access(v!1, w!1)
[2]    val(g)(w!1)
```

The interpretation of a PVS sequent is that the conjunction of formulas above the |—- turnstile line should entail the disjunction of those below. Terms such as `w!1` are Skolem constants, `access` is the accessibility relation on possible worlds, and `val` is the valuation function for propositional constants, so `val(g)(v!1)` is the valuation of `g` in world `v!1`. Here, we see that the proof would be completed if line −1 implied line 1: that is, if the accessibility relation were symmetric.

Thus, we recognize that the conclusion to the argument should be modified to mention symmetry, and this explains the final form for ERC shown in Fig. 3, which is proved by the same proof steps as above.

Alternatively, we could cite the standard modal axiom B (i.e., $P \supset \Box\Diamond P$), which characterizes symmetric accessibility relations, as an additional premise. In this case, ERC_tmp is proved by the following commands. Notice that the instantiation into B is not trivial (observe the negation).

```
(lemma "ER1") (lemma "ER2")                          PVS proof steps
(lemma "B") (inst -1 "~val(g)")
(grind :if-match nil)
(inst? -2)
(grind :polarity? t)
```

As can be inferred from the PVS sequent shown on a previous page, these proofs operate directly on the possible world semantics of modal logic: essentially they expand the definitions given in Fig. 2 and then perform quantificational reasoning over the possible worlds. This is automated effectively and efficiently by the standard proof mechanization of PVS and the intermediate proof states, displayed as sequents, are fairly easy to interpret.[4] However, it does not reproduce the style of semi-formal proofs that typically accompany journal presentations. Accordingly, we now illustrate how PVS can be used to examine those kinds of proofs.

Reconstructing a manual proof

Eder and Ramharter (2015, Sect. 4.2) provide a typical semi-formal manual proof that is presented below in a slightly reorganized form.

Step 1: Applying Modal Axiom 5 to $\neg g$ and interpreting $\Diamond$ as $\neg\Box\neg$ we obtain $\neg\Box g \supset \Box\neg\Box g$; standard propositional logic then gives $\Box g \vee \Box\neg\Box g$.

Step 2: By contraposing Anselm's principle ER1 (i.e., $\neg\Box g \supset \neg g$) and instantiating Modal Axiom K with $P = \neg\Box g$ and $Q = \neg g$ we arrive at $\Box\neg\Box g \supset \Box\neg g$.

Step 3: The previous steps combine to give $\Box g \vee \Box\neg g$ which can be rewritten as $\Box g \vee \neg\Diamond\neg\neg g$ and then simplified to $\Box g \vee \neg\Diamond g$.

Conclusion: The previous step combined with God's possibility ER2 allows us to conclude $\Box g$ from which Modal Axiom T allows us to infer g.

We can reproduce this level of reasoning in PVS by stating the conclusion of each step as a lemma as shown below.

```
step1: LEMMA □g \/ □ ~□g                               PVS text

step2: LEMMA □ ~□g => □ ~g

step3: LEMMA □g \/ □ ~g
```

---

[4] Greater automation is possible using modern SMT solvers (Ge and de Moura 2009) but that can be less helpful when, as here, a proof fails and we want to understand why.

We then invite PVS to prove each of the lemmas and the conclusion `ERC_tmp` without expanding the modal qualifiers. Here is the beginning of the proof for `step1`.

```
(lemma "five") (inst - "~g")                         PVS proof steps
(grind :if-match nil :exclude ("<>" "□")
   :rewrites "dexpand")
(inst?)
```

The `:exclude` construct prevents expansion of the modal qualifiers, but note that `dexpand` will automatically rewrite $\diamond$ as $\neg\square\neg$. This brings us to the following sequent, which represents the instantiation of Modal Axiom 5 (the formula above the turnstile line) and allows us to see how this will discharge the conclusion to `step1` (the two formulas below the line). That final step is accomplished by the proof command `(grind)`. Note that the Skolem constant on worlds `w!1` comes from the definition of validity for modal constructs as true in all worlds.

