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The valuation of forward-start rainbow options

Chun-Ying Chen · **Hsiao-Chuan Wang** ·
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Abstract This paper studies the valuation and hedging problems of forward-start rainbow options (FSROs). By combining the characteristics of both multiple assets and forward-start feature, this new type of derivative has many potential applications, for instance, to incorporate the reset provision in rainbow options for investors or hedgers or design more effective executive compensation plans. The main contribution of this paper is a novel martingale pricing technique for options whose payoffs are associated with multiple assets and time points. Equipped with this technique, the analytic pricing formula and the formulae of the delta and gamma of the FSRO are first derived. We conduct numerical experiments to verify these formulae and examine the characteristics of the FSRO's price and Greek letters. To demonstrate the importance and general applicability of the proposed technique, we also apply it to deriving the pricing formula for the discrete-sampling lookback rainbow options.

Keywords Discrete-sampling path-dependent option · Rainbow option ·
Forward-start option · Reset option · Lookback option

JEL Classification G13

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1 Introduction

Since the forward-start/reset and rainbow options are both desirable for investors, we are motivated to study rainbow options with the forward-start/reset feature. However, the valuation of options with both forward-start/reset feature and multiple underlying assets has been an ongoing problem. The difficulty of deriving analytic pricing formulae arises when combining different pricing methods for each feature. The main contribution of this paper is to propose a technique to evaluate options with both these features. This technique is also crucial in pricing a wide class of discrete-sampling path-dependent rainbow options.

Forward-start options (also known as delayed options) are similar to standard options except that the decision about a contractual term, such as the strike price, is postponed until a prespecified date. When the strike price is the only contractual term to be determined, forward-start options are called delayed-strike options. Forward-start options are often combined in series to form a ratchet option (also called a cliquet option) in equity or interest rate swap markets such that each option commences with an at-the-money strike price when the previous option expires. This provides a convenient way to hedge risks or lock in profits periodically. Rubinstein (1991) evaluates forward-start options where the strike price is determined on the forward-start date so as to make the option at the money at that time point. He combines the homogeneous property with respect to the asset and strike prices and the no-arbitrage argument to derive the pricing formula for forward-start options.

A reset option is in essence a delayed-strike forward-start option with a prespecified initial strike price. For a reset call (put) option on reset dates, a common strike-updating rule is to set the new strike price to be the minimum (maximum) of the current strike price and underlying asset price. This rule is attractive for option holders since it transforms out-of-the-money options to at-the-money options on reset dates. Several types of single-asset reset options are explored in the literature. Gray and Whaley (1999) introduce the analytic pricing formula for conventional reset put options. Cheng and Zhang (2000) propose the analytic-form solution for reset options with multiple reset dates. Both of Gray and Whaley (1999) and Cheng and Zhang (2000) evaluate directly the integrals over normal or lognormal distributions for solving the pricing formulae. On the other hand, Liao and Wang (2003) apply the technique for probability measure change to derive the closed-form pricing formula for reset options with a stepped reset of the strike price on prespecified reset dates.

Options involving two or more risky assets are generally referred to as rainbow options. The multi-asset feature has been applied to derivative products to benefit from the diversification, making them attractive to option holders. Stulz (1982) derives the analytic pricing formula for options on the maximum or minimum of two assets via solving partial differential equations. Options on the maximum or minimum of multiple assets are investigated by Johnson (1987), who derives the analytic-form formulae by exploiting the pricing formula and the corresponding change-of-numeraire interpretation for exchange options in Margrabe (1978). Following this methodology, Ouwehand and West (2006) show the details of using the change-of-numeraire technique to derive pricing formulae for various rainbow options.

Due to the benefit of the multiple assets and forward-start/reset feature in designing derivatives, this paper investigates rainbow options with the forward-start/reset feature, which are referred to as the forward-start rainbow options (abbreviated as FSROs for simplicity hereafter). There are many possible applications based on FSROs. For instance, it may not be the best arrangement for long-term rainbow option holders to have a fixed strike price specified on the issue day. The delay of the determination of the strike price gives rainbow option holders more flexibility, which is particularly desirable if the whole market is involved in some events during the option life. In addition, it is also possible to apply FSROs to the design of executive compensation plans. It is generally believed that the performance of a manager should be evaluated by several indexes, such as the equity price, sales volume, and net profit. The rainbow call options on the minimum of these indexes can encourage the manager to improve the firm in many aspects, and thus solve the problems arising from linking compensation plans only to equity prices. Since most compensation plans are in effect for multiple years, it is unreasonable to decide a fixed benchmark, i.e., the strike price of the rainbow call option on the minimum of multiple indexes, in the beginning. The introduction of the forward-start/reset feature to determine the benchmark, for example annually, may provide a satisfied solution for both firms and executive managers.

Although attractive as the FSRO may be, however, its pricing problem has never been discussed in the literature. One possible reason is that it is not a simple task to combine the pricing techniques for forward-start/reset options and rainbow options. Even though the change-of-probability-measure technique in Liao and Wang (2003) (for pricing reset options) and the change-of-numeraire technique in Ouwehand and West (2006) (for pricing rainbow options) are similar,[1] it is not straightforward to apply any one or the combination of these two techniques to evaluate the expected value of an asset price at a specified time point conditional on the comparison results between individual asset prices evolving up to different time points,[2] which is a problem arising exclusively when pricing FSROs or other discrete-sampling path-dependent rainbow options. This is because neither of these two techniques specifies how to transform individual asset prices (or the corresponding Brownian motions) evolving up to different time points for an auxiliary probability measure which may be related to another time point. To overcome the difficulty, this paper proposes a never explored way to use the change-of-probability-measure technique and find this novel martingale pricing technique is useful and convenient for pricing many discrete-sampling path-dependent rainbow options.

Based on this technique, we are able to derive the analytic formulae for FSROs and their delta and gamma. We consider a general payoff for FSROs, which encompass the rainbow options (Johnson 1987), the single-asset reset options (Gray and Whaley 1999), and the plain vanilla options (Black and Scholes 1973) as special cases. In addition to the analytical results, numerical experiments are conducted to examine the characteristics of the FSRO's price and Greek letters. We discover that there are

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[1] Theoretically speaking, both of these two techniques can be classified as martingale pricing methods.

[2] More concretely, to evaluate $E^Q[S_a(s) \cdot I(S_b(u) \leq S_c(v))]$, where Q denotes the risk-neutral probability measure, $S_i(z)$ denotes the price of the asset i at the time point z, and $I(\cdot)$ is defined as an indicator function.

significant differences between the benefit of the forward-start feature in the single-asset case and that in the multiple-asset case. Similar to the findings in the literature, there are jumps appearing for the delta and gamma as time passes the forward-start date. Finally, to demonstrate this pricing technique's general applicability, this paper also presents the pricing formula for the discrete-sampling lookback rainbow option, which is another example of path-dependent rainbow options that can be evaluated with the proposed technique.

This paper is organized as follows. We introduce the valuation framework, the novel martingale pricing technique, and the FSRO pricing formula in Sect. 2. Applications of the proposed pricing formula, which cover several well-known results in the literature, are discussed in Sect. 3. In Sect. 4, numerical experiments are conducted to validate the FSRO formula and to investigate various properties of FSROs. Sect. 5 concludes the paper.

2 Valuation of forward-start rainbow options

2.1 Basic framework

Consider a forward-start rainbow put option (hereafter abbreviated as FSRPO) on n non-dividend-paying assets,[3] whose price processes at the time point z are denoted as $S_i(z)$ for $i = 1, \ldots, n$. The current time is 0. On the forward-start date t, the forward-start strike price will be determined as the maximum among $S_1(t), \ldots, S_n(t)$, and K, which is the guaranteed minimum strike price specified initially. At maturity T, the cheapest underlying asset among $S_1(T), \ldots, S_n(T)$ can be sold at the forward-start strike price. The payoff of the FSRPO can be expressed as

$$P(T) = (\max[K, S_1(t), \ldots, S_n(t)] - \min[S_1(T), \ldots, S_n(T)])^+. \qquad (2.1)$$

Note that the considered payoff function is general and encompasses the payoffs of the rainbow put option on the minimum of multiple assets ($t = 0$), the single-asset reset put option ($n = 1$), and the plain vanilla put option ($t = 0, n = 1$) as special cases.

The underlying asset prices are assumed to follow geometric Brownian motions.[4] Under the risk-neutral measure Q, the dynamics of the underlying asset prices are posited as the following stochastic differential equations:

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[3] Although this paper considers only the non-dividend-paying case for simplicity, all results are straightfoward to be extended to underlying assets with constant dividend yields.

[4] Although it is widely accepted that jumps are able to explain the empirical regularities of derivative pricing, this paper focuses on pure diffusion processes. This is because incorporating additional jumps not only complicates the problem substantially but also obscures the contribution of this paper, which proposes a technique to tackle the evaluation of the expected value of an asset price at a specified time point conditional on the comparison results between individual asset prices (or the corresponding Brownian motions) evolving up to different time points. Even so, we highly appreciate the anonymous referee to mention this point. Nevertheless, we can analyze qualitatively the possible impacts of adding jumps on the values of FSRPOs. Since the reset provision can turn out-of-the-money options to become at-the-money options on reset dates, the reset provision can partially eliminate the effect of unfavorable jump movements. As for put options on the minimum of multiple assets, through the diversification effect, additional jumps could lower the

$$\frac{dS_i(z)}{S_i(z)} = r\,dt + \sigma_i\,dW_i^Q(z), \quad \text{for } i = 1, \ldots, n, \tag{2.2}$$

where the risk-free rate during the option life is assumed to be r, the volatility of the i-th asset's return process is denoted by σ_i, and the standard Brownian motion processes $W_i^Q(z)$ and $W_j^Q(z)$ for the i-th and j-th assets are correlated with the correlation coefficient ρ_{ij}.

2.2 The martingale pricing approach

When applying the martingale pricing approach to the FSRPO, note first that the pay-off function in Eq. (2.1) can be further decomposed into multiple components which are more easily evaluated. There are $n + 1$ possible scenarios for the forward-start strike price $\max[K, S_1(t), \ldots, S_n(t)]$: The first is that K turns out to be the maximum, that is, $\max[K, S_1(t), \ldots, S_n(t)] = K$; in the rest of the n possible scenarios $\max[K, S_1(t), \ldots, S_n(t)] = S_M(t)$, for $M = 1, \ldots, n$, where M denotes the asset index of the maximum price at time t. Similarly, the cheapest asset at maturity T has n possible cases: $\min[S_1(T), \ldots, S_n(T)] = S_m(T)$, for $m = 1, \ldots, n$, where m denotes the asset index of the minimum price at time T. By such a categorization, the FSRPO's payoff $P(T)$ (Eq. 2.1) is decomposed as

$$\begin{aligned}
P(T) = {}& \sum_{m=1}^{n} (K - S_m(T)) \cdot I\left(\{K \geq S_i(t)\}_{1 \leq i \leq n}, \left\{S_j(T) \geq S_m(T)\right\}_{1 \leq j \neq m \leq n}, K \geq S_m(T)\right) \\
& + \sum_{1 \leq m, M \leq n} (S_M(t) - S_m(T)) \cdot I\left(\{S_M(t) \geq S_{it}\}_{1 \leq i \leq n}, \left\{S_j(T) \geq S_m(T)\right\}_{1 \leq j \neq m \leq n}, \right. \\
& \left. S_M(t) \geq S_m(T)\right), \tag{2.3}
\end{aligned}$$

where $I(\cdot)$ is the indicator function, $S_{it} = K$ if $i = M$, and $S_{it} = S_i(t)$ otherwise.

