Review of Derivatives Research (2021) 24:79–94
https://doi.org/10.1007/s11147-020-09170-4

Uncertain strike lookback options pricing with floating interest rate

Lidong Zhang[1,3] · Yanmei Sun[2] · Ziping Du[2,3] · Xiangbo Meng[1]

Published online: 13 August 2020

Abstract

Considering the floating interest rate and the uncertainty of the strike price, we derive the pricing formulas of lookback options including lookback call option and lookback put option. Furthermore, we give the numerical algorithms to illustrate our results and analyze the relationships between the price of lookback options and all the parameters.

Keywords Uncertain process · Floating interest rate · Uncertain strike price · Lookback option

JEL Classification G23 · C13

This research was supported in part by The Youth Project of Humanities and Social Sciences of Ministry of Education (Grant No. 19YJCZH251) and in part by The Scientific Research Program of Tianjin Education Commission (Grant No. 2018KJ113).

✉ Lidong Zhang
zhanglidong1999@126.com

Yanmei Sun
sunymss@163.com

Ziping Du
dusx@tust.edu.cn

Xiangbo Meng
mengxb@tust.edu.cn

1 School of Science, Tianjin University of Science and Technology, Tianjin 300457, People's Republic of China

2 College of Economics and Management, Tianjin University of Science and Technology, Tianjin 300222, People's Republic of China

3 Financial Engineering and Risk Management Research Center, Tianjin University of Science and Technology, Tianjin 300222, People's Republic of China

1 Introduction

Early researches on option pricing were based on probability theory. As we all know, when using probability theory, a basic premise is that the estimated probability distribution is close enough to the long-run cumulative frequency. Otherwise, the law of large numbers is no longer valid and probability theory is no longer applicable. However, in many situations, there are not enough (or even no) historical data. For example, there are no historical data when new stocks are issued.

Furthermore, many assumptions on Black–Scholes model are too idealistic, many scholars later have made various amendments and promotions. Lots of studies have shown that the distribution of stock returns is not consistent with the assumption of the normal distribution. In many situations, researchers pointed that stochastic differential equation is inappropriate to model the actual financial market. Many empirical studies showed the price of underlying asset does not behave like randomness, and it is often affected by the belief degrees of investors in Kahneman and Tversky (1979) and Tversky and Kahneman (1992). Liu (2013b) also gave some paradoxes about stochastic finance theory and suggested that uncertain differential equation may be a potential mathematical foundation of finance theory.

In order to deal with this case, we have to invite some domain experts to evaluate their belief degree that each event will occur. For modeling the uncertainty of human beings, uncertainty theory was founded by Liu (2007) and refined by Liu (2009). Liu (2007) gave the definitions of uncertainty distribution, inverse uncertainty distribution, expected value and variance. Liu (2008) introduced the concept of uncertain process, which can describe dynamic uncertain systems, and presented a type of differential equations driven by canonical Liu process.

After that, uncertainty theory was used to investigate options pricing. Liu (2009) proposed an uncertain differential equation to describe the stock price, and obtained the European option pricing formula. Later, Chen (2011) gave the corresponding analytic solution of American option price. In the same year, Peng and Yao (2011) proposed an uncertain mean-reversion stock model and presented the pricing formulas of European and American options. Yu (2012) studied the option pricing formula under the uncertain stock model with jump. In the following year, Zhang and Liu (2014) and Sun and Chen (2015) derived the pricing formulas of geometric average Asian options and arithmetic average Asian options respectively. Under an uncertain Ornstein–Uhlenbeck model, Gao et al. (2018) obtained the pricing formulas of lookback options. Furthermore, see more details on other options pricing in Gao et al. (2019), Zhang and Sun (2020), and Zhang et al. (2020).

It is well known that a lookback option is a path-dependent option with payoff determined not only by the settlement but also by the maximum or minimum price of the asset within the life of the option. In this paper, we are concerned on lookback options with floating interest rate and uncertain strike price. We extend the lookback options pricing problems discussed in Gao et al. (2018) to an uncertain strike lookback options pricing problems with floating interest rate. Compared with existing models, our contributions are as follows. As is known to all, interest rate and strike price are also important indicators of economic measurement in options pricing and they are often fluctuated by uncertain factors such as the financial markets and economic policies. Thus, it is suitable to regard the interest rate and the strike price as uncertain variables when we study the options pricing problem. On one hand, interest rate is a changing quantity rather than a constant. On the other hand, the strike price is uncertain and the strike price is determined by another asset which is described by an uncertain differential equation. In addition, we also conducted empirical research and sensitivity analysis on the parameters in the model in order to better describe the true situation of the financial market.

The rest of this paper is organized as follows. In Sect. 2, the elementary concepts and theorems on uncertainty theory are introduced. In Sect. 3, we derive lookback call/put options pricing formulas with floating interest rate and uncertain strike price, and give the numerical algorithm to calculate the prices. Finally, a brief conclusion is given in Sect. 4.