```
{-1}  NOT (□~~val(g))(w!1)                            PVS sequent
      IMPLIES (□~□~~val(g))(w!1)
   |-------
{1}   (□val(g))(w!1)
{2}   (□~□val(g))(w!1)
```

Preference for one or the other style of mechanically assisted proof depends on the purpose of the exercise. If it is to examine validity and soundness of a particular argument (i.e., selection of axioms), then I think the automated approach is preferable: it is generally fast and simple. But if the purpose is to examine a specific proof (i.e., chain of inferences) then a mechanically assisted reconstruction will be the necessary choice. I will examine some specific proofs in Sect. 4 but will favor automated verification for the remainder of this section.

Returning to the automated verification of Sect. 3.1, PVS assures us that this specification of the argument is valid, so we should now consider whether the premises and assumptions are reasonable and sound.

Analysis and interpretation

We focus first on the assumption of a symmetric accessibility relation or, alternatively, the standard axiom B.

The argument makes sense only in an Alethic logic, that is one where $\square$ and $\diamond$ are interpreted as *necessary* and *possibly*, respectively, and Alethic modal logics are characterized by having accessibility relations that are equivalence relations. Hence, the assumption of symmetry in `ERC` presents no difficulty. Dually, Alethic logics are also characterized by the standard axioms T, 4, and 5—a combination that is generally referred to as S5[5]—from which B can be derived as a theorem. We conclude

---

[5] The modal formula K is often mentioned, too, and the logic is then sometimes referred to as KT45; however, K is a theorem, true in all modal logics (alternatively, it can be seen to characterize what is a modal logic), so its mention is redundant; furthermore, T and 5 imply 4, so mention of 4 is also redundant.

that the assumption of symmetry or modal axiom B is uncontroversial, provided one accepts that the Alethic logic needed here is correctly characterized by S5—which few would dispute. If in doubt, Ladstaetter (2012) provides a clear description of *logical*, *metaphysical*, and *physical* notions of necessity, and explains why logical necessity is required here, and why S5 is the right formalization.

We note that the manual proof of Sect. 3.1 used Modal Axioms 5, K, and T instead of B. However, K is true of all modal logics, and 5 and T are included in S5, and therefore this combination is acceptable in an Alethic logic.

Next, we consider the premise ER1. This seems, and is generally considered to be, a reasonable premise for a truly perfect or greatest being, such as God: if such a being existed, it would surely not do so merely contingently. Notice that this premise is specific to the constant g with the interpretation used here: the generalization $P \supset \Box P$, where P is a metavariable ranging over all propositional constants is not a plausible premise, nor is the instance where g is interpreted as a perfect island, as in Gaunilo's (1079) refutation.

The premise ER2 is discussed at length in some philosophical papers (e.g., Hartshorne 1961). One concern is that the concept of a perfect or greatest being must be non self-contradictory (otherwise it could not possibly exist), and this is not straightforward as there are properties of a perfect being that seem to conflict (e.g., omnibenevolence and perfect justice). Another concern is that non self-contradiction may be insufficient to ensure the possibility of existence. However, as we are concerned with logic and mechanization, we accept ER2 as otherwise the proof cannot proceed.

The overall argument seems fairly satisfying: we have one premise about a hypothesized God's necessary existence, another about his possible existence, and we are able to conclude his actual existence, which is something of a surprise and therefore satisfying.

However, we have the background assumption of a symmetric accessibility relation or the modal axiom B and, in the presence of this assumption, the premise ER1 reduces to $\Diamond g \supset g$. This is stated in its general form as ER_triv in Fig. 3; its proof is simply (grind :polarity? t). But now the argument is exposed as trivial:

$$\Diamond g \supset g, \Diamond g, \text{ hence } g.$$

We will examine several variants on this argument; mostly they adjust the first premise, sometimes also the conclusion (using $\Box g$ rather than simply g), and sometimes they assume modal axioms other than B. But in all cases, we will see that the argument reduces to the same trivial form.

The version by Kane

Kane (1984) presents a version of the argument that uses a slightly more complex formulation of the first premise. The interpretation of g, the second premise, and the conclusion are the same as in Eder and Ramharter's version from the previous section.

Premise K1: $\Box(g \supset \Box g)$.

Kane reads this as "Necessarily, if a perfect being exists, then necessarily a perfect being exists." He states that Hartshorne and others assume that it follows from a principle he names N: "By definition, anything which is perfect is such that, if it exists, it exists necessarily," where "the first $\Box$ in K1 corresponds to the 'by definition' of N" (Kane 1984, p. 337).

Premise K2: $\Diamond g$ (i.e., God's existence is possible).

Conclusion KC: g (i.e., God exists in a classical sense).

The heart of the PVS specification for this version is shown below.