Strictly speaking, the payoff decomposition of Eq. (2.3) holds for almost all events except when two or more underlying prices attain the same extremum values. Since such exceptions have zero probability, they do not affect the validity of the pricing formula developed in the following context. According to the martingale pricing theory, the arbitrage-free price of the FSRPO based on today's information set $\mathcal{F}_0$ is

$$P(0) = e^{-rT} E^Q [P(T)|\mathcal{F}_0]. \tag{2.4}$$

Footnote 4 continued

expected value of $\min[S_1(T), \ldots, S_n(T)]$. This is because if there is any asset with a net negative jump movement, this harmful effect will be retained by the minimum function. On the other hand, the realized value of $\min[S_1(T), \ldots, S_n(T)]$ rises only when all assets are with net positive jump movements. Since the FSRPO considered in this paper is essentially based on a minimum put options plus the reset option for the strike price, due to the above analysis, it can be expected that the additional jump processes will exhibit a generally positive impact on the value of the FSRPO.

After substituting the payoff decomposition in Eq. (2.3) into Eq. (2.4), we obtain

$$
\begin{aligned}
P(0) = & \sum_{m=1}^{n} K e^{-rT} E^{Q}\left[I\left(\{K \geq S_{i}(t)\}_{1 \leq i \leq n},\left\{S_{j}(T) \geq S_{m}(T)\right\}_{1 \leq j \neq m \leq n}, K \geq S_{m}(T)\right) | \mathcal{F}_{0}\right] \\
& -\sum_{m=1}^{n} e^{-rT} E^{Q}\left[S_{m}(T) \cdot I\left(\{K \geq S_{i}(t)\}_{1 \leq i \leq n},\left\{S_{j}(T) \geq S_{m}(T)\right\}_{1 \leq j \neq m \leq n}, K \geq S_{m}(T)\right) | \mathcal{F}_{0}\right] \\
& +\sum_{1 \leq m, M \leq n} e^{-rT} E^{Q}\left[S_{M}(t) \cdot I\left(\{S_{M}(t) \geq S_{it}\}_{1 \leq i \leq n},\left\{S_{j}(T) \geq S_{m}(T)\right\}_{1 \leq j \neq m \leq n}, S_{M}(t) \geq S_{m}(T)\right) | \mathcal{F}_{0}\right] \\
& -\sum_{1 \leq m, M \leq n} e^{-rT} E^{Q}\left[S_{m}(T) \cdot I\left(\{S_{M}(t) \geq S_{it}\}_{1 \leq i \leq n},\left\{S_{j}(T) \geq S_{m}(T)\right\}_{1 \leq j \neq m \leq n}, S_{M}(t) \geq S_{m}(T)\right) | \mathcal{F}_{0}\right].
\end{aligned}
\tag{2.5}
$$

2.3 Change of measure with respect to different assets and time points

In Eq. (2.5), each expectation inside the first summation can be simply evaluated as a multivariate standard normal cumulative distribution function (CDF). However, when we proceed with the expectations inside the other summations, different situations develop.

Take the expectations inside the third summation for example. Since $S_{M}(t) = S_{M}(0) \exp\left((r - \sigma_{M}^{2}/2)t + \sigma_{M} W_{M}^{Q}(t)\right)$, we can rewrite each expectation as

$$
\begin{aligned}
S_{M}(0) e^{rt} E^{Q}&\left[\exp(\sigma_{M} W_{M}^{Q}(t) - \sigma_{M}^{2} t/2)\right. \\
&\left.\cdot I\left(\{S_{M}(t) \geq S_{it}\}_{1 \leq i \leq n},\left\{S_{j}(T) \geq S_{m}(T)\right\}_{1 \leq j \neq m \leq n}, S_{M}(t) \geq S_{m}(T)\right) | \mathcal{F}_{0}\right].
\end{aligned}
$$

In order to evaluate the above expectation, we now introduce a new Radon–Nikodym derivative:

$$
\frac{d R_{Mt}}{d Q} = \exp\left(\sigma_{M} W_{M}^{Q}(t) - \sigma_{M}^{2} t/2\right),
$$

with which an equivalent probability measure R_{Mt} can be determined. Note that to define the probability measure R_{Mt}, we specifically take both an asset and a time point into account, i.e., the probability measure R_{Mt} is defined with respect to both the M-th asset and the time point t. Our approach is different from the common change-of-probability-measure technique used for option pricing, in which the auxiliary probability measure is associated with only a specified asset. Due to the consideration of the time dimension, our approach is more appropriate to evaluate options whose payoffs are associated with multiple assets and time points.

This approach raises a natural question: How is the Cameron–Martin–Girsanov theorem (or Girsanov theorem for short) applied with this Radon–Nikodym derivative? The answer of this question leads to a rather general solution that transforms Q-Brownian motions into R_{is}-Brownian motions, where i is the index of the underlying asset and s can be any time point. The theorem we proposed to solve this problem is stated as follows.

Theorem 2.1 *Let $W_j^Q(z)$ and $W_i^Q(z)$ be two correlated Q-Brownian motions, and ρ_{ji} is their correlation coefficient. By defining the Radon–Nikodym derivative as $dR_{is}/dQ = \exp(\sigma_i W_i^Q(s) - \sigma_i^2 s/2)$, we can obtain that*

$$W_j^{R_{is}}(z) = W_j^Q(z) - \rho_{ji}\sigma_i \min(s, z)$$

is an R_{is}-Brownian motion for any time points s and z.

Proof See "Appendix 1".   □

Theorem 2.1 states that, given the Radon–Nikodym derivative dR_{is}/dQ and the corresponding R_{is} probability measure, the transformation from the Q-Brownian motion $W_j^Q(z)$ to the R_{is}-Brownian motion $W_j^{R_{is}}(z)$ is simply achieved by subtracting a drift term $\rho_{ji}\sigma_i \min(s, z)$, where i and j are any two asset indexes, and s and z can be any two time points. More specifically, if the considered time point z is before s, $W_j^{R_{is}}(z) = W_j^Q(z) - \rho_{ji}\sigma_i z$; if the considered time point z is after s, $W_j^{R_{is}}(z) = W_j^Q(z) - \rho_{ji}\sigma_i s$. Note that when $z > s$, although both $W_j^{R_{is}}(z)$ and $W_j^Q(z)$ are Brownian motions evolving up to the time point z, the adjustment term is $\rho_{ji}\sigma_i s$ rather than $\rho_{ji}\sigma_i z$. In "Appendix 1", we not only prove Theorem 2.1 but also elaborate this specialty regarding the time dimension through evaluating an illustrative example.

It should be noted that to prove Theorem 2.1, we consider the kernel of the Girsanov theorem to be $\sigma_i \cdot I(z \leq s)$ rather than a constant σ_i, which is commonly used in the literature for option pricing, such as Ouwehand and West (2006) for pricing rainbow options and Liao and Wang (2003) for pricing reset options.[5] Comparing Theorem 2.1 with this commonly used method, one can tell that Theorem 2.1 is more general due to the consideration of the additional time dimension. By analyzing the different kernels used in the Girsanov theorem, it is straightforward to infer that Theorem 2.1 can reduce to the commonly used method if the specified time s is fixed to be the maturity date T.

To the best of our knowledge, Theorem 2.1 proposes a never explored way to use the Girsanov theorem for option pricing. In fact, Theorem 2.1 is crucial for not only pricing new path-dependent rainbow options but also unifying existing results. While we apply Theorem 2.1 to pricing FSRPOs in the next subsection, we also employ it in "Appendix 4" to obtain the pricing formula for the discrete-sampling lookback rainbow option, which is another new type of path-dependent rainbow options. In addition, Sect. 3 will demonstrate that the pricing formula of the

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[5] For example, Eqs. (A.5) and (A.6) in Liao and Wang (2003) can be rewritten with our notation system as follows.

$$\frac{dR_1}{dQ} = \exp\left(\sigma_1 W_1^Q(T) - \sigma_1^2 T/2\right), \text{ and } dW_1^{R_1}(z) = dW_1^Q(z) - \sigma_1 dt.$$

Since they price a single-asset reset option, the volatility of the only asset is denoted as σ_1. With σ_1 as the kernel in the Girsanov theorem, an equivalent probability measure R_1 and the Brownian motion under it, $dW_1^{R_1}(z)$, can be defined. Note that the probability measure R_1 is defined with respect to the single asset but independent of any specified time point.

FSRPO based on Theorem 2.1 can reduce to the pricing formulae of rainbow options, single-asset reset options, and plain vanilla options that are already derived in the literature.

2.4 Analytical formulae for FSRPO and its delta and gamma

Equipped with Theorem 2.1, we can now derive the FSRPO pricing formula by the martingale pricing approach. The detailed derivation to obtain the explicit solution is presented in "Appendix 2", and the FSRPO pricing formula is stated as follows:

$$
\begin{aligned}
P(0) = {} & \sum_{m=1}^{n} K e^{-rT} N_{2n}\left(\left\{d_{K,it}^{Q}\right\}_{1\le i\le n}, \left\{d_{jT,mT}^{Q}\right\}_{1\le j\ne m\le n}, d_{K,mT}^{Q}; R^{K,m}\right) \\
& - \sum_{m=1}^{n} S_{m}(0) N_{2n}\left(\left\{d_{K,it}^{R_{mT}}\right\}_{1\le i\le n}, \left\{d_{jT,mT}^{R_{mT}}\right\}_{1\le j\ne m\le n}, d_{K,mT}^{R_{mT}}; R^{K,m}\right) \\
& + \sum_{1\le m,M\le n} S_{M}(0) e^{-r(T-t)} N_{2n}\left(\left\{d_{it}^{R_{Mt}}\right\}_{1\le i\le n}, \left\{d_{jT,mT}^{R_{Mt}}\right\}_{1\le j\ne m\le n}, d_{Mt,mT}^{R_{Mt}}; R^{M,m}\right) \\
& - \sum_{1\le m,M\le n} S_{m}(0) N_{2n}\left(\left\{d_{it}^{R_{mT}}\right\}_{1\le i\le n}, \left\{d_{jT,mT}^{R_{mT}}\right\}_{1\le j\ne m\le n}, d_{Mt,mT}^{R_{mT}}; R^{M,m}\right), \quad (2.6)
\end{aligned}
$$

where $N_{2n}(d_{\bullet,\bullet}^{\bullet}; R^{\bullet,\bullet})$ denotes a $2n$-variate standard normal CDF with $2n$ parameters $d_{\bullet,\bullet}^{\bullet}$ and a $2n \times 2n$ correlation matrix $R^{\bullet,\bullet}$. In addition, we also define σ_{ab} as $\sqrt{\sigma_a^2 - 2\rho_{ab}\sigma_a\sigma_b + \sigma_b^2}$ and $\sigma_{as,b\tau}$ as $\sqrt{\sigma_a^2 s - 2\rho_{ab}\sigma_a\sigma_b \min(s,\tau) + \sigma_b^2 \tau}$. With the above notations, $d_{\bullet,\bullet}^{\bullet}$ and $R^{\bullet,\bullet}$ are explicitly expressed as follows.