2 Preliminaries

Uncertainty theory is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms, which was founded by Liu (2007) and refined by Liu (2009). In this section, we will introduce some basic definitions and results on uncertainty theory. As for the detailed applications of uncertainty theory, please see the references Gao (2013) and Liu (2013a, 2014, 2015).

2.1 Uncertain variables

Definition 1 (Liu 2007, 2009) Let L be a σ-algebra on a nonempty set Γ. A set function $M : L \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $\mathrm{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2. (Duality Axiom) $\mathrm{M}\{\Lambda\} + \mathrm{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathrm{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathrm{M}\{\Lambda_i\}.$$

Axiom 4. (Product Axiom) Let $(\Gamma_k, L_k, \mathrm{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$, the product uncertain measure M is an uncertain measure satisfying

$$\mathrm{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathrm{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, \ldots$, respectively.

Definition 2 (Liu 2007) An uncertain variable is a function from an uncertainty space (Γ, L, M) to the set of real numbers, such that, for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event.

Next, we will define the uncertainty distribution of an uncertain variable and the independence of multi-variables.

Definition 3 The uncertainty distribution $\Phi(x)$ of an uncertain variable ξ is defined by $\Phi(x) = \mathrm{M}\{\xi \le x\}$ for any real number x. An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and $\lim_{x \to -\infty} \Phi(x) = 0$, $\lim_{x \to +\infty} \Phi(x) = 1$.

If ξ has a regular uncertainty distribution $\Phi(x)$, then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

Definition 4 (Liu 2009) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$\mathrm{M}\left\{\bigcap_{i=1}^{m} \{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^{m} \mathrm{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

By virtue of the operation law of uncertain variables, Liu (2010) gave the inverse uncertainty distribution of strictly monotone function.

Theorem 1 (Liu 2010) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with $x_{m+1}, x_{m+2}, \ldots, x_n$, then*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_m, \xi_{m+1}, \xi_{m+2}, \ldots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right).$$

Definition 5 (Liu 2007) The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{M}\{\xi \ge x\}dx - \int_{-\infty}^{0} \mathrm{M}\{\xi \le x\}dx$$

provided that at least one of the two integrals exists.

Liu (2010) showed that the expected value of an uncertain variable with a regular uncertainty distribution was given by the following theorem.

Theorem 2 (Liu 2010) *Assume the uncertain variable* ξ *has a regular uncertainty distribution* Φ*, then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha.$$

2.2 Uncertain differential equations

In this section, we will give the concepts and theorems on uncertain differential equation. For more details on them, please see the references Liu (2008, 2009) and Yao and Chen (2013).

Definition 6 (Liu 2008) Let T be an index set and let (Γ, L, M) be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, L, \mathrm{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B,

$$\{X_t \in B\} = \{\gamma \in \Gamma | X_t(\gamma) \in B\}$$

is an event.

Definition 7 (Liu 2009) An uncertain process C_t is said to be a Liu process if

(i) $C_t = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in R.$$

Definition 8 (Liu 2009) Let X_t be an uncertain process and let C_t be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|,$$

then Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_t \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite.

Definition 9 (Liu 2008) Suppose C_t is a Liu process, and f and g are two functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

is called an uncertain differential equation. A solution is an uncertain process X_t that satisfies the equation identically in t.

Definition 10 (Yao and Chen 2013) Let α be a number with $0 < \alpha < 1$. An uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation

$$dX_t^\alpha = f(t, X_t^\alpha)dt + \left|g(t, X_t^\alpha)\right| \Phi^{-1}(\alpha)dt,$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of standard normal uncertain variable,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Theorem 3 (Yao and Chen 2013) *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

respectively. Then

$$\begin{aligned} &\mathrm{M}\left\{X_t \le X_t^\alpha, \forall t\right\} = \alpha, \\ &\mathrm{M}\left\{X_t > X_t^\alpha, \forall t\right\} = 1 - \alpha. \end{aligned}$$

Theorem 4 (Yao 2013) *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

respectively. Then for any time $s > 0$ and strictly increasing (decreasing) function $J(x)$, the supremum $\sup\limits_{0 \le t \le s} J(x_t)$ has an inverse uncertainty distribution

$$\Phi_s^{-1}(\alpha) = \sup_{0 \le t \le s} J(x_t^\alpha) \left(\Phi_s^{-1}(\alpha) = \sup_{0 \le t \le s} J(x_t^{1-\alpha})\right)$$

and the time integral $\int_0^s J(x_t)dt$ has an α-path

$$Y_s^\alpha = \int_0^s J(x_t^\alpha)dt \left(Y_s^\alpha = \int_0^s J(x_t^{1-\alpha})dt\right).$$

2.3 Some results on uncertain Square–Root process and uncertain exponential Ornstein–Uhlenbeck process

In this section, we will introduce uncertain Square–Root process and uncertain exponential Ornstein–Uhlenbeck process, which are used to describe the interest rate and the stock price respectively and recall their inverse uncertainty distributions. The uncertain Square–Root process satisfies the following uncertain equation

$$dr_t = a\,(b - r_t)\,dt + \sigma_0\sqrt{r_t}dC_t,$$

where a, b, σ_0 are some positive real numbers and C_t is an independent canonical Liu processe.