```
 K1: LEMMA □ (g => □ g)                          Kane's version in PVS

 K_triv: LEMMA symmetric?(access)
   IMPLIES □(P => □P) IMPLIES (<>P => P)

 K2: AXIOM <> g

 KC: THEOREM symmetric?(access) => g
```

We label the PVS rendition of K1 as a LEMMA rather than an AXIOM because it can be proved from ER1; the proof is simply (grind :polarity? t). Again, proof of the conclusion requires modal axiom B or, equivalently, a symmetric accessibility relation. Given that K1 is entailed by ER1, it is unsurprising that it also reduces to $\Diamond g \supset g$, which is stated as K_triv in the PVS specification. Thus, at bottom, this version is based on the same trivial reasoning as the previous one.

Kane devotes considerable space to discussing the acceptability of modal axiom B. But, as explained in the previous section, B is a theorem of S5, which is the standard formalization for the Alethic logic that is needed to provide the desired interpretations for $\Box$ and $\Diamond$, so Kane's agonizing seems otiose.

The version of Malcolm as reported by Hartshorne

Hartshorne (1961) presents a version of the argument that he attributes to Malcolm (1960). This has the same premises as Eder and Ramharter's version, but the conclusion is $\Box g$ rather than simply g. Furthermore, rather than modal Axiom B, this version uses the modal Axiom 5, which is $\Diamond P \supset \Box \Diamond P$. Axiom 5 corresponds to a Euclidean accessibility relation[6] and is assumed in an Alethic logic.

The core of the formal specification in PVS is shown below,

```
 M1: AXIOM g => □ g                              Malcolm's version in PVS

 M_triv: LEMMA Euclidean?(access)
       IMPLIES (P => □P) IMPLIES (<>P => □P)

 M2: AXIOM <> g

 MC: THEOREM Euclidean?(access) => □ g

 MC_alt: THEOREM symmetric?(access) => □ g
```

---

[6] A relation R is Euclidean if $\forall u, v, w : R(u, v) \land R(u, w) \supset R(v, w)$.

PVS easily proves the conclusion MC, but it also proves the lemma M_triv and so again the reasoning is too trivial to claim our interest.

The conclusion can also be proved assuming modal axiom B, or alternatively a symmetric accessibility relation as stated in MC_alt: we first use the method of Eder and Ramharter's version to establish g, then a second application of the premise M1 delivers $\square g$.

The version of Adams as reported by Matthews

Adams (1971) derives a modal argument, not from *Proslogion* Chapter III, but from Anselm's reply (1079, Chapter I, paragraph 6) reply to Gaunilo (1079). Despite its different origin, this argument is the same as Kane's. However, Matthews (2005), who reports Adam's version, gives the conclusion as $\square g$. I will use this latter version as it completes the set of permutations on the first premise and conclusion.

This version of the argument in PVS is shown below.

```
A1: LEMMA □(g => □ g)                          Adams' Version in PVS

A_triv: LEMMA
    Euclidean?(access)
       IMPLIES □(P => □P) IMPLIES (<>P => □P)