$$
\begin{aligned}
d_{K,it}^{Q} &= \frac{\ln(K/S_i(0)) - (r - \sigma_i^2/2)t}{\sigma_i\sqrt{t}}, \\
d_{jT,mT}^{Q} &= \frac{\ln(S_j(0)/S_m(0)) + (\sigma_m^2 - \sigma_j^2)T/2}{\sigma_{jm}\sqrt{T}}, \\
d_{K,mT}^{Q} &= \frac{\ln(K/S_m(0)) - (r - \sigma_m^2/2)T}{\sigma_m\sqrt{T}}, \\
d_{K,it}^{R_{mT}} &= \frac{\ln(K/S_i(0)) - (r - (\sigma_{im}^2 - \sigma_i^2)/2)t}{\sigma_i\sqrt{t}}, \\
d_{jT,mT}^{R_{mT}} &= \frac{\ln(S_j(0)/S_m(0)) - \sigma_{jm}^2 T/2}{\sigma_{jm}\sqrt{T}}, \\
d_{K,mT}^{R_{mT}} &= \frac{\ln(K/S_m(0)) - (r + \sigma_m^2/2)T}{\sigma_m\sqrt{T}},
\end{aligned}
$$

$$d_{it}^{R_{Mt}} = \begin{cases} d_{Mt,K}^{R_{Mt}} = \frac{\ln(S_M(0)/K)+(r+\sigma_M^2/2)t}{\sigma_M\sqrt{t}} & i = M \\ d_{Mt,it}^{R_{Mt}} = \frac{\ln(S_M(0)/S_i(0))+\sigma_{iM}^2 t/2}{\sigma_{iM}\sqrt{t}} & i \neq M \end{cases},$$

$$d_{jT,mT}^{R_{Mt}} = \frac{\ln(S_j(0)/S_m(0)) + (\sigma_{Mt,mT}^2 - \sigma_{Mt,jT}^2)/2}{\sigma_{jm}\sqrt{t}},$$

$$d_{Mt,mT}^{R_{Mt}} = \frac{\ln(S_M(0)/S_m(0)) + r(t-T) + \sigma_{Mt,mT}^2/2}{\sigma_{Mt,mT}},$$

$$d_{it}^{R_{mT}} = \begin{cases} d_{Mt,K}^{R_{mT}} = \frac{\ln(S_M(0)/K)+(r-(\sigma_{Mm}^2-\sigma_m^2)/2)t}{\sigma_M\sqrt{t}} & i = M \\ d_{Mt,it}^{R_{mT}} = \frac{\ln(S_M(0)/S_i(0))+(\sigma_{im}^2-\sigma_{Mm}^2)t/2}{\sigma_{iM}\sqrt{t}} & i \neq M \end{cases},$$

$$d_{jT,mT}^{R_{mT}} = \frac{\ln(S_j(0)/S_m(0)) - \sigma_{jm}^2 T/2}{\sigma_{jm}\sqrt{T}},$$

$$d_{Mt,mT}^{R_{mT}} = \frac{\ln(S_M(0)/S_m(0)) + r(t-T) - \sigma_{Mt,mT}^2/2}{\sigma_{Mt,mT}},$$

$$R^{K,m} = \begin{pmatrix} (I)_{n\times n} & (II)_{n\times(n-1)} & (III)_{n\times 1} \\ (II)'_{(n-1)\times n} & (IV)_{(n-1)\times(n-1)} & (V)_{(n-1)\times 1} \\ (III)'_{1\times n} & (V)'_{1\times(n-1)} & 1 \end{pmatrix}_{2n\times 2n},$$

with

$$(I)_{n\times n} = (\rho_{ik})_{\substack{1\le i\le n \\ 1\le k\le n}},$$

$$(II)_{n\times(n-1)} = \left(\frac{\rho_{im}\sigma_m - \rho_{il}\sigma_l}{\sigma_{lm}}\sqrt{\frac{t}{T}} \right)_{\substack{1\le i\le n \\ 1\le l\neq m\le n}},$$

$$(III)_{n\times 1} = \left(\rho_{im}\sqrt{\frac{t}{T}} \right)_{1\le i\le n},$$

$$(IV)_{(n-1)\times(n-1)} = \left(\frac{\sigma_m^2 - \rho_{lm}\sigma_m\sigma_l - \rho_{jm}\sigma_j\sigma_m + \rho_{jl}\sigma_j\sigma_l}{\sigma_{jm}\sigma_{lm}} \right)_{\substack{1\le j\neq m\le n \\ 1\le l\neq m\le n}},$$

$$(V)_{(n-1)\times 1} = \left(\frac{\sigma_m - \rho_{jm}\sigma_j}{\sigma_{jm}} \right)_{1\le j\neq m\le n},$$

and

$$R^{M,m} = \begin{pmatrix} (VI)_{n\times n} & (VII)_{n\times(n-1)} & (VIII)_{n\times 1} \\ (VII)'_{(n-1)\times n} & (IX)_{(n-1)\times(n-1)} & (X)_{(n-1)\times 1} \\ (VIII)'_{1\times n} & (X)'_{1\times(n-1)} & 1 \end{pmatrix}_{2n\times 2n},$$

with

$$(VI)_{n \times n} = \left(\frac{\rho_{ik}\sigma_i\sigma_k - \rho_{kM}\sigma_k\sigma_M - \rho_{iM}\sigma_i\sigma_M + \sigma_M^2}{\sigma_{iM}\ \sigma_{kM}} \right)_{\substack{1 \le i \le n \\ 1 \le k \le n}},$$

$$(VII)_{n \times (n-1)} = \left(\frac{\rho_{im}\sigma_i\sigma_m - \rho_{il}\sigma_i\sigma_l - \rho_{mM}\sigma_m\sigma_M + \rho_{lM}\sigma_l\sigma_M}{\sigma_{iM}\ \sigma_{lm}} \sqrt{\frac{t}{T}} \right)_{\substack{1 \le i \le n \\ 1 \le l \ne m \le n}},$$

$$(VIII)_{n \times 1} = \left(\frac{\rho_{im}\sigma_i\sigma_m - \rho_{iM}\sigma_i\sigma_M - \rho_{mM}\sigma_m\sigma_M + \sigma_M^2}{\sigma_{iM}\ \sigma_{Mt,mT}} \sqrt{t} \right)_{1 \le i \le n},$$

$$(IX)_{(n-1) \times (n-1)} = \left(\frac{\sigma_m^2 - \rho_{jm}\sigma_j\sigma_m - \rho_{lm}\sigma_l\sigma_m + \rho_{jl}\sigma_j\sigma_l}{\sigma_{jm}\ \sigma_{lm}} \right)_{\substack{1 \le j \ne m \le n \\ 1 \le l \ne m \le n}},$$

$$(X)_{(n-1) \times 1} = \left(\frac{\sigma_m^2 T - \rho_{mM}\sigma_m\sigma_M t - \rho_{jm}\sigma_j\sigma_m T + \rho_{jM}\sigma_j\sigma_M t}{\sigma_{jm}\sqrt{T}\ \sigma_{Mt,mT}} \right)_{1 \le j \ne m \le n}.$$

Note that we assume $S_{it} = K$ if $i = M$ in Eq. (2.3); thus in $R^{M,m}$, we set $\sigma_i = 0$ when $i = M$. Consequently, $\sigma_{iM} = \sqrt{\sigma_i^2 - 2\rho_{iM}\sigma_i\sigma_M + \sigma_M^2} = \sqrt{\sigma_M^2} = \sigma_M$ when $i = M$.

In addition to the pricing formula, the delta and gamma of the FSRPO can be derived analytically. The delta of the a-th underlying asset is

$$\frac{\partial P}{\partial S_a} = -N_{2n}\left(\left\{ d_{K,it}^{R_{aT}} \right\}_{1 \le i \le n}, \left\{ d_{jT,aT}^{R_{aT}} \right\}_{1 \le j \ne a \le n}, d_{K,aT}^{R_{aT}};\ R^{K,a} \right)$$

$$+ \sum_{a'=1}^{n} e^{-r(T-t)} N_{2n}\left(\left\{ d_{it}^{R_{at}} \right\}_{1 \le i \le n}, \left\{ d_{jT,a'T}^{R_{at}} \right\}_{1 \le j \ne a' \le n}, d_{at,a'T}^{R_{at}};\ R^{a,a'} \right)$$

$$- \sum_{a'=1}^{n} N_{2n}\left(\left\{ d_{it}^{R_{aT}} \right\}_{1 \le i \le n}, \left\{ d_{jT,aT}^{R_{aT}} \right\}_{1 \le j \ne a \le n}, d_{a't,aT}^{R_{aT}};\ R^{a',a} \right),$$

where a' is merely an index of the underlying asset. As for the gamma $\frac{\partial^2 P}{\partial S_a^2}$ and the cross gamma $\frac{\partial^2 P}{\partial S_a \partial S_b}$, the lengthy results are omitted in order to streamline this paper. "Appendix 3" shows how to analytically derive the formulae of the delta, gamma, and cross gamma of the FSRPO.

3 Applications of the pricing formula

This section shows that the pricing formula of the FSRPO (Eq. 2.6) encompasses many well-known option pricing formulae as special cases, such as rainbow options in Johnson (1987), single-asset reset options in Gray and Whaley (1999), and plain vanilla options in Black and Scholes (1973). These results demonstrate the wide potential and applications of Theorem 2.1, which can be used to derive the pricing formulae for not

only discrete-sampling path dependent rainbow options but also many existent options in the literature.

3.1 Rainbow puts on the minimum of multiple assets

The FSRPO can be reduced to the rainbow put option on the minimum of multiple assets (or rainbow put option for short) if we set the forward-start date to be today, that is, $t = 0$. In this scenario, the strike price at time 0, i.e., $\max[K, S_1(0), \ldots, S_n(0)]$, should be a known constant. For the reason of notation brevity, this strike price is still denoted K. With this new notation, we simplify Eqs. (2.1) and (2.6) and obtain, respectively, the payoff and the pricing formula of the rainbow put option:

$$P_m(T) = (K - \min[S_1(T), \ldots, S_n(T)])^+,$$

and

$$P_m(0) = \sum_{m=1}^{n} K e^{-rT} N_n \left(\left\{ d_{jT,mT}^Q \right\}_{1 \le j \ne m \le n}, d_{K,mT}^Q; R^{K,m} \right)$$

$$- \sum_{m=1}^{n} S_m(0) N_n \left(\left\{ d_{jT,mT}^{R_{mT}} \right\}_{1 \le j \ne m \le n}, d_{K,mT}^{R_{mT}}; R^{K,m} \right),$$

where

$$d_{jT,mT}^Q = \frac{\ln \left(S_j(0)/S_m(0) \right) + \left(\sigma_m^2 - \sigma_j^2 \right) T/2}{\sigma_{jm} \sqrt{T}},$$

$$d_{K,mT}^Q = \frac{\ln(K/S_m(0)) - (r - \sigma_m^2/2)T}{\sigma_m \sqrt{T}},$$

$$d_{jT,mT}^{R_{mT}} = \frac{\ln(S_j(0)/S_m(0)) - \sigma_{jm}^2 T/2}{\sigma_{jm} \sqrt{T}},$$

$$d_{K,mT}^{R_{mT}} = \frac{\ln(K/S_m(0)) - (r + \sigma_m^2/2)T}{\sigma_m \sqrt{T}},$$

$$R^{K,m} = \begin{pmatrix} (I)_{(n-1)\times(n-1)} & (II)_{(n-1)\times 1} \\ (II)'_{1\times(n-1)} & 1 \end{pmatrix}_{n \times n},$$

with

$$(I)_{(n-1)\times(n-1)} = \left(\frac{\sigma_m^2 - \rho_{lm}\sigma_m\sigma_l - \rho_{jm}\sigma_j\sigma_m + \rho_{jl}\sigma_j\sigma_l}{\sigma_{jm}\sigma_{lm}} \right)_{\substack{1 \le j \ne m \le n \\ 1 \le l \ne m \le n}},$$

$$(II)_{(n-1)\times 1} = \left(\frac{\sigma_m - \rho_{jm}\sigma_j}{\sigma_{jm}} \right)_{1 \le l \ne m \le n}.$$

By the put-call parity, the above formula is consistent with Johnson's (1987) results for rainbow call options.

3.2 Reset options

If our model takes only one asset into account, that is, $n = 1$, the payoff in Eq. (2.1) becomes

$$P_R(T) = (\max[K, S_1(t)] - S_1(T))^+,$$

It is indeed the payoff of the single-asset reset put option whose strike price is reset to be the maximum of K and $S_1(t)$ at time t.