Theorem 5 (Jiao and Yao 2015) *Suppose that an uncertain process satisfies the following uncertain equation*

$$dr_t = a\,(b - r_t)\,dt + \sigma_0\sqrt{r_t}dC_t,$$

then the inverse uncertainty distribution $r_t^{-1}(\alpha)$ of r_t satisfies the following ordinary differential equation

$$dr_t^{-1}(\alpha) = a\left(b - r_t^{-1}(\alpha)\right)dt + \sigma_0\sqrt{r_t^{-1}(\alpha)}\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}dt.$$

Dai et al. (2017) firstly gave the uncertain exponential Ornstein–Uhlenbeck process which was determined by the following uncertain differential equation

$$dY_t = \mu\,(1 - c\ln Y_t)\,Y_t dt + \sigma Y_t dC_t,$$

where $c > 0$, $\sigma > 0$ and μ are constants. Subsequently, Sun et al. (2018) gave the analytic solution of this model and Dai et al. (2017) derived the inverse uncertainty distribution of the uncertain exponential Ornstein–Uhlenbeck process.

Theorem 6 (Sun et al. 2018) *Suppose that an uncertain process satisfies the following uncertain equation*

$$dY_t = \mu\,(1 - c\ln Y_t)\,Y_t dt + \sigma Y_t dC_t,$$

then

$$Y_t = \exp\left(\frac{1}{c} + \left(\ln Y_0 - \frac{1}{c}\right)\exp(-\mu c t) + \sigma\int_0^t \exp(c\mu(s-t))dC_s\right).$$

Theorem 7 (Dai et al. 2017) *Suppose an uncertain process satisfies the following uncertain equation*

$$dY_t = \mu(1 - c\ln Y_t)Y_t dt + \sigma Y_t dC_t,$$

then the inverse uncertainty distribution of Y_t is

$$Y_t^{-1}(\alpha) = \exp\left(\exp\left(-\mu ct\right)\ln Y_0 + \frac{1}{c}(1-\exp(-\mu ct))\left(1+\frac{\sqrt{3}\sigma}{\mu\pi}\ln\frac{\alpha}{1-\alpha}\right)\right).$$

3 Lookback options pricing model with floating interest and uncertain strike

In this paper, we assume the interest rate r_t satisfies uncertain Square–Root process, and uncertain strike price k_t and the stock price X_t follow uncertain exponential Ornstein–Uhlenbeck process, which are stated as below

$$\begin{cases} dr_t = a\left(b - r_t\right)dt + \sigma_0\sqrt{r_t}dC_t, \\ dX_t = \mu_1(1 - c_1\ln X_t)X_t dt + \sigma_1 X_t dC_{1t}, \\ dk_t = \mu_2(1 - c_2\ln k_t)k_t dt + \sigma_2 k_t dC_{2t}, \end{cases} \tag{1}$$

where $a, b, \mu_1, \mu_2, \sigma_0, \sigma_1$ and σ_2 are positive real numbers and $c_1, c_2 \neq 0$. Furthermore, C_t, C_{1t} and C_{2t} are independent canonical Liu processes.

By virtue of Theorems 5 and 7, the inverse uncertainty distributions of the interest rate r_t, the stock price X_t and uncertain strike price k_t are given by the following equation

$$\begin{aligned} dr_t^{-1}(\alpha) &= a\left(b - r_t^{-1}(\alpha)\right)dt + \sigma_0\sqrt{r_t^{-1}(\alpha)}\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}dt, \\ X_t^{-1}(\alpha) &= \exp\left(\exp(-\mu_1 c_1 t)\ln X_0 + \frac{1}{c_1}(1-\exp(-\mu_1 c_1 t))\left(1+\frac{\sqrt{3}\sigma_1}{\mu_1\pi}\ln\frac{\alpha}{1-\alpha}\right)\right), \\ k_t^{-1}(\alpha) &= \exp\left(\exp(-\mu_2 c_2 t)\ln k_0 + \frac{1}{c_2}(1-\exp(-\mu_2 c_2 t))\left(1+\frac{\sqrt{3}\sigma_2}{\mu_2\pi}\ln\frac{\alpha}{1-\alpha}\right)\right). \end{aligned}$$

The following two subsections will give uncertain strike lookback call/put options pricing formulas with the floating interest rate.