A2: AXIOM <> g

AC: THEOREM Euclidean?(access) => □ g

AC_alt: THEOREM symmetric?(access) => □ g
```

As we expect, based on the other variants we have examined, this version can be proved assuming either a symmetric or Euclidean accessibility relation (i.e., standard axioms B or 5, respectively). And with a Euclidean accessibility relation, the lemma A_triv reveals that Premise A1 reduces to $\lozenge g \supset \square g$ so the conclusion follows trivially from Premise A2.

The version by Hartshorne

Hartshorne (1962, pp. 50–51) presented the first formal reconstruction of the modal argument. This uses a different and more complex formulation of the first premise than the others we have examined. The interpretation of g, the second premise, and the conclusion are the same as in the versions of Eder and Ramharter, and Kane.

Premise H1: $g \prec \square g$, where $\prec$ denotes *strict* implication.
Premise H2: $\lozenge g$ (i.e., God's existence is possible).
Conclusion HC: g (i.e., God exists in a classical sense).

We say that P strictly implies Q if it is not possible for P to be true and Q false: that is, in an Alethic logic

$$P \prec Q \stackrel{\text{def}}{=} \neg\lozenge(P \wedge \neg Q).$$

It is a theorem that strict implication is the same as necessary material implication:

$$P \Rrightarrow Q = \Box(P \supset Q).$$

This equality is a theorem of all modal logics (i.e., it requires no axioms) but it carries the intended interpretation only in Alethic logics. Thus, premise `H1` is equivalent to Kane's premise `K1` of Sect. 3.2 and the rest of the argument is also the same as Kane's. It follows that the conclusion requires the assumption of a symmetric accessibility relation or, equivalently, modal axiom B, and that under this assumption the premise `H1` reduces to $\Diamond g \supset g$ and the argument is thereby seen to be trivial.

In PVS, this is written as follows Rushby (2017), where we use $|>$ for strict implication.

```
|>(P, Q): pmlformulas =  ~ <>(P & ~ Q)            Hartshorne's version in PVS

H1: AXIOM g |> □ g

H_triv: LEMMA symmetric?(access)
  IMPLIES (P |> □P) IMPLIES (<>P => P)

H2: AXIOM <> g

HC: THEOREM symmetric?(access) IMPLIES g
```

The conclusion `HC` can be proved by the following PVS commands, whose length is due to weak quantifier reasoning in PVS.

```
(lemma "H1") (lemma "H2")                          PVS proof
(grind :if-match nil)
(inst?) (skosimp) (inst?)
(grind :polarity? t)
```

Similarly, `H_triv` is proved as follows.

```
(grind :if-match nil) (inst? -2)                  PVS proof
  (grind :polarity? t)
```

We know from Sect. 3.4 that Adams' version is the same as Kane's with the conclusion changed to $\Box g$ and the modal assumption changed to a Euclidean accessibility relation or, equivalently, modal axiom 5. Since Hartshorne's version is equivalent to Kane's we can perform the same transformations here to yield the following variant.

```
H_triv_alt: LEMMA                                Alternative Hartshorne PVS
   Euclidean?(access)
    IMPLIES (P |> □P) IMPLIES (<>P => □P)