Setting $n = 1$, the FSRPO's pricing formula (Eq. 2.6) reduces to the formula of the reset put option:

$$
\begin{aligned}
P_R(0) = {} & K e^{-rT} N_2 \left(d^Q_{K,1t}, d^Q_{K,1T}; \ R^{K,1} \right) \\
& - S_1(0) \ N_2 \left(d^{R_{1T}}_{K,1t}, d^{R_{1T}}_{K,1T}; \ R^{K,1} \right) \\
& + S_1(0) e^{-r(T-t)} \ N_2 \left(d^{R_{1t}}_{1t,K}, d^{R_{1t}}_{1t,1T}; \ R^{1,1} \right) \\
& - S_1(0) \ N_2 \left(d^{R_{1T}}_{1t,K}, d^{R_{1T}}_{1t,1T}; \ R^{1,1} \right),
\end{aligned}
$$

where

$$d^Q_{K,1t} = \frac{\ln(K/S_1(0)) - (r - \sigma_1^2/2)t}{\sigma_1 \sqrt{t}},$$

$$d^Q_{K,1T} = \frac{\ln(K/S_1(0)) - (r - \sigma_1^2/2)T}{\sigma_1 \sqrt{T}},$$

$$d^{R_{1T}}_{K,1t} = \frac{\ln(K/S_1(0)) - (r + \sigma_1^2/2)t}{\sigma_1 \sqrt{t}},$$

$$d^{R_{1T}}_{K,1T} = \frac{\ln(K/S_1(0)) - (r + \sigma_1^2/2)T}{\sigma_1 \sqrt{T}},$$

$$d^{R_{1t}}_{1t,K} = \frac{\ln(S_1(0)/K) + (r + \sigma_1^2/2)t}{\sigma_1 \sqrt{t}},$$

$$d^{R_{1t}}_{1t,1T} = -(r/\sigma_1 - \sigma_1/2)\sqrt{T - t},$$

$$d^{R_{1T}}_{1t,K} = \frac{\ln(S_1(0)/K) + (r + \sigma_1^2/2)t}{\sigma_1 \sqrt{t}},$$

$$d^{R_{1T}}_{1t,1T} = -(r/\sigma_1 + \sigma_1/2)\sqrt{T - t},$$

$$R^{K,1} = \begin{pmatrix} 1 & \sqrt{t/T} \\ \sqrt{t/T} & 1 \end{pmatrix}_{2\times 2},$$

$$R^{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}.$$

The above pricing formula is identical to that in Gray and Whaley (1999).

3.3 Plain vanilla options

The FSRPO becomes the vanilla put option when $n = 1$ and $t = 0$. Under these assumptions, the strike price of $\max[K, S_1(0)]$ should be a known constant at the present time point. We denote this strike price as K for the brevity of notations. Therefore, the payoff (Eq. 2.1) becomes

$$P_{BS}(T) = (K - S_1(T))^+.$$

Accordingly, setting $(n, t) = (1, 0)$ in the FSRPO's pricing formula (Eq. 2.6), we obtain the famous Black–Scholes formula of the plain vanilla put option:

$$P_{BS}(0) = Ke^{-rT} N_1(d_{K,1T}^Q) - S_1(0) \, N_1(d_{K,1T}^{R_{1T}}),$$

where

$$d_{K,1T}^Q = \frac{\ln(K/S_1(0)) - (r - \sigma_1^2/2)T}{\sigma_1 \sqrt{T}},$$

$$d_{K,1T}^{R_{1T}} = \frac{\ln(K/S_1(0)) - (r + \sigma_1^2/2)T}{\sigma_1 \sqrt{T}}.$$

4 Numerical analysis and option characteristics

This section conducts numerical experiments to study the characteristics of the FSRPO. In order to verify the FSRPO's pricing formula, the analytical option prices and the results generated according to the Monte Carlo simulations under various parameter settings are first compared. Second, we contrast the characteristics of the FSRPO with those of rainbow put and reset options since the FSRPO is in essence the combination of these two types of options. Finally, this paper investigates the price behavior as well as the delta, gamma, and cross gamma of the FSRPO. We examine an FSRPO with two underlying assets in this section. Its payoff is reduced from Eq. (2.1) as follows.

$$P(T) = (\max[K, S_1(t), S_2(t)] - \min[S_1(T), S_2(T)])^+. \tag{4.1}$$

4.1 Validity test for the FSRPO pricing formula

To demonstrate the validity of the proposed closed-form pricing formula, this section compares analytical option values computed via the pricing formula (2.6) with (experimental) CIs obtained through the Monte Carlo simulations. The parameters of interest are as follows. The initial underlying prices $S_1(0)$ and $S_2(0)$ and the guaranteed minimum strike price K are chosen from the set {90, 100, 110}. Due to the symmetric roles of the two underlying assets, we set σ_1, the volatility of the return of the first asset,

at 10, 30, and 50 % while fixing σ_2, the volatility of the return of the second asset, at 30 % for simplicity. Four different values—0, 0.25, 0.5, and 0.75—are considered for the forward-start date t. The option maturity T is fixed at one year, and the risk-free rate r is fixed at 5 %.

Table 1 lists the numerical results for all possible combinations of the parameter values when $\rho_{12} = -0.5$. With respect to each combination, the FSRPO's price is computed based on the pricing formula (2.6), and the 95 % CI, reported in parentheses under each option price, is computed by the Monte Carlo method with 100,000 simulations. Almost all theoretical option values are within the 95 % CIs. The results of these numerical experiments demonstrate the validity of our pricing formula (2.6) for the FSRPOs. More numerical results for $\rho_{12} = 0$ and $\rho_{12} = 0.5$ are not reported for the purpose to streamline this paper. Nevertheless, these results are available upon request.

Note that as $t = 0$, the forward-start strike price $\max[K, S_1(t), S_2(t)]$ is known to be the result of $\max[K, S_1(0), S_2(0)]$ and the FSRPO is identical to the rainbow put option with the strike price of $\max[K, S_1(0), S_2(0)]$ in Stulz (1982). To avoid confusion about the meaning of the strike price K in this paper and in Stulz (1982), when comparing the numerical results for $t = 0$, we report only the results for $K \geq \max[S_1(0), S_2(0)]$. This is becuase in these scenarios, the effective strike price considered in our model is $\max[K, S_1(0), S_2(0)] = K$, and one can simply input the value of K as the strike price in Stulz's (1982) option pricing formula to obtain identical results as those in the column of $t = 0$ in Table 1.

4.2 Compare FSRPOs with rainbow and forward-start/reset options

Although the FSRO is designed essentially as the combination of the rainbow and forward-start/reset options, we are interested to investigate whether the FSRO owns distinct characteristics from those in rainbow and forward-start/reset options. If it does, we can conclude that there exists interaction between the rainbow option and the forward-start/reset feature. To achieve it, we examine the impact of different forward-start dates and different correlations on the price of the FSRPO in this section.

The base case of the parameter values in this section is as follows: $S_1(0) = S_2(0) = K = 100$, $r = 0.05$, $\rho_{12} = 0$, $\sigma_1 = \sigma_2 = 0.3$, $t = 0.25$, and $T = 1$.

4.2.1 Characteristics with respect to varying the forward-start date

The solid curve in Fig. 1 shows how the FSRPO value varies with respect to the forward-start date t while the other parameters remain fixed. Because of the benefit of the forward-start feature, an FSRPO is more valuable than a rainbow put option, which is represented by the dashed line in Fig. 1. In addition, as the forward-start date t tends towards the present date, this benefit disappears and the price of the FSRPO converges towards that of the two-asset rainbow put option.

Moreover, Fig. 1 shows a positive relationship between the value of the FSRPO and the forward-start date t, which means the benefit of the forward-start feature increases with the delay of the forward-start date. However, this price behavior contrasts sharply with that of the single-asset case, which is explained as follows.

**Table 1** Option Values of the Forward-Start Rainbow Puts as $\rho_{12} = -0.5$

(S_1, S_2)	K	$\sigma_1 = 10\%$				$\sigma_1 = 30\%$				$\sigma_1 = 50\%$			
		$t=0$	$t=0.25$	$t=0.5$	$t=0.75$	$t=0$	$t=0.25$	$t=0.5$	$t=0.75$	$t=0$	$t=0.25$	$t=0.5$	$t=0.75$
(90,90)	90	9.87 (9.82, 9.98)	16.38 (16.23, 16.48)	19.99 (19.81, 20.02)	23.12 (23.01, 23.32)	15.63 (15.51, 15.70)	25.18 (25.10, 25.28)	30.11 (29.97, 30.22)	34.29 (34.10, 34.46)	21.77 (21.72, 21.77)	34.68 (34.63, 34.72)	41.06 (40.95, 41.07)	46.36 (46.33, 46.46)
	100	18.13 (18.02, 18.20)	20.37 (20.23, 20.45)	23.04 (23.01, 23.24)	25.65 (25.43, 25.72)	24.08 (23.94, 24.18)	28.24 (28.09, 28.38)	32.71 (32.66, 32.86)	36.69 (36.59, 36.80)	30.36 (30.32, 30.40)	37.60 (37.59, 37.66)	43.69 (43.62, 43.77)	48.90 (48.82, 48.99)
	110	27.52 (27.41, 27.60)	28.23 (28.17, 28.33)	29.85 (29.71, 29.94)	31.63 (31.45, 31.69)	33.22 (33.14, 33.30)	34.62 (34.48, 34.66)	37.75 (37.53, 37.86)	41.07 (40.95, 41.24)	39.48 (39.43, 39.53)	43.21 (43.19, 43.29)	48.28 (48.21, 48.30)	53.03 (53.03, 53.13)
(90,100)	90	-	18.69 (18.62, 18.79)	22.27 (22.15, 22.46)	25.46 (25.27, 25.64)	-	27.24 (27.06, 27.35)	32.17 (31.97, 32.28)	36.45 (36.22, 36.62)	-	36.79 (36.74, 36.84)	43.22 (43.12, 43.25)	48.67 (48.52, 48.71)
	100	14.54 (14.44, 14.58)	20.79 (20.64, 20.93)	24.11 (23.98, 24.14)	27.10 (26.94, 27.24)	20.94 (20.85, 21.00)	28.70 (28.58, 28.83)	33.67 (33.53, 33.92)	37.89 (37.77, 37.99)	27.57 (27.54, 27.64)	38.17 (38.10, 38.20)	44.74 (44.61, 44.80)	50.28 (50.19, 50.40)
	110	23.81 (23.73, 23.91)	26.54 (26.46, 26.66)	29.26 (29.17, 29.36)	31.72 (31.72, 31.82)	29.80 (29.71, 29.95)	33.13 (33.09, 33.22)	37.28 (37.20, 37.43)	41.20 (41.14, 41.44)	36.40 (36.34, 36.44)	41.96 (41.92, 42.01)	47.98 (47.96, 48.07)	53.30 (53.25, 53.36)
(90,110)	90	-	23.75 (23.68, 23.82)	26.92 (26.84, 27.02)	29.98 (29.80, 30.08)	-	31.52 (31.39, 31.74)	36.12 (35.98, 36.21)	40.33 (40.10, 40.52)	-	40.80 (40.69, 40.79)	47.00 (46.97, 47.13)	52.43 (52.36, 52.55)
	100	-	24.61 (24.54, 24.67)	27.91 (27.83, 28.16)	30.96 (30.68, 31.03)	-	32.05 (31.95, 32.16)	36.88 (36.81, 37.06)	41.22 (41.10, 41.56)	-	41.28 (41.23, 41.39)	47.76 (47.68, 47.82)	53.37 (53.32, 53.45)
	110	21.09 (21.13, 21.27)	28.06 (27.88, 28.16)	31.43 (31.30, 31.51)	34.30 (34.11, 34.46)	27.16 (26.97, 27.17)	34.50 (34.25, 34.61)	39.19 (39.03, 39.30)	43.43 (43.38, 43.60)	33.96 (33.89, 33.98)	43.30 (43.25, 43.37)	49.79 (49.71, 49.86)	55.43 (55.37, 55.56)
(100,90)	90	-	18.76 (18.71, 18.86)	22.21 (22.07, 22.36)	25.36 (25.32, 25.53)	-	27.24 (27.08, 27.47)	32.17 (31.89, 32.26)	36.45 (36.31, 36.47)	-	36.95 (36.87, 36.98)	43.48 (43.44, 43.60)	48.97 (48.92, 49.11)
	100	15.19 (15.09, 15.27)	19.28 (19.23, 19.38)	22.73 (22.50, 22.80)	25.87 (25.81, 26.08)	20.94 (20.82, 21.12)	28.70 (28.50, 28.89)	33.67 (33.55, 33.95)	37.89 (37.88, 38.20)	27.30 (27.25, 27.32)	38.86 (38.78, 38.88)	45.42 (45.32, 45.45)	50.95 (50.84, 51.03)
	110	23.89 (23.81, 23.95)	24.69 (24.57, 24.78)	26.75 (26.64, 26.83)	29.12 (28.94, 29.25)	29.80 (29.72, 29.93)	33.13 (33.04, 33.28)	37.28 (37.09, 37.40)	41.20 (41.01, 41.24)	36.26 (36.22, 36.31)	43.18 (43.16, 43.26)	49.13 (49.04, 49.23)	54.39 (54.33, 54.54)
(100,100)	90	-	17.99 (17.90, 18.06)	21.94 (21.87, 22.13)	25.40 (25.41, 25.60)	-	27.36 (27.32, 27.53)	32.65 (32.40, 32.82)	37.19 (37.09, 37.55)	-	37.70 (37.59, 37.69)	44.55 (44.49, 44.63)	50.29 (50.26, 50.41)
	100	10.97 (10.94, 11.13)	18.20 (18.10, 18.30)	22.21 (22.15, 22.39)	25.69 (25.53, 25.71)	17.37 (17.28, 17.42)	27.98 (27.73, 27.94)	33.46 (33.29, 33.62)	38.10 (38.04, 38.37)	24.19 (24.20, 24.23)	38.53 (38.47, 38.57)	45.62 (45.55, 45.62)	51.51 (51.50, 51.59)
	110	19.14 (19.06, 19.15)	21.91 (21.78, 22.06)	25.03 (24.94, 25.10)	28.03 (27.95, 28.07)	25.77 (25.65, 25.92)	30.87 (30.88, 31.06)	35.93 (35.68, 36.03)	40.41 (40.21, 40.65)	32.74 (32.68, 32.78)	41.32 (41.24, 41.31)	48.16 (48.08, 48.19)	53.97 (53.88, 54.02)