3.1 Lookback call option pricing formula with the uncertain strike price

A lookback call option provides the holder with the right to sell a certain asset at the highest price during a certain period. Here, we suppose that a lookback call option has an expiration time T, then the payoff of selling a lookback call option is

$$\left(\max_{0\le t\le T} X_t - k_T\right)^+.$$

Taking into account the time value of the stock return, the present value of the payoff is

$$\exp\left(-\int_0^T r_t dt\right)\left(\max_{0\le t\le T} X_t - k_T\right)^+.$$

For the lookback call option is the expected present value of the payoff, then the lookback call option has a price

$$f_{call} = E\left[\exp\left(-\int_0^T r_t dt\right)\left(\max_{0\le t\le T} X_t - k_T\right)^+\right].$$

Theorem 8 *Suppose that a lookback call option for the stock model* (1) *has an expiration time T, then the lookback call option pricing formula is*

$$f_{call} = \int_0^1 \exp\left(-\int_0^T r_t^{-1}(1-\alpha)dt\right)\max_{0\le t\le T}\left(X_t^{-1}(\alpha) - k_T^{-1}(1-\alpha)\right)^+ d\alpha, \quad (2)$$

where

$$\begin{aligned} dr_t^{-1}(1-\alpha) &= a\left(b - r_t^{-1}(1-\alpha)\right)dt + \sigma_0\sqrt{r_t^{-1}(1-\alpha)}\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}dt, \\ X_t^{-1}(\alpha) &= \exp\left(\exp(-\mu_1 c_1 t)\ln X_0 + \frac{1}{c_1}(1-\exp(-\mu_1 c_1 t))\left(1 + \frac{\sqrt{3}\sigma_1}{\mu_1\pi}\ln\frac{\alpha}{1-\alpha}\right)\right), \\ k_T^{-1}(1-\alpha) &= \exp\left(\exp(-\mu_2 c_2 T)\ln k_0 + \frac{1}{c_2}(1-\exp(-\mu_2 c_2 T))\left(1 + \frac{\sqrt{3}\sigma_2}{\mu_2\pi}\ln\frac{1-\alpha}{\alpha}\right)\right). \end{aligned}$$

Proof According to Theorem 4, the time integral $\int_0^T r_t dt$ has an inverse uncertainty distribution

$$\varphi^{-1}(\alpha) = \int_0^T r_t^{-1}(\alpha)dt.$$

Since $\exp(-x)$ is a strictly decreasing function, the uncertain variable

$$\exp\left(-\int_0^T r_t dt\right)$$

has an inverse uncertainty distribution

$$\gamma^{-1}(\alpha) = \exp\left(-\varphi^{-1}(1-\alpha)\right) = \exp\left(-\int_0^T r_t^{-1}(1-\alpha)dt\right)$$

Thus, the inverse distribution function of

$$\exp\left(-\int_0^T r_t dt\right)\left(\max_{0\le t\le T} X_t - k_T\right)^+$$

is given by the following equation

$$\Psi^{-1}(\alpha) = \exp\left(-\int_0^T r_t^{-1}(1-\alpha)dt\right)\max_{0\le t\le T}\left(X_t^{-1}(\alpha) - k_T^{-1}(1-\alpha)\right)^+.$$

So the price of lookback call option can be obtained

$$\begin{aligned} f_{call} &= E\left[\exp\left(-\int_0^T r_t dt\right)\left(\max_{0\le t\le T} X_t - k_T\right)^+\right] \\ &= \int_0^1 \Psi^{-1}(\alpha)d\alpha \\ &= \int_0^1 \exp\left(-\int_0^T r_t^{-1}(1-\alpha)dt\right)\max_{0\le t\le T}\left(X_t^{-1}(\alpha) - k_T^{-1}(1-\alpha)\right)^+ d\alpha. \end{aligned}$$

The pricing formula of lookback call option is derived. □

From $\left(\max_{0\le t\le T} X_t - k_T\right)^+ = \max_{0\le t\le T}(X_t - k_T)^+$ and Theorem 8, the algorithm of calculating lookback call option price is given as follows.