HC_alt: THEOREM Euclidean?(access) IMPLIES □ g
```

These are proved in the same way as the originals.

The utility of these methods

In my view, the utility of the methods illustrated here, and of computational philosophy more generally, is not so much that they enable verification of hard logical problems (those abound in logic and mathematics, not so much in philosophy) but that they enable rapid and error-free exploration of multiple formulations of the same or similar problems. This allows discovery of equivalences and subtle differences, not to mention checking of validity and discovery of economical and persuasive formulations.

In the case of the five specific formulations of the Modal Ontological Argument examined in Sect. 3, we found that all of them are variations on a simple pattern. Their authors and subsequent commentators did not seem to recognize this similarity. Hartshorne's (1962) formulation uses strict implication whereas his earlier (1961) treatment of Malcolm's (1960) original exposition does not. Eder and Ramharter (2015, Sect. 4.1) state that they follow Hartshorne's (1962) formulation but use material implication. Kane (1984) states that his formulation is from Hartshorne (1962) but "readers of that work and subsequent journal literature on the modal OA may not recognize the adaptation." He attributes it to an unpublished simplification due to C. Anthony Anderson (Kane 1984, p. 339). Adams (1971) treatment is derived not from the *Proslogion*, but from Anselm's reply to Gaunilo.

Despite the apparent differences in their original presentations, all of the reconstructions have a first premise that asserts, in one way or another, that if a perfect or greatest being (i.e., God) exists, then he exists necessarily (i.e., $g \supset \Box g$, $\Box(g \supset \Box g)$, or $g \dashv3 \Box g$); a second premise that asserts his existence is possible (i.e., $\Diamond g$); and a conclusion that he exists or necessarily exists (i.e., g or $\Box g$).

Proof that the conclusion follows from the premises requires certain properties of the modal logic concerned; these can be expressed either by citing one of the standard modal axioms (i.e., B or 5) or the corresponding property of the accessibility relation on possible worlds (i.e., symmetric or Euclidean, respectively). The argument requires an Alethic logic (one where the modalities express necessity and possibility) and these have accessibility relations that are equivalence relations or, alternatively, assume modal axioms T and 5, and have B as a theorem; consequently, there can be no objection to the modal properties required in the proofs.

In addition to verifying the different formulations of the argument, we showed that in each case, the modal axiom that was assumed is sufficient to prove that the first premise entails a formula that states that the second premise directly entails the conclusion, that is:

$$\textit{Modal axiom} \supset \textit{Premise 1} \supset (\textit{Premise 2} \supset \textit{Conclusion}).$$

Thus, the reasoning is essentially trivial.

Mechanization is needed in this kind of examinations for the same reasons that automated calculation is used in other branches of engineering and science: it is simply too tedious and too error-prone to conduct repetitive analyses by hand. Some may concede this for hard problems but dismiss it for the Modal Ontological Argument: one reviewer wrote "the results of the analysis have been common knowledge in the field for at least the past 40 years (at least among those with even minimal competence

in propositional modal logic)." I would respectfully disagree that these results have been common knowledge and would also be cautious about widespread reliability, if not competence, in modal logic. To quote Lewis (1970, p. 175):

> Philosophy abounds in troublesome modal arguments—endlessly debated, perennially plausible, perennially suspect. The standards of validity for modal reasoning have long been unclear; they become clear only when we provide a semantic analysis of modal logic by reference to possible worlds…

For illustration, this journal published a paper by Jacquette (1997) in which he criticizes Hartshorne's (1962) formulation. Jacquette presents a semi-formal 10-line proof that he attributes to Hartshorne and finds it defective on several grounds. Most damagingly, he claims there is a "logical difficulty in a key assumption of the proof that renders the entire inference unsound. Proposition (5), which Hartshorne says follows from (1) as a modal form of *modus tollens*, is clearly false." Here, Proposition (1) is Anselm's Principle $g \supset \Box g$ and (5) is $\Box \neg \Box g \supset \Box \neg g$ which, contrary to Jacquette, is perfectly valid as can be verified by adding it to any of the PVS specifications presented in Sect. 3.

Jacquette attributes what he considers Hartshorne's error to his use of a "modal form of *modus tollens*" that Jacquette writes as $(\alpha \rightarrow \beta) \rightarrow (\Box \sim \beta \rightarrow \Box \sim \alpha)$ and states "is not generally true." He presents a counterexample in which α is "snow is red" and β is "2+2 = 5." This single formula highlights a couple of interesting pitfalls in semi-formal use of modal logic. First of all, Hartshorne uses *strict* implication, which we write as ⥽ to distinguish it from the $\supset$ of material implication. Jacquette makes no mention of this and it is not clear whether his $\rightarrow$ is intended to represent strict or material implication.

Second, it is often necessary to be careful about which parts of a sentence are to be interpreted modally, and which are conventional propositional logic. Specifically, a correct statement of modal *modus tollens* is expressed in PVS as follows (i.e., two modal sentences connected propositionally),

```
   tollens: LEMMA (P => Q) IMPLIES (□~Q => □~P)          PVS text
```

whereas the following (i.e., a single modal sentence) is invalid.