Table 1 continued

(S_1, S_2)	K	$\sigma_1 = 10\%$				$\sigma_1 = 30\%$				$\sigma_1 = 50\%$			
		$t=0$	$t=0.25$	$t=0.5$	$t=0.75$	$t=0$	$t=0.25$	$t=0.5$	$t=0.75$	$t=0$	$t=0.25$	$t=0.5$	$t=0.75$
(100,110)	90	-	20.29 (20.10, 20.33)	24.24 (24.22, 24.39)	27.76 (27.54, 27.80)	-	29.71 (29.55, 29.81)	35.03 (34.97, 35.25)	39.66 (39.59, 40.01)	-	40.27 (40.21, 40.32)	47.18 (47.09, 47.25)	53.04 (52.92, 53.10)
	100	-	20.35 (20.27, 20.55)	24.36 (24.30, 24.55)	27.91 (27.72, 27.97)	-	29.91 (29.67, 30.08)	35.41 (35.36, 35.51)	40.16 (39.75, 40.32)	-	40.54 (40.46, 40.59)	47.69 (47.63, 47.75)	53.74 (53.64, 53.90)
	110	15.52 (15.40, 15.63)	22.39 (22.33, 22.58)	26.12 (25.94, 26.19)	29.48 (29.42, 29.60)	22.63 (22.48, 22.74)	31.37 (31.31, 31.64)	36.90 (36.76, 36.99)	41.68 (41.59, 41.91)	29.96 (29.93, 30.04)	41.93 (41.88, 42.01)	49.23 (49.15, 49.29)	55.37 (55.33, 55.49)
(110,90)	90	-	23.92 (23.84, 24.11)	26.94 (26.71, 27.01)	29.93 (29.71, 30.01)	-	31.52 (31.39, 31.70)	36.12 (35.93, 36.30)	40.33 (40.24, 40.52)	-	40.87 (40.84, 40.95)	47.26 (47.21, 47.36)	52.81 (52.78, 52.97)
	100	-	23.93 (23.81, 24.04)	26.96 (26.74, 26.96)	29.97 (29.82, 30.23)	-	32.05 (31.88, 32.15)	36.88 (36.75, 37.17)	41.22 (40.97, 41.37)	-	42.02 (41.94, 42.06)	48.64 (48.54, 48.70)	54.32 (54.26, 54.52)
	110	21.63 (21.52, 21.77)	24.86 (24.74, 24.98)	27.84 (27.81, 28.09)	30.80 (30.63, 30.95)	27.16 (27.11, 27.32)	34.50 (34.39, 34.71)	39.19 (38.98, 39.41)	43.43 (43.19, 43.49)	33.58 (33.56, 33.62)	45.10 (45.09, 45.19)	51.54 (51.42, 51.56)	57.12 (57.07, 57.29)
(110,100)	90	-	20.42 (20.28, 20.45)	24.27 (24.08, 24.27)	27.77 (27.59, 27.81)	-	29.71 (29.55, 29.88)	35.03 (34.80, 34.97)	39.66 (39.58, 40.04)	-	40.25 (40.18, 40.30)	47.24 (47.16, 47.29)	53.15 (53.02, 53.24)
	100	-	20.42 (20.33, 20.57)	24.28 (24.15, 24.32)	27.79 (27.64, 27.93)	-	29.91 (29.87, 30.05)	35.41 (35.40, 35.65)	40.16 (39.91, 40.30)	-	40.71 (40.68, 40.79)	47.97 (47.96, 48.05)	54.06 (53.95, 54.22)
	110	16.23 (16.13, 16.34)	20.95 (20.83, 21.05)	24.82 (24.73, 24.89)	28.32 (28.24, 28.46)	22.63 (22.39, 22.62)	31.37 (31.28, 31.50)	36.90 (36.69, 37.01)	41.68 (41.48, 41.80)	29.68 (29.61, 29.70)	42.60 (42.49, 42.61)	49.89 (49.78, 49.93)	56.02 (56.01, 56.12)
(110,110)	90	-	19.79 (19.64, 19.91)	24.14 (24.02, 24.30)	27.93 (27.77, 28.19)	-	30.06 (29.87, 30.16)	35.79 (35.70, 35.92)	40.69 (40.43, 40.88)	-	41.37 (41.30, 41.39)	48.72 (48.63, 48.77)	54.90 (54.84, 55.03)
	100	-	19.79 (19.63, 19.90)	24.14 (24.02, 24.30)	27.94 (27.68, 28.03)	-	30.11 (29.98, 30.28)	35.95 (35.78, 36.11)	40.96 (40.78, 41.15)	-	41.50 (41.49, 41.58)	49.06 (48.99, 49.11)	55.40 (55.32, 55.58)
	110	12.07 (11.94, 12.11)	20.02 (20.01, 20.21)	24.43 (24.32, 24.58)	28.26 (28.08, 28.32)	19.11 (19.01, 19.19)	30.78 (30.56, 30.81)	36.80 (36.69, 37.01)	41.91 (41.65, 42.11)	26.61 (26.54, 26.63)	42.38 (42.34, 42.44)	50.18 (50.04, 50.24)	56.66 (56.55, 56.68)

This table reports the prices of FSRPOs according to the analytic formula and the Monte Carlo simulation. The parameters not specified in the table are summarized as follows: The number of assets n is two, the time to maturity T is one year, the volatility of the second asset σ_2 is 30%, and the risk-free interest rate r is 5%. Each FSRPO price (the number above in each cell) is computed with Eq. (2.6). The multivariate standard normal CDF is computed by Mathematica 5.2, which adopts a general method based on Genz (1992). Below each analytic option price is the 95% CIs generated by the Monte Carlo method with 100,000 simulations. In our notation system, as $t = 0$ and $K \geq \max[S_1(0), S_2(0)]$, the strike price is known to be K at $t = 0$, and the FSRPO is identical to the rainbow put option in Stulz (1982) with the strike price K. To avoid confusion with the strike price of the rainbow put option in Stulz (1982), option prices are not considered when $t = 0$ and $K < \max[S_1(0), S_2(0)]$ and represented here by a dash

Option Value
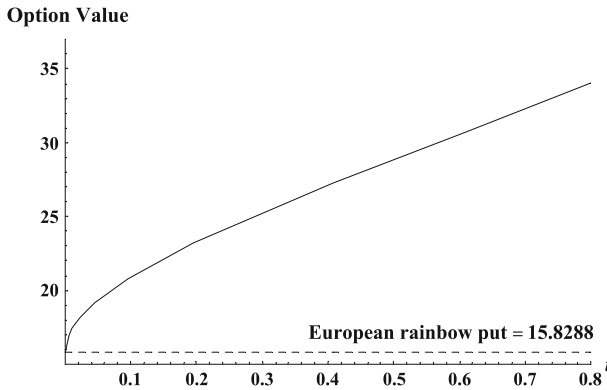
Fig. 1 Values of the FSRPO with respect to different forward-start date t. This figure depicts the relationship between the FSRPO value and the forward-start date t. The forward-start date t varies from 0 to 0.8, and the values of the other parameters are as follows: $S_1(0) = S_2(0) = K = 100$, $r = 0.05$, $\rho_{12} = 0$, $\sigma_1 = \sigma_2 = 0.3$, and $T = 1$. It is clear to find a positive relationship between the FSRPO price and the forward-start date t. The *dashed line* corresponds to the option value of the European rainbow put option on the minimum of two assets derived by the pricing formula in Johnson (1987). When t tends to zero, the *solid curve* converges to the *dashed line*, which demonstrates that the pricing formula for the classic rainbow option is a special case of our general pricing formula for the FSRO

Consider the single-asset reset put option whose payoff is

$$P_R(T) = (\max[K, S_1(t)] - S_1(T))^+. \tag{4.2}$$

The value of this option, based on the base parameter values, attains an interior maximum 12.1154 at $t = 0.557$ shown in Fig. 2.[6] Note that for $t = T$, the payoff function in Eq. (4.2) reduces to that of the plain vanilla put option,

$$P_R(T) = (\max[K, S_1(T)] - S_1(T))^+ = (K - S_1(T))^+.$$

This observation shows that the benefit of the forward-start feature also diminishes as t approaches the maturity date. Since the benefit of the forward-start feature disappears at both $t = 0$ and $t = T$ in the single-asset case, it results in the dome-like curve in Fig. 2.

Compared with the single-asset case, the FSRPO generally retains the benefit of the forward-start feature when $t = T$. This conclusion simply follows from contrasting the payoff in Eq. (4.1) when $t = 0$ with that when $t = T$:

$$P(T) = \begin{cases} (\max[K, S_1(0), S_2(0)] - \min[S_1(T), S_2(T)])^+ & \text{if } t = 0, \\ (\max[K, S_1(T), S_2(T)] - \min[S_1(T), S_2(T)])^+ & \text{if } t = T. \end{cases}$$

The strike price for $t = T$, $\max[K, S_1(T), S_2(T)]$, is expected to be higher than the strike price for $t = 0$, $\max[K, S_1(0), S_2(0)]$, due to the positive growth rates

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[6] The results in this figure are consistent with those in Fig. 5 of Gray and Whaley (1999).