Step 0: Set $\alpha_i = i/N$ and $t_j = jT/M$, $i = 1, 2, \ldots, N-1$, $j = 1, 2, \ldots, M$, where N and M are two large numbers.
Step 1: Set $i = 0$.
Step 2: Set $i \leftarrow i + 1$.
Step 3: Set $j = 0$.
Step 4: Set $j \leftarrow j + 1$.
Step 5: Set $G_{t_0}^{\alpha_i} = 0$.
Step 6: Calculate the interest rate, the difference between stock price and the terminal strike price at the time t_j are

$$\begin{aligned} r_{t_j}^{-1}(1-\alpha_i) &= r_{t_{j-1}}^{-1}(1-\alpha_i) + a\left(b - r_{t_{j-1}}^{-1}(1-\alpha_i)\right)\Delta t_j \\ &\quad + \sigma_0\sqrt{r_{t_{j-1}}^{-1}(1-\alpha_i)}\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha_i}{\alpha_i}\Delta t_j, \\ G_{t_j}^{\alpha_i} &= \exp\left(\exp(-\mu_1 c_1 t_j)\ln X_0 + \frac{1}{c_1}(1-\exp(-\mu_1 c_1 t_j))\left(1 + \frac{\sqrt{3}\sigma_1}{\mu_1\pi}\ln\frac{\alpha_i}{1-\alpha_i}\right)\right) \\ &\quad - \exp\left(\exp(-\mu_2 c_2 T)\ln k_0 + \frac{1}{c_2}(1-\exp(-\mu_2 c_2 T))\left(1 + \frac{\sqrt{3}\sigma_2}{\mu_2\pi}\ln\frac{1-\alpha_i}{\alpha_i}\right)\right). \end{aligned}$$

Step 7: Calculate the discount rate

$$\exp\left(-\int_0^T r_t^{-1}(1-\alpha_i)dt\right) \leftarrow \exp\left(-\frac{T}{M}\sum_{j=1}^{M} r_{t_j}^{1-\alpha_i}\right)$$

and the maximum difference between stock price and the terminal strike price over the time interval $[0, T]$

$$\max_{0\le t\le T}\left(X_t^{-1}(\alpha_i) - k_T^{-1}(1-\alpha_i)\right)^+ \leftarrow max(G_{t_1}^{\alpha_i}, \ldots, G_{t_M}^{\alpha_i}, 0).$$

Step 8: Set

$$f^{\alpha_i} \leftarrow \exp\left(-\frac{T}{M}\sum_{j=1}^{M} r_{t_j}^{-1}(1-\alpha_i)\right) max(G_{t_1}^{\alpha_i}, \ldots, G_{t_M}^{\alpha_i}, 0).$$

If $i < N-1$, then return to *Step 2*.
Step 9: Calculate the lookback call option price

$$f_{call} \leftarrow \frac{1}{N-1}\sum_{i=1}^{N-1} f^{\alpha_i}.$$

Example 1 Assume that the parameters of the stock price are $X_0 = 1, c_1 = 1, \mu_1 = \sqrt{3}, \sigma_1 = \pi$, the parameters of the interest rate is $r_0 = 0.03, a = 1, b = 0.5, \sigma_0 = 0.02$, the parameters of the strike price is $k_0 = 3, c_2 = 0.9, \mu_2 = 1.5, \sigma_2 = 3.1$ and $M = 999, N = 9999$, then, the price of the lookback call option with a maturity date $T = 5$ is 3.4549.

Finally, we present a graph of the lookback call option pricing formula with different parameters as shown below. Figure 1a represents the price with respect to the floating interest rate; Fig. 1b represents the price with respect to the expiration time; Fig. 1c represents the price with respect to the starting value of stock price; Fig. 1d represents the price with respect to the original uncertain strike price.

3.2 Lookback put option pricing formula with the uncertain strike price

Suppose that a lookback put option has an expiration time T, then the payoff of buying a lookback put option is