```
   tollens_bad: CLAIM (P => Q) => (□~Q => □~P)          PVS text
```

Jacquette seems to be using the invalid form, but we cannot be sure because his notation lacks the necessary distinctions. The key difference is that the correct form quantifies possible worlds separately for each side of the implication, whereas the invalid form requires the whole sentence to be true in the same worlds. This is similar to the reason that the deduction theorem is generally invalid in modal logic.

These errors in Jacquette's analysis are easily detected using truly formal and mechanically checked specifications. Furthermore, we or Jacquette could easily satisfy ourselves that even if we have qualms about Hartshorne's proof, his formulation of the argument (i.e., selection of premises and conclusion) is valid.

Although Jacquette's charge of invalidity for Hartshorne's formulation of the argument is mistaken, he raises other points that have merit. In particular, he observes that

Anselm's presentation has the form of a *reductio ad absurdum* whereas Hartshorne's (and we might add, all others we have considered) does not. He also notes that Anselm uses the notion of "greater than" (i.e., an ordering) on beings and that he uses "conceivable" as a modality that seems different than the "possible" of alethic logic. These points are all abstracted away in conventional formalizations of the argument. Accordingly, Jacquette (1997, Sect. 4) uses ideas that he attributes to Priest (2002) to construct a formalization of the argument that remedies these deficiencies. His treatment begins as follows.

1. [Definition of g:] $g = \delta x : \neg\exists y : \diamond(y > x)$.
   Here, δ is "indifferently a definite or indefinite descriptor" (i.e., "the" or "an") and I have substituted $\diamond$ for Jacquette's τ, which he intends as "an intensional modality" (since no semantics are given, this shift to a more familiar notation has no impact).

An informal reading is that God is some being than which no greater is conceivable. The construction $\neg\exists y : \diamond(y > x)$ is also used in formalizations using first order modal logic for the traditional, Proslogion Chapter II, Ontological Argument (Eder and Ramharter 2015, Sect. 4.2). These modal formalizations of the traditional argument have been mechanized in PVS (Rushby 2019, Sect. 3.1), as has a version in classical first-order logic that uses definite descriptions (Rushby 2013, where indefinite descriptions or choice functions, and Hilbert's ε, are also discussed).

Incidentally, we might wonder whether the given construction is equivalent to $\neg\diamond\exists y : (y > x)$ (i.e., interchange the modality and quantifier) and the answer is: it depends on whether we have constant or varying domains and, if varying, whether they are nonincreasing or nondecreasing, thereby highlighting some of the subtlety lurking here (Rushby 2017, Sect. 3).

The next interesting line in Jacquette's treatment is the following.

3. [Conceivable greatness:] $\forall x : (\neg E!x \supset \exists y : \diamond(y > x))$.
   Here, $E!x$ is a predicate indicating the "real existence" of x.

Jacquette uses the definition 1 and premise 3 in a *reductio* proof that establishes $E!g$: that is, the real existence of God.

Now, we could attempt to verify his proof in PVS but I am reluctant to extend, here and at this stage, the treatment of modal logic from propositional to first order, so I will make my point using Jacquette's informal notation, but the following derives from experience in formal verification of the Proslogion Chapter II argument with PVS using both classical and first order modal reasoning (Rushby 2020, 2019).

Nowhere in Jacquette's proof is the construction $\exists y : \diamond(y > x)$ opened up to reveal either y or the $>$ relation. Thus, we can replace the whole thing by an uninterpreted function $F(x)$. Then 1 and 3 above become the following "primed" variants;

   $1'$: $g = \delta x : \neg F(x)$,
   $3'$: $\forall x : \neg E!x \supset F(x)$, and we can replace this by its contrapositive
   $3''$: $\forall x : \neg F(x) \supset E!x$.

But now the deduction is revealed as trivial: we have $\neg F(g)$ from 1' and instantiate 3" to give $\neg F(g) \supset E!g$ and the conclusion $E!g$ is immediate. Furthermore, the proof

has lost all trace of modal reasoning. Jacquette's goal was to make explicit more of Anselm's reasoning, yet he ends up trivializing it.