Option Value
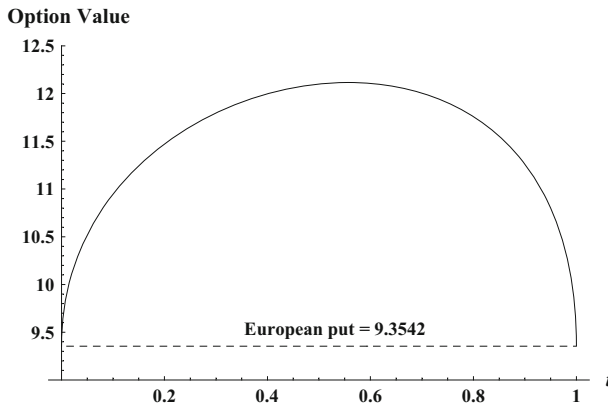
Fig. 2 Values of single-asset reset puts with respect to different reset date t. This figure examines the relation between the values of single-asset reset puts and different reset dates t ranging from 0 to T. In addition to $n = 1$ in the FSRPO's pricing formula, the values of the parameters are $S_1(0) = K = 100$, $r = 0.05$, $\sigma_1 = 0.3$, and $T = 1$. The *dashed line* represents the option value of the European plain vanilla put option. Since both the payoffs for $t = 0$ and $t = T$ for the single-asset reset put option are the same as for $(K - S_1(T))^+$, which equals the payoff of the European plain vanilla put option, the value of the single-asset reset put option converges to the value of the European put option at $t = 0$ and $t = T$. The *dome-like curve* reflects the different magnitudes of the benefits of the reset feature with respect to different reset dates. The maximum option value for the single-asset reset put option is about 12.1154 at $t = 0.557$. The results in this figure are consistent with those in Fig. 5 of Gray and Whaley (1999)

of the underlying assets, that is, $r > 0$. As a result, the benefit of the forward-start feature persists, and even grows when $t = T$ in the multi-asset case. Moreover, since the terminal payoff of FSRPOs depends on the relative levels of the underlying asset prices at both t and T, we further anticipate that the benefit of the forward-start feature should be affected by the correlation between the underlying assets.

Due to the above results, we conclude that the rainbow and reset options in the FSRPO could interact with each other such that the effect of the forward-start/reset feature of FSRPOs is significantly different from that of single-asset reset put options. Moreover, the magnitude of this difference depends on both the growth rate and the correlations of the underlying assets.

4.2.2 Characteristics with respect to varying the correlation

Since the correlations between underlying assets are important to measure the diversification effect in the rainbow option and thus determine its value, this section examines the impact of the correlations between underlying assets (or said the diversification effect) on FSROs. Similar to classic rainbow put options, Fig. 3 shows that the value of the FSRPO increases when the correlation ρ_{12} decreases. When the two underlying assets are negatively correlated, due to the diversification effect, one can anticipate that the expected strike price determined at t will increase and the expected cheapest asset price at T will decrease in the payoff (4.1). Both effects benefit the option holders and are strengthened when the underlying assets are more negatively

Option Value
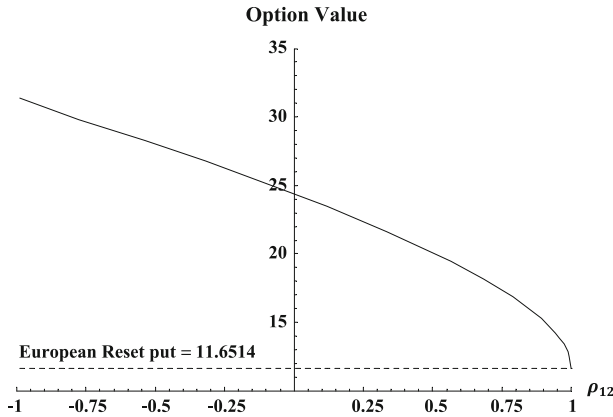
Fig. 3 Values of the FSRPO with respect to different correlation ρ_{12}. This figure shows the FSRPO values for different values of correlation ρ_{12}, ranging from -1 to 1. The values of the other parameters are $S_1(0) = S_2(0) = K = 100$, $r = 0.05$, $\sigma_1 = \sigma_2 = 0.3$, $t = 0.25$ and $T = 1$. The *dashed line* corresponds to the option value of the classic European reset put option in the case of a single underlying asset. The convergence to the value of the single-asset reset put option when ρ_{12} tends to one confirms that the single-asset reset option is a special case of the FSRO for $\rho_{12} = 1$. In addition, similar to the classic rainbow put option, the FSRPO's value increases with the decrease of the correlation ρ_{12}

correlated. On the other hand, these two kinds of benefits are weakened if the underlying assets become more positively correlated. In the most extreme case, when the two underlying assets are perfectly positively correlated and with the same initial price and volatility, the two-asset FSRPO behaves identically to the single-asset reset put.

4.3 Price behavior and greek letters of FSRPOs

This section examines the behavior of the option values and Greek letters of the FSRPO. Since the most important characteristics of the FSRPO are associated with the delayed strike and multiple sources of uncertainty, we always report the option values and Greek letters along the dimensions of the forward-start date t and the correlation ρ_{12}. In practice, three representative values for t (0.25, 0.5, and 0.75) and three representative values for ρ_{12} (-0.5, 0, and 0.5) are selected when analyzing the price behaviors as well as the Greek letters of the FSRPO.

4.3.1 Prices of FSRPOs for different initial asset prices and volatilities

Figure 4 plots the option price surfaces with respect to the initial asset prices $S_1(0)$ and $S_2(0)$. In Panels (a), (b), and (c) of Fig. 4, the examined values of ρ_{12} are -0.5, 0, and 0.5, respectively. By comparing these three panels, it can be noted that the value of the FSRPO decreases with the increase of the correlation coefficient between the two underlying assets, even when we consider various combinations of t, K, $S_1(0)$, and $S_2(0)$. These results demonstrate that the diversification effect can influence FSROs as well as classic rainbow options in a similar way. Moreover, it can be observed that the patterns of the diagrams are highly analogous given different values of ρ_{12} in
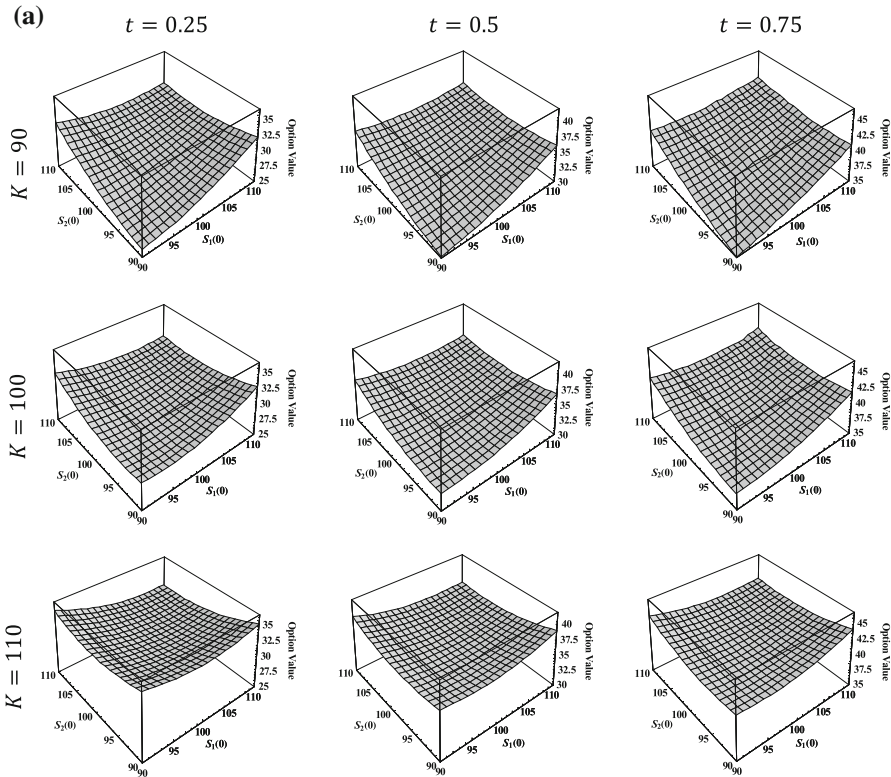
Fig. 4 Option values under different initial underlying prices. This figure depicts the values of the FSRPO with respect to $S_1(0)$ and $S_2(0)$. We examine $\rho_{12} = -0.5, 0$, and 0.5 in (**a**), (**b**), and (**c**), respectively. Each panel shows nine diagrams corresponding to different combinations of the guaranteed minimum strike price K at 90, 100, and 110 and the forward-start date t at 0.25, 0.5, and 0.75. The values of the other parameters are $r = 0.05$, $\sigma_1 = \sigma_2 = 0.3$, and $T = 1$. Similar to the results in Fig. 1, it can be observed that the option value of the FSRPO increases with t. In addition, as expected, the FSRPO becomes more valuable with an increase of K. Furthermore, there is one interesting phenomenon that when the initial price of one asset is relatively small, e.g., $S_2(0) = 90$, different from plain vanilla options, the option value is not a monotonic function of the initial price of the other asset, e.g., $S_1(0)$, in the case of a relatively large K (see the diagram corresponding to $t = 0.25$ and $K = 110$ for example). Also note this phenomenon is weakened when the forward-start date t is relatively late, e.g., comparing the diagrams corresponding to $t = 0.25$ and $t = 0.75$ given K equal to 110. Finally, across these three panels, it can be noticed that although the option price levels decrease with the increase of ρ_{12}, the price behaviors of the FSRPO are highly similar given different values of ρ_{12}. **a** $\rho_{12} = -0.5$, **b** $\rho_{12} = 0$, **c** $\rho_{12} = 0.5$

Panels (a), (b), and (c). Due to the similarity among these three panels, we take Panels (a) ($\rho_{12} = -0.5$) for example to investigate the price behavior of the FSRPO in the following paragraphs.

First, via comparing the individual columns corresponding to different forward-start date t, it is clear that the FSRPO value exhibits a positive relationship with the forward-start date. According to the explanation associated with Fig. 1, we know that this phenomenon is due to the impact of the rainbow option on the forward-start/reset feature. As for the guaranteed minimum strike price K, the
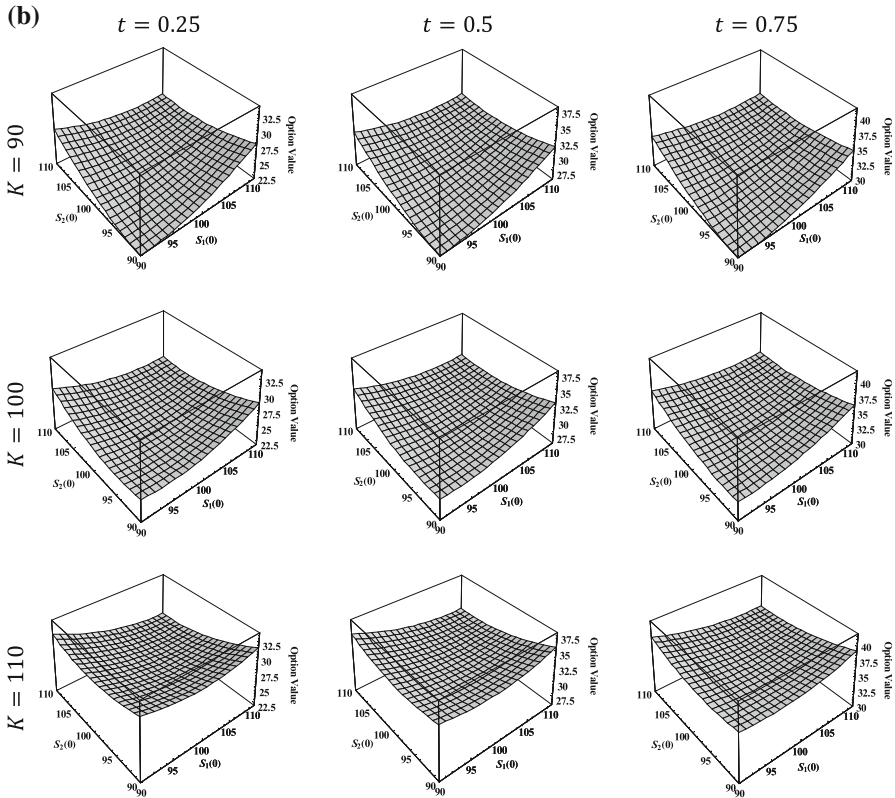
(b)

Fig. 4 continued

FSRPO value rises as K increases, especially when both $S_1(0)$ and $S_2(0)$ are relatively low. The underlying intuition is that the larger the guaranteed minimum strike price K, the more likely the option will be in the money at maturity, and this phenomenon is more pronounced when K is significantly larger than $S_1(0)$ and $S_2(0)$.