$$\left(k_T - \min_{0\le t\le T} X_t\right)^+.$$

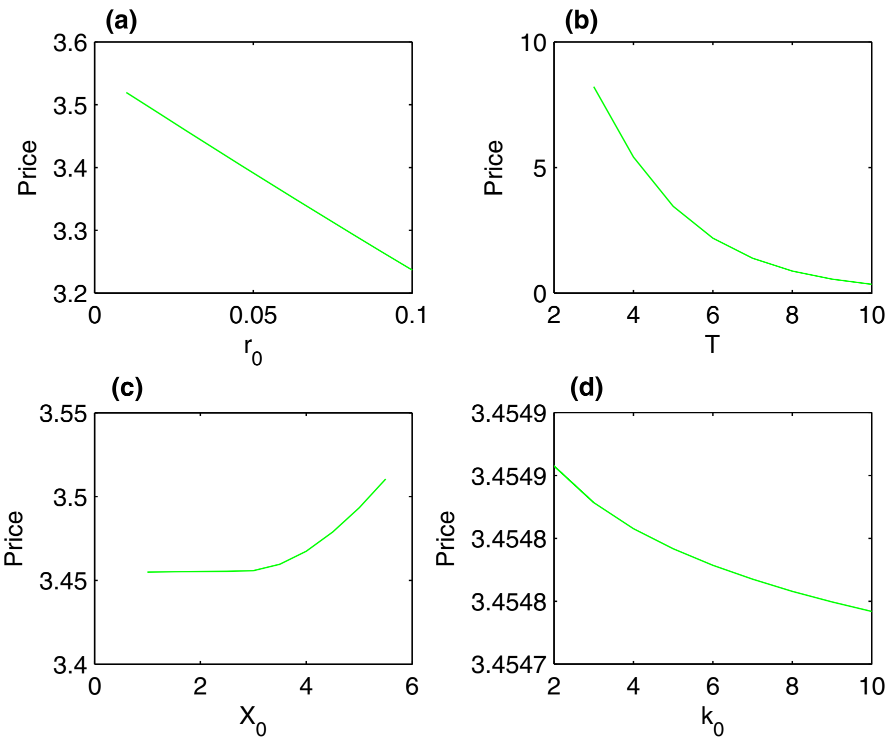

Fig. 1 Lookback call option price with respect to different parameters

Taking into account the time value of money, the present value of this payoff is

$$\exp\left(-\int_0^T r_t dt\right)\left(k_T - \min_{0\le t\le T} X_t\right)^+.$$

For the lookback put option is the expected present value of the payoff, then the lookback put option has a price

$$f_{put} = E\left[\exp\left(-\int_0^T r_t dt\right)\left(k_T - \min_{0\le t\le T} X_t\right)^+\right].$$

Theorem 9 *Suppose that a lookback put option for the stock model has an expiration time T. Then the pricing formula lookback put option is*

$$f_{put} = \int_0^1 \exp\left(-\int_0^T r_t^{-1}(1-\alpha)dt\right) \max_{0\le t\le T}\left(k_T^{-1}(\alpha) - X_t^{-1}(1-\alpha)\right)^+ d\alpha, \quad (3)$$

where

$$
\begin{aligned}
dr_t^{-1}(1-\alpha) &= a\left(b - r_t^{-1}(1-\alpha)\right)dt + \sigma_0\sqrt{r_t^{-1}(1-\alpha)}\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}dt,\\
X_t^{-1}(1-\alpha) &= \exp\left(\exp(-\mu_1 c_1 t)\ln X_0 + \frac{1}{c_1}(1-\exp(-\mu_1 c_1 t))\left(1+\frac{\sqrt{3}\sigma_1}{\mu_1\pi}\ln\frac{1-\alpha}{\alpha}\right)\right),\\
k_T^{-1}(\alpha) &= \exp\left(\exp(-\mu_2 c_2 T)\ln k_0 + \frac{1}{c_2}(1-\exp(-\mu_2 c_2 T))\left(1+\frac{\sqrt{3}\sigma_2}{\mu_2\pi}\ln\frac{\alpha}{1-\alpha}\right)\right).
\end{aligned}
$$

Proof From Theorem 4, it is easily to derive the inverse distribution function of $\exp\left(-\int_0^T r_t dt\right)\left(k_T - \min_{0\le t\le T} X_t\right)^+$ is

$$
\Psi^{-1}(\alpha) = \exp\left(-\int_0^T r_t^{-1}(1-\alpha)dt\right)\left(k_T^{-1}(\alpha) - \min_{0\le t\le T} X_t^{-1}(1-\alpha)\right)^+,
$$

where $r_t^{-1}(1-\alpha)$, $X_t^{-1}(1-\alpha)$, $K_T^{-1}(\alpha)$ are given in Theorem 9. Thus

$$
\begin{aligned}
f_{put} &= E\left[\exp\left(-\int_0^T r_t dt\right)\left(k_T - \min_{0\le t\le T} X_t\right)^+\right]\\
&= \int_0^1 \Psi^{-1}(\alpha)d\alpha\\
&= \int_0^1 \exp\left(-\int_0^T r_t^{-1}(1-\alpha)dt\right)\left(k_T^{-1}(\alpha) - \min_{0\le t\le T} X_t^{-1}(1-\alpha)\right)^+ d\alpha.
\end{aligned}
$$

The pricing formula of the lookback put option is verified. □

From $\left(k_T - \min_{0\le t\le T} X_t\right)^+ = \max_{0\le t\le T}(k_T - X_t)^+$ and Theorem 9, the algorithm of calculating the lookback put option price is as follows.

Step 0: Set $\alpha_i = i/N$ and $t_j = jT/M$, $i = 1, 2, \ldots, N-1$, $j = 1, 2, \ldots, M$, where N and M are two large numbers.
Step 1: Set $i = 0$.
Step 2: Set $i \leftarrow i + 1$.
Step 3: Set $j = 0$.
Step 4: Set $j \leftarrow j + 1$.
Step 5: Set $H_{t_0}^{\alpha_i} = 0$.
Step 6: Calculate the interest rate, the difference between the terminal strike price and stock price at the time t_j are