Although his execution is defective, we can agree with Jacquette's motivation that a faithful reconstruction of Anselm's argument should represent his language explicitly. A reviewer for this paper makes the same point and suggests that PVS could be used systematically to explore different renditions for "conceive," "nothing greater" etc. I am sympathetic to this but suspect that the defective renditions would exhibit triviality in the style of Jacquette's example rather than invalidity, and this is more difficult to detect automatically. I hope others may wish to explore some of the alternatives and that verification tools such as PVS will assist their endeavors.

Conclusion

We have examined several reconstructions of the modal ontological argument from Anselm's Proslogion Chapter III and his reply to Gaunilo. When fully formalized, all of the reconstructions have a very similar form, and are proved in the same way.

The proofs require certain properties of the modal logic employed, and these can be specified either by citing one of the standard modal axioms (i.e., B or 5), or the corresponding property of the accessibility relation on possible worlds (i.e., symmetric or Euclidean, respectively). The argument requires an Alethic logic (one where the modalities express necessity and possibility) and these have accessibility relations that are equivalence relations or, alternatively, assume modal axioms T and 5, and have B as a theorem; consequently, there can be no objection to the modal properties required in the proofs.

We expressed the various reconstructions in PVS and mechanically verified their validity. We also showed that in each case, the modal axiom that was assumed is sufficient to prove that the first premise entails a formula that states that the second premise directly entails the conclusion, that is:

$$\textit{Modal axiom} \supset \textit{Premise 1} \supset (\textit{Premise 2} \supset \textit{Conclusion}).$$

Thus, the reasoning is essentially trivial and standard readings of the modal ontological argument should not retain our interest.

We also showed how mechanically assisted formalization and analysis could expose flawed treatments, including one published in this journal, and also vacuous formalizations that present the appearance of Anselm's argument but are logically trivial.

Campbell (2018, 2019) provides an alternative reading in which Chapter III of the Proslogion does not stand alone but continues the argument of Chapter II. Campbell interprets Anselm differently than the scholars considered here and constructs a different translation into English. Formal analysis of his and related readings will require separate treatment.

This paper is the fifth is a series where I apply mechanized verification to analysis of formal reconstructions of Anselm's Ontological arguments. In the first (Rushby 2013), I examined a formalization by Oppenheimer and Zalta (2011) for the traditional Proslogion II argument that uses definite descriptions. I showed that this formalization

is very close to circular (i.e., begs the question) in an informal sense. In the second (Rushby 2018), I gave formal definitions for question begging in formal proofs and I showed that all the examined reconstructions of the Proslogion II argument in classical (i.e., first- and higher-order) logic were guilty of begging the question. In a revision and extension to that paper (Rushby 2020) I additionally showed that all these reconstructions entail variants that apply no interpretation to "something than which there is no greater" and are therefore vacuous and vulnerable to Gaunilo's (1079) refutation, and I argued that the basic reconstructions inherit these charges. In the fourth (Rushby 2019), I extended these analyses to reconstructions of the Proslogion II argument in quantified modal logic and found they suffer the same defects. In the present paper, we have examined reconstructions of the Proslogion III argument in propositional modal logic and found it to be trivial once the assumed modal axiom is taken into account.

Mechanization is necessary for these analyses because they require detailed scrutiny of small variations in premises and assumptions. This is tedious and error-prone to do by hand, but fast, simple, and even enjoyable to accomplish with mechanized assistance. I hope these exercises may encourage others to apply modern formal verification tools to examination of other suitable philosophical and theological topics.

Compliance with ethical standards

Conflict of interest The author declares he has no conflict of interest.

References

Adams, R. M. (1971). The logical structure of Anselm's arguments. *The Philosophical Review*, *80*(1), 28–54.