Certain subtle phenomena in Fig. 4 deserve more analysis. When $K = 110$, the option price curve starting from $(S_1(0), S_2(0)) = (90, 90)$ and along the positive direction of either $S_1(0)$ or $S_2(0)$ is not monotonic but like a valley, and this phenomenon is more pronounced when the forward-start date t is relatively early. First note the downward portion of the option price curve in the diagram for $K = 110$ and $t = 0.25$. Given that both $S_1(0)$ and $S_2(0)$ are much smaller than K and the period $[0, t]$ is relatively short, the result of $\max[K, S_1(t), S_2(t)]$ is likely to be K. Therefore, an increase in either $S_1(0)$ or $S_2(0)$ is expected to little affect the result of $\max[K, S_1(t), S_2(t)]$ but to raise the result of $\min[S_1(T), S_2(T)]$ more pronouncedly, which results in a downward tendency in option prices. For a later forward-start date t, the result of $\max[K, S_1(t), S_2(t)]$ is more likely to be determined by either $S_1(t)$ or $S_2(t)$ due to the positive growth rates of the underlying assets. As a result, the expected result of $\max[K, S_1(t), S_2(t)]$ increases correspond-
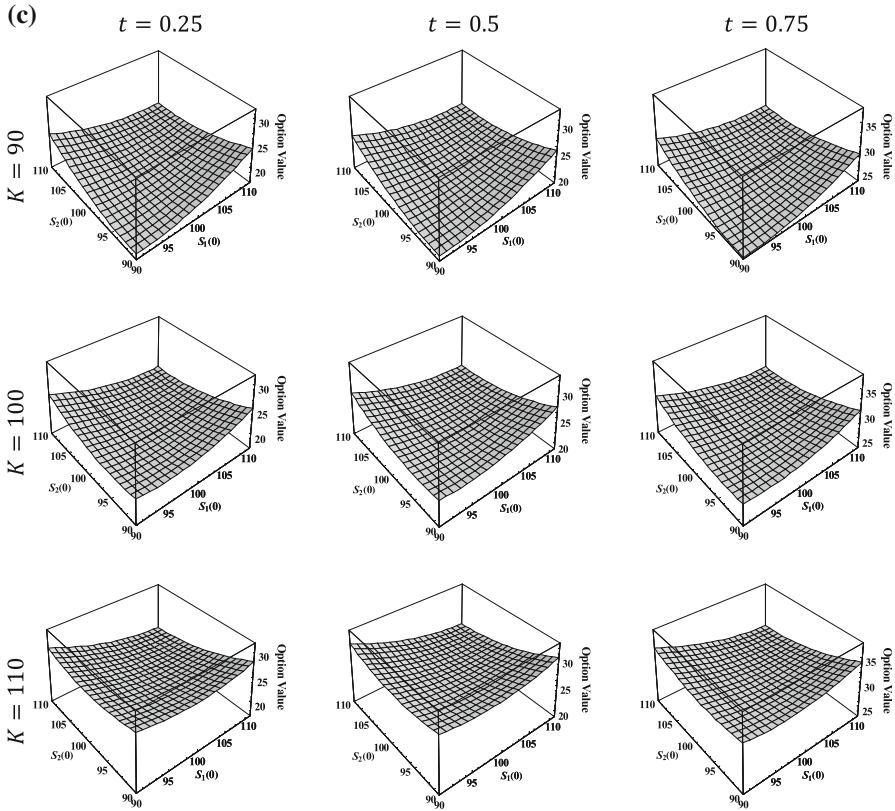
(c)

Fig. 4 continued

ing to an increase in either $S_1(0)$ or $S_2(0)$. This effect is strengthened with the delay of the forward-start date, and when the forward-start date t approaches the maturity date T, the expected increase in $\max[K, S_1(t), S_2(t)]$ will eventually dominate the expected increase in $\min[S_1(T), S_2(T)]$ due to the diversification effect. Thus, the downward curve of the option price disappears given a later forward-start date t, for example, see the most in front corner of the diagrams for $K = 110$ and $t = 0.75$.

On the other hand, let us consider that one of the initial prices is high and the other is low, for instance, suppose $S_1(0)$ is close to 110 and $S_2(0)$ is about 90. In this case, the forward-start strike price computed via $\max[K, S_1(t), S_2(t)]$ is very likely to be $S_1(t)$, and there is a high probability that $\min[S_1(T), S_2(T)]$ will turn out to be $S_2(T)$. Consequently, the price curve increases along the positive direction of $S_1(0)$. When $S_1(0)$ is small and $S_2(0)$ is large, the argument is similar, and the upward portion of the option price curve is thus explained.

Figure 5 plots the FSRPO values with respect to different volatilities of the underlying assets. The results given $\rho_{12} = -0.5, 0$, and 0.5 are presented in Panels (a), (b), and (c), respectively. In all these three panels, the option value surfaces with respect
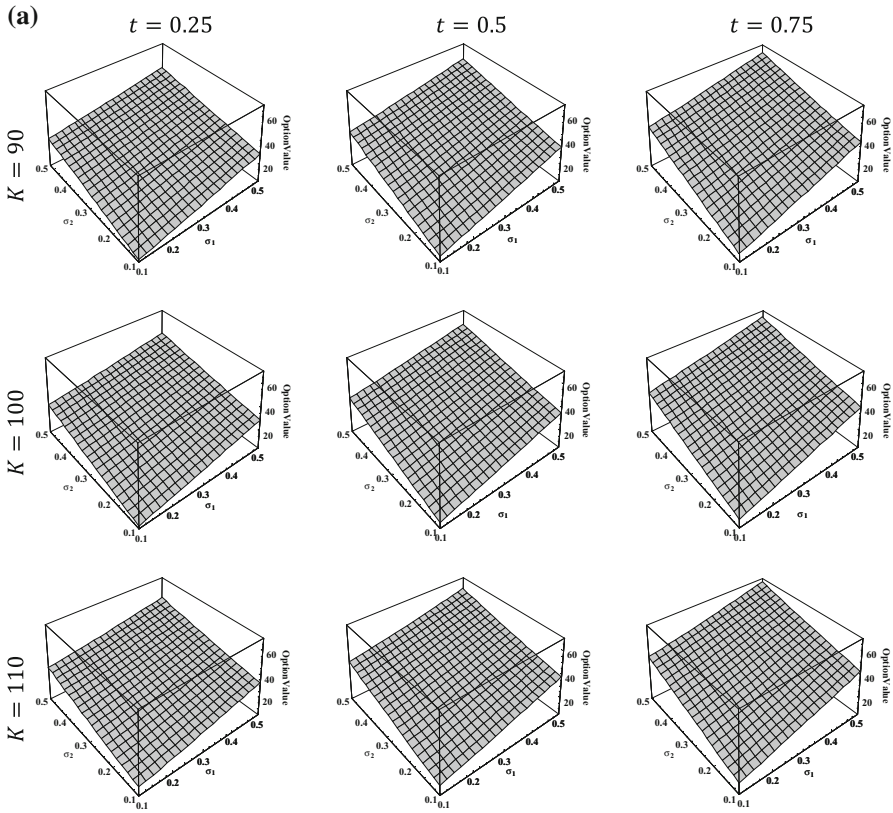
Fig. 5 Option values under different volatilities. This figure shows the values of the FSRPO with respect to different volatilities σ_1 and σ_2. **a**, **b**, and **c** examine $\rho_{12} = -0.5$, 0, and 0.5, respectively, and inside each of them, there are nine diagrams corresponding to the combination of three values of the guaranteed minimum strike price K: 90, 100, and 110 and three value of the forward-start date t: 0.25, 0.5, and 0.75. The values of the other parameters are $S_1(0) = S_2(0) = 100$, $r = 0.05$, and $T = 1$. Different from Fig. 4, the FSRPO's value is monotonically increasing with respect to both the volatilities σ_1 and σ_2. Note also that the value of the FSRPO exhibits positive relationships with respect to both the forward-start date t and the guaranteed minimum strike price K. By comparing Panels (a), (b), and (c), it is obvious that the value of the FSRPO shows a negative relationship with respect to the value of ρ_{12}. **a** $\rho_{12} = -0.5$, **b** $\rho_{12} = 0$, **c** $\rho_{12} = 0.5$

to different combinations of volatilities are quite flat and the option value increases monotonically with the underlying volatilities. In addition, it conforms to our intuition that a higher value of K leads to a higher value of the FSRPO. Note also that the value of the FSRPO exhibits positive relationship with respect to the forward-start date t, which is consistent with the results in Figs. 1 and 4. Therefore, we can conclude that the increasing benefit of the forward-start feature with the delay of the forward-start date t is a robust property of the FSRPO under various parameter values. Finally, by comparing Panels (a), (b), and (c), it is obvious that the value of the FSRPO shows a negative relationship with respect to the value of ρ_{12}. It again demonstrates that the diversification effect influences the FSRPO as well as the rainbow put option in a similar manner.
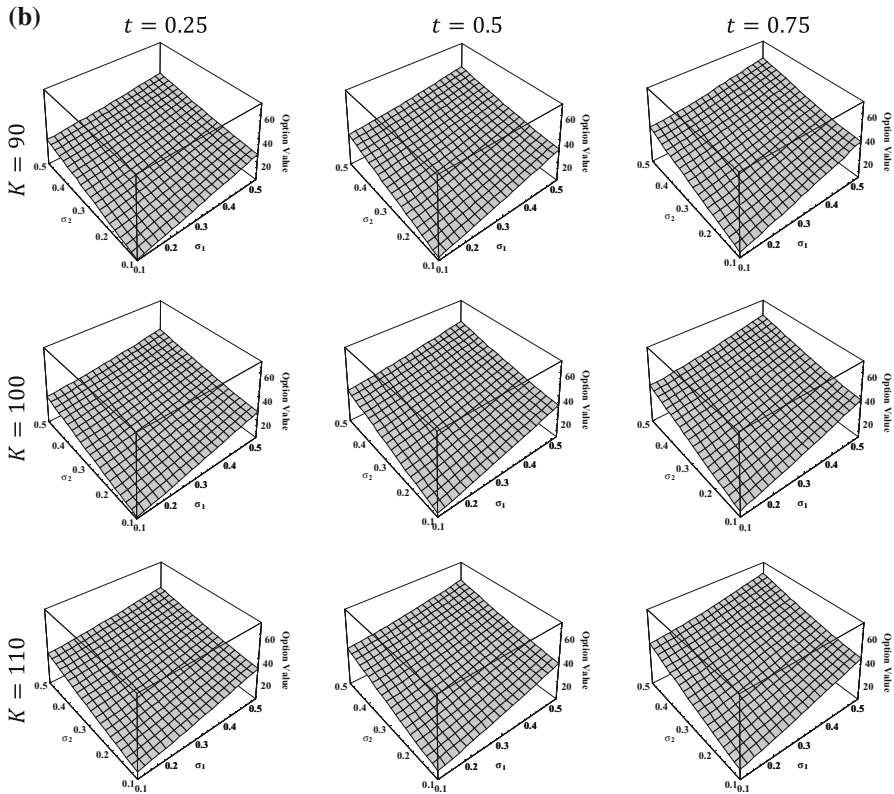
Fig. 5 continued

4.3.2 Delta, gamma, and cross gamma

This subsection is devoted to analyze the delta, gamma, and cross gamma of the FSRPO. Since the analogous price behaviors under different ρ_{12} in Fig. 4 imply similar patterns of the delta, gamma, and cross gamma under different ρ_{12}, we report the results of the delta, gamma, and cross gamma only given $\rho_{12} = -0.5$ in Figs. 6, 7, 8 and 9. Nevertheless, the diagrams of the delta, gamma, and cross gamma for ρ_{12} equal to 0 and 0.5 are available upon request.