$$
\begin{aligned}
r_{t_j}^{-1}(1-\alpha_i) = r_{t_{j-1}}^{-1}(1-\alpha_i) + a\left(b - r_{t_{j-1}}^{-1}(1-\alpha_i)\right)\Delta t_j\\
+\sigma_0\sqrt{r_{t_{j-1}}^{-1}(1-\alpha_i)}\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha_i}{\alpha_i}\Delta t_j,
\end{aligned}
$$

$$H_{t_j}^{\alpha_i} = \exp\left(\exp(-\mu_2 c_2 T)\ln k_0 + \frac{1}{c_2}(1-\exp(-\mu_2 c_2 T))\left(1+\frac{\sqrt{3}\sigma_2}{\mu_2\pi}\ln\frac{\alpha_i}{1-\alpha_i}\right)\right)$$
$$-\exp\left(\exp(-\mu_1 c_1 t_j)\ln X_0 + \frac{1}{c_1}(1-\exp(-\mu_1 c_1 t_j))\left(1+\frac{\sqrt{3}\sigma_1}{\mu_1\pi}\ln\frac{1-\alpha_i}{\alpha_i}\right)\right).$$

Step 7: Calculate the discount rate

$$\exp\left(-\int_0^T r_t^{-1}(1-\alpha_i)dt\right) \leftarrow \exp\left(-\frac{T}{M}\sum_{j=1}^{M} r_{t_j}^{1-\alpha_i}\right)$$

and the maximum difference between stock price and the terminal strike price over the time interval $[0, T]$

$$\max_{0\le t\le T}\left(k_T^{-1}(\alpha_i) - X_t^{-1}(1-\alpha_i)\right)^+ \leftarrow max(H_{t_1}^{\alpha_i}, \ldots, H_{t_M}^{\alpha_i}, 0).$$

Step 8: Set

$$f^{\alpha_i} \leftarrow \exp\left(-\frac{T}{M}\sum_{j=1}^{M} r_{t_j}^{-1}(1-\alpha_i)\right) max(H_{t_1}^{\alpha_i}, \ldots, H_{t_M}^{\alpha_i}, 0).$$

If $i < N-1$, then return to *Step 2*.
Step 9: Calculate the lookback put option price is

$$f_{put} \leftarrow \frac{1}{N-1}\sum_{i=1}^{N-1} f^{\alpha_i}.$$

Example 2 Assume that the parameters of the stock price are $X_0 = 1, c_1 = 1, \mu_1 = \sqrt{3}, \sigma_1 = \pi$, the parameters of the interest rate is $r_0 = 0.03, a = 1, b = 0.5, \sigma_0 = 0.02$, the parameters of the strike price is $k_0 = 3, c_2 = 0.9, \mu_2 = 1.5, \sigma_2 = 3.1$, then, the price of lookback put option with a maturity date $T = 5$ is 21.1753.

Figure 2 shows the relationship of the lookback put option pricing formula and the parameters. Figure 2a represents the price with respect to the floating interest rate; Fig. 2b represents the price with respect to the expiration time; Fig. 2c represents the price with respect to the original value; Fig. 2d represents the price with respect to the original uncertain strike price.

4 Conclusion

This paper was concerned on lookback options pricing problem within the framework of uncertainty theory. We extend the pricing problem discussed by Gao et al. (2018) to the lookback option pricing with uncertain strike price in a floating interest rate