Anselm, S. (1078). Proslogion. In *Internet medieval sourcebook*. Fordham University (in English, the original is in Latin).

Anselm, S. (circa 1079). Apologetic in reply to Gaunilo. In *Internet medieval sourcebook*. Fordham University (in English, the original is in Latin).

Benzmüller, C., & Woltzenlogel-Paleo, B. (2014). Automating Gödel's ontological proof of God's existence with higher-order automated theorem provers. In *ECAI 2014: 21st European conference on artificial intelligence* (pp. 93–98). Prague: Czech Republic.

Campbell, R. (2018). *Rethinking Anselm's arguments: A vindication of his proof of the existence of god*. Leiden: Brill.

Campbell, R. (2019). That Anselm's god exists and Gaunilo's Island does not. In *Considering religions, rights and bioethics: For Max Charlesworth, Sophia studies in cross-cultural philosophy of traditions and cultures*, vol. 30 (pp. 115–137). Springer.

Cresswell, M. J., & Hughes, G. E. (1996). *A new introduction to modal logic*. Abingdon: Routledge.

Eder, G., & Ramharter, E. (2015). Formal reconstructions of St. Anselm's ontological argument. *Synthese*, *192*(9), 2795–2825.

Garbacz, P. (2012). Prover9's simplifications explained away. *Australasian Journal of Philosophy*, *90*(3), 585–592.

Gaunilo, a Monk of Marmoutier (circa 1079) In behalf of the fool. In *Internet medieval sourcebook*. Fordham University (in English, the original is in Latin).

Ge, Y., & de Moura, L. (2009). Complete instantiation for quantified formulas in satisfiabiliby modulo theories. In *Computer-Aided verification, CAV '09* (pp. 306–320). Grenoble: Springer.

Grim, P., & Singer, D. (2020). Computational philosophy. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy, spring* (2020th ed.). Stanford: Metaphysics Research Lab, Stanford University.

Hartshorne, C. (1961). The logic of the ontological argument. *The Journal of Philosophy*, *58*(17), 471–473.

Hartshorne, C. (1962). *The logic of perfection*. La Salle, IL: Open Court.

Hartshorne, C. (1965). *Anselm's discovery: A re-examination of the ontological argument for god's existence*. La Salle, IL: Open Court.

Jacquette, D. (1997). Conceivability, intensionality, and the logic of Anselm's modal argument for the existence of god. *International Journal for Philosophy of Religion*, *42*, 163–173.

Kane, R. (1984). The modal ontological argument. *Mind*, *93*(371), 336–350.

Ladstaetter, K. (2012). On tracy Lupher's "A logical choice: the role of modal logics in the modal ontological argument". *Southwest Philosophy Review*, *28*(2), 101–106.

Lewis, D. (1970). Anselm and actuality. *Noûs*, *4*(2), 175–188.

Malcolm, N. (1960). Anselm's ontological arguments. *The Philosophical Review*, *69*(1), 41–62.

Matthews, G. B. (2005). The ontological argument. In W. E. Mann (Ed.), *The blackwell guide to the philosophy of religion* (pp. 81–102). Hoboken: Blackwell Publishing.

Oppenheimer, P. E., & Zalta, E. N. (2011). A computationally-discovered simplification of the ontological argument. *Australasian Journal of Philosophy*, *89*(2), 333–349.

Priest, G. (2002). *Beyond the limits of thought*. Oxford: Oxford University Press.

PVS (constantly updated) PVS home page. http://pvs.csl.sri.com/.

Rushby, J. (2013). The ontological argument in PVS. In: Shilov N. (eds) *Fun with formal methods*. St. Petersburg, Russia, workshop in association with CAV'13.

Rushby, J. (2017). PVS embeddings of propositional and quantified modal logic. In *Technical reports on, computer science laboratory*. Menlo Park, CA: SRI International.

Rushby, J. (2018). A mechanically assisted examination of begging the question in Anselm's ontological argument. *Journal of Applied Logics—IFCoLog Journal of Logics and their Applications*, *5*(7), 1473–1497.

Rushby, J. (2019). Mechanized analysis of modal reconstructions of Anselm's traditional ontological argument. In *Technical reports on computer science laboratory*. Menlo Park, CA: SRI International.

Rushby, J. (2020). A mechanically assisted examination of vacuity and question begging in Anselm's ontological argument. In R. Silvestre, P. Gocke, J. Y. Beziau, & P. Bilimoria (Eds.), *Beyond faith and rationality: Essays on logic, religion and philosophy*. Berlin: Springer. to appear.

Smith, A. D. (2014). *Anselm's other argument*. Cambridge, MA: Harvard University Press.

Sobel, J. H. (2003). *Logic and Theism: Arguments for and Against Beliefs in God*. Cambridge, MA.

Wikipedia (retrieved 2019) Standard Translation (of modal logic into first-order logic). https://en.wikipedia.org/wiki/Standard_translation.