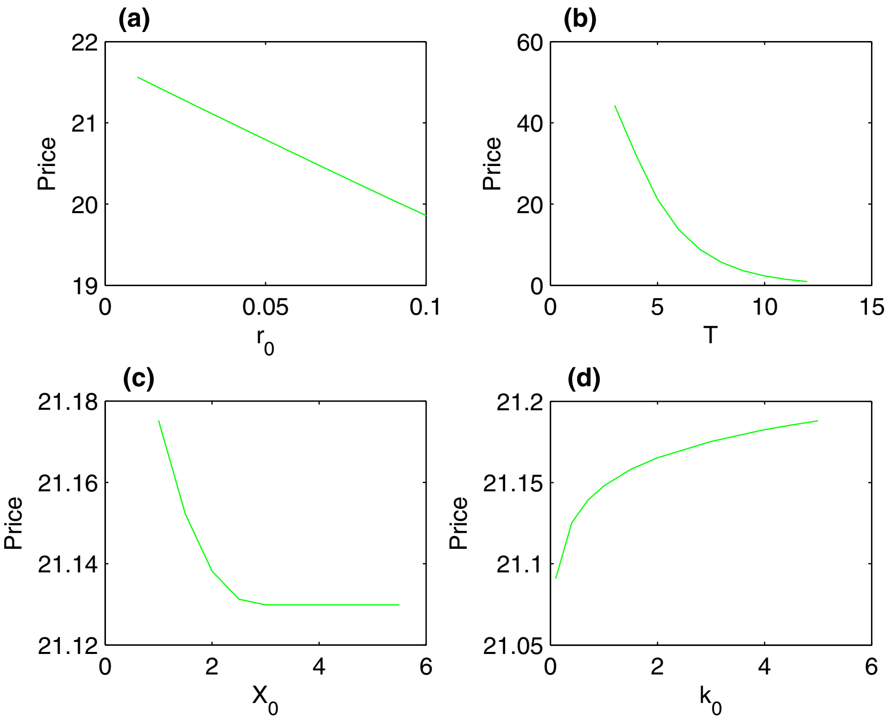

Fig. 2 Lookback put option price with respect to different parameters

environment. Based on the assumption that floating interest rate and uncertain strike price follow the uncertain differential equations respectively, the lookback call option and lookback put option pricing formulas were derived by the method of uncertain calculus. In addition, some numerical algorithms and examples are given to illustrate the pricing formulas. Besides, the relationship between the option price and the parameter were discussed. Further research could establish the uncertain-stochastic integral and investigate the integral formulas. Our future research is to investigate the multi-assets options pricing in random-uncertain environment.

Acknowledgements The authors would like to thank the anonymous referee and the editor for their helpful comments and valuable suggestions that led to several important improvements.

References

Chen, X. (2011). American option pricing formula for uncertain financial market. *International Journal of Operations Research*, *8*(2), 32–37.

Dai, L., Fu, Z., & Huang, Z. (2017). Option pricing formulas for uncertain financial market based on the exponential Ornstein–Uhlenbeck model. *Journal of Intelligent Manufacturing*, *28*(3), 597–604.

Gao, J. (2013). Uncertain bimatrix game with applications. *Fuzzy Optimization and Decision Making*, *12*(1), 65–78.

Gao, R., Liu, K., Li, Z., et al. (2019). American barrier option pricing formulas for stock model in uncertain environment. *IEEE Access*, *7*, 97846–97856.
Gao, Y., Yang, X., & Fu, Z. (2018). Lookback option pricing problem of uncertain exponential Ornstein–Uhlenbeck model. *Soft Computing*, *22*(17), 5647–5654.
Jiao, D., & Yao, K. (2015). An interest rate model in uncertain environment. *Soft Computing*, *19*(3), 775–780.
Kahneman, D., & Tversky, A. (1979). Prospect theory: Analysis of decision under risk. *Econometrica*, *47*(2), 263–291.
Liu, B. (2007). *Uncertainty theory* (2nd ed.). Berlin: Springer.
Liu, B. (2008). Fuzzy process, hybrid process and uncertain process. *Journal of Uncertain systems*, *2*(1), 3–16.
Liu, B. (2009). Some research problems in uncertainty theory. *Journal of Uncertain systems*, *3*(1), 3–10.
Liu, B. (2010). *Uncertainty theory: A branch of mathematics for modeling human uncertainty*. Berlin: Springer.
Liu, B. (2013a). A new definition of independence of uncertain sets. *Fuzzy Optimization and Decision Making*, *12*(4), 451–461.
Liu, B. (2013b). Toward uncertain finance theory. *Journal of Uncertainty, Analysis and Applications.*, *1*(1), 1.
Liu, B. (2014). Uncertain random graph and uncertain random network. *Journal of Uncertain systems*, *8*(1), 3–12.
Liu, B. (2015). *Uncertainty theory*. Berlin: Springer.
Peng, J., & Yao, K. (2011). A new option pricing model for stocks in uncertainty markets. *International Journal of Operations Research*, *8*(2), 18–26.
Sun, J., & Chen, X. (2015). Asian option pricing formula for uncertain financial market. *Journal of Uncertainty Analysis and Applications*, *3*(1), 11.
Sun, Y., Yao, K., & Fu, Z. (2018). Interest rate model in uncertain environment based on exponential Ornstein–Uhlenbeck equation. *Soft Computing*, *22*(2), 465–475.
Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, *5*(4), 297–323.
Yao, K. (2013). Extreme values and integral of solution of uncertain differential equation. *Journal of Uncertainty Analysis and Applications*, *1*(1), 1–21.
Yao, K., & Chen, X. (2013). A numerical method for solving uncertain differential equations. *Journal of Intelligent and Fuzzy Systems*, *25*(3), 825–832.
Yu, X. (2012). A stock model with jumps for uncertain markets. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *20*(3), 421–432.
Zhang, L., & Sun, Y. (2020). Power options pricing in uncertain environment. *Acta Scientiarum Naturalium Universitatis Nankaiensis*, *53*(2), 1–6.
Zhang, L., Sun, Y., & Meng, X. (2020). European spread option pricing with the floating interest rate for uncertain financial market. *Mathematical Problems in Engineering*, *2020*, 1–8.
Zhang, Z., & Liu, W. (2014). Geometric average asian option pricing for uncertain financial market. *Journal of Uncertain Systems*, *8*(4), 317–320.