Figure 6 depicts $\frac{\partial P}{\partial S_1}$, the FSRPO's delta for the first underlying asset. It shows that $\frac{\partial P}{\partial S_1}$ increases as $S_1(0)$ increases and $S_2(0)$ decreases, and the FSRPO's delta can be negative as well as positive. Figure 7 shows that the gamma with respect to $S_1(0)$, $\frac{\partial^2 P}{\partial S_1^2}$, is positive, which is in accordance with Fig. 4 where the option value is a convex function of the initial underlying asset prices. However, the cross gamma with respect to $S_1(0)$ and $S_2(0)$ in Fig. 8, $\frac{\partial^2 P}{\partial S_1 \partial S_2}$, is negative. In contrast to the positive gamma, a characteristic that favors option holders, the negative cross gamma indicates an additional dimension of risk for the holders of the FSRPO.
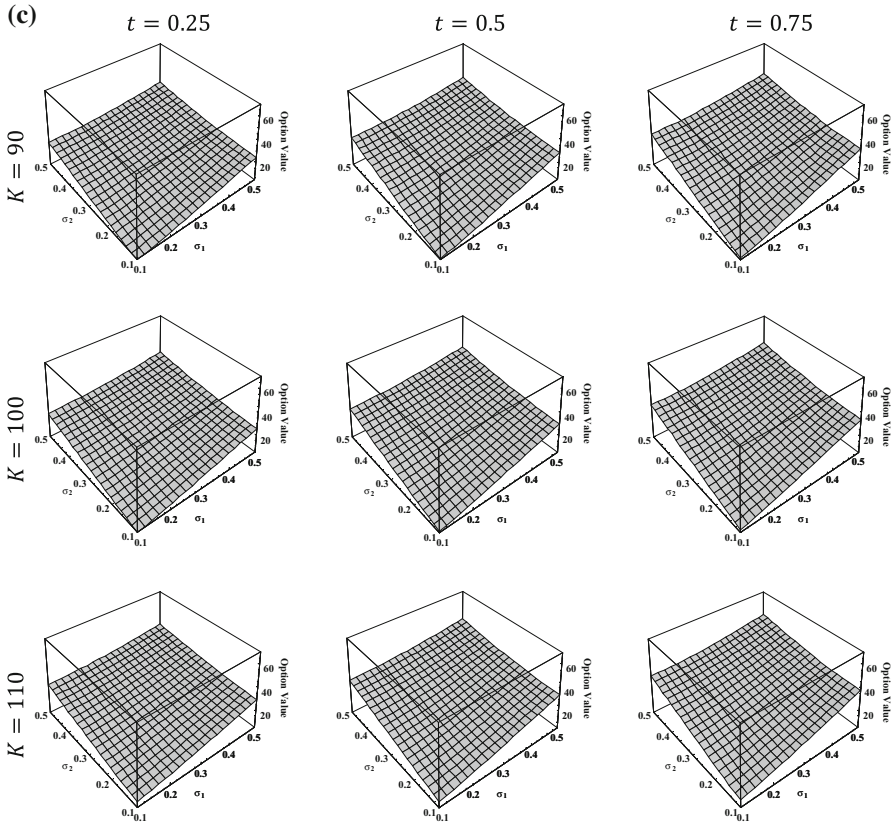
Fig. 5 continued

The intuitions behind the behaviors of the Greeks presented in Figs. 6, 7 and 8 are elaborated as follows. First, suppose $S_1(0)$ is much smaller than K and $S_2(0)$. With a high probability, the payoff (4.1) is around to be $\max[K, S_2(t)] - S_1(T)$, which implies the option value decreases as $S_1(0)$ increases. Thus the delta $\frac{\partial P}{\partial S_1}$ is negative. When $S_1(0)$ increases (but remains smaller than K and $S_2(0)$), the option value decreases, but not so much as it did, and the delta is less negative than it was. On the other hand, $\frac{\partial P}{\partial S_1}$ is positive when $S_1(0)$ is much larger than K and $S_2(0)$ since the payoff (4.1) is very likely to be $S_1(t) - S_2(T)$ in this scenario. This argument explains why $\frac{\partial P}{\partial S_1}$ increases from negative values to positive values as $S_1(0)$ increases, and the gamma with respect to $S_1(0)$, $\frac{\partial^2 P}{\partial S_1^2}$, is positive.

Next, suppose $S_1(0)$ and $S_2(0)$ both increase by one unit. The option value then declines since the expected value of $\max[K, S_1(t), S_2(t)]$ increases by less than that of $\min[S_1(T), S_2(T)]$. The main reason is that the former value (which increases up to t) is less than the latter one (which increases up to T) according to the positive growth rate, that is, $r > 0$. This explains why the cross gamma $\frac{\partial^2 P}{\partial S_1 \partial S_2}$ is negative in Fig. 8 and equivalently the delta $\frac{\partial P}{\partial S_1}$ increases as $S_2(0)$ decreases in Fig. 6.
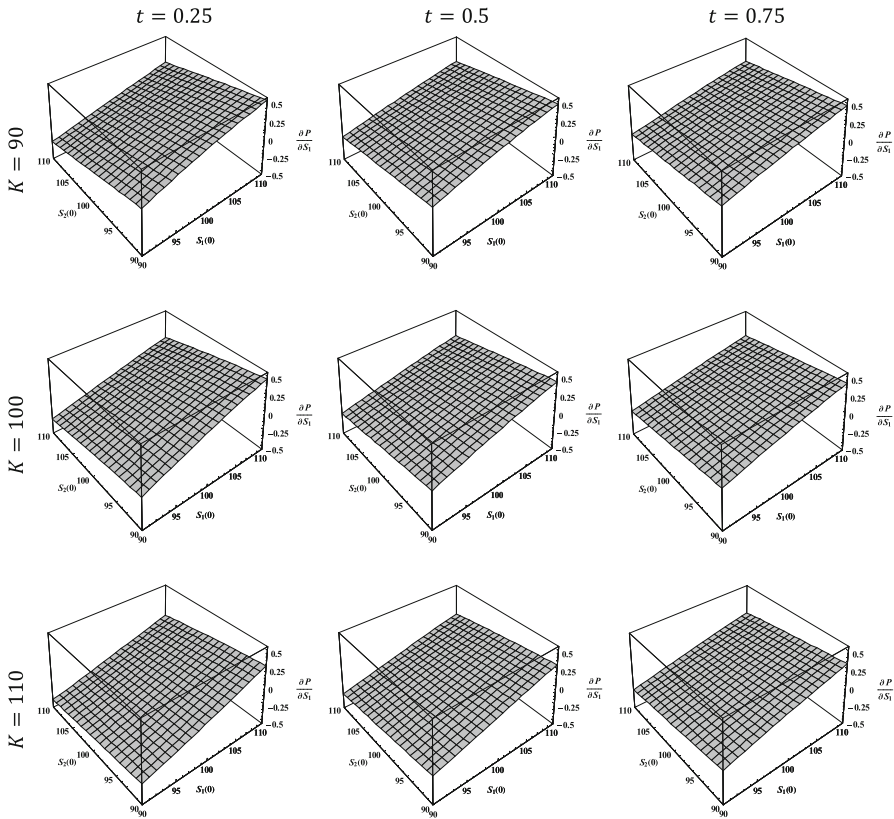
Fig. 6 Values of delta with respect to the initial underlying price of the first asset. This figure shows the values of the delta of the FSRPO with respect to $S_1(0)$ under different combinations of $S_1(0)$ and $S_2(0)$. The examined values of parameters are $r = 0.05$, $\rho_{12} = -0.5$, $\sigma_1 = \sigma_2 = 0.3$, and $T = 1$. In addition, three representative values for the guaranteed minimum strike price K are considered: 90, 100, and 110, and three representative values for the forward-start date t are considered: 0.25, 0.5, and 0.75. Due to the valley-like surface for the option value with respect to $S_1(0)$ and $S_2(0)$ in Fig. 4, unlike the negative delta for the plain vanilla put option, the FSRPO's delta can be negative as well as positive

To hedge the FSRPO, it should be worth noting the jumps of delta, gamma, and cross gamma across the forward-start date. The values of delta, gamma, and cross gamma near the forward-start date are shown in Fig. 9, where all the parameters examined are the same as those in Figs. 6, 7 and 8, except considering a present date which is near the forward-start date. Specifically, we consider $t = 0.001$ and $T = 1$ in Fig. 9. In this scenario, the surfaces of delta, gamma, and cross gamma with respect to $S_1(0)$ and $S_2(0)$ are different from the smooth ones in Figs. 6, 7 and 8, respectively.

As shown in Fig. 9, the discontinuities in the delta and gamma with respect to $S_1(0)$ occur near the points $\{(S_1(0), S_2(0)) : K \leq S_1(0) = S_2(0) \text{ or } S_2(0) \leq S_1(0) = K\}$. The discontinuity points for the delta and gamma with respect to $S_2(0)$ can be simply obtained by interchanging the roles of $S_1(0)$ and $S_2(0)$ in the above point set, i.e., $\{(S_1(0), S_2(0)) : K \leq S_2(0) = S_1(0) \text{ or } S_1(0) \leq S_2(0) = K\}$. As for the cross
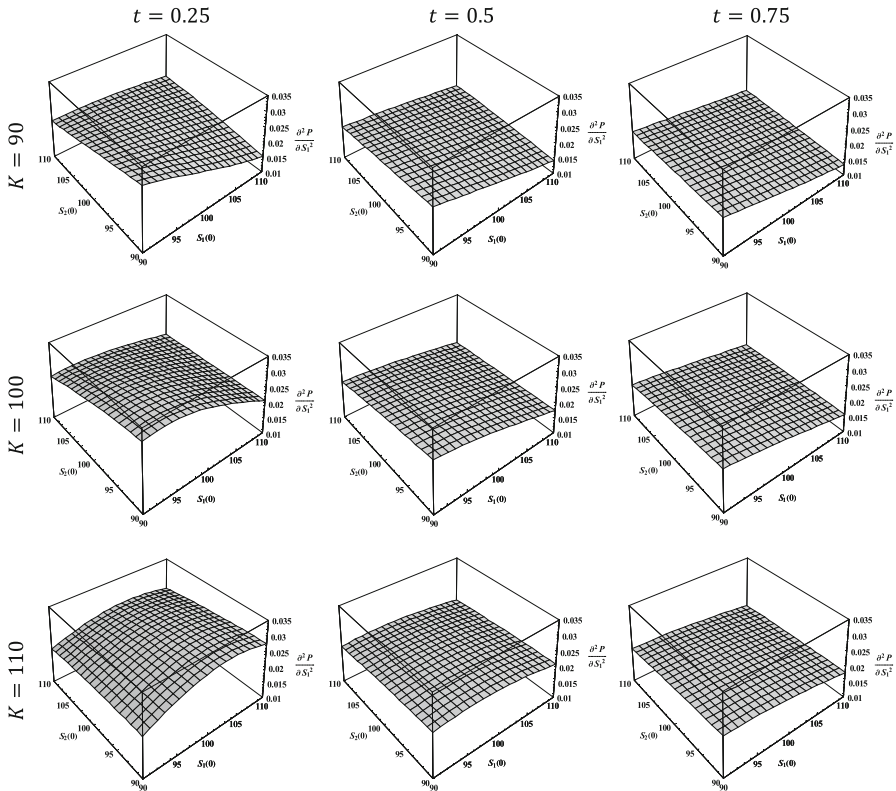
Fig. 7 Values of gamma with respect to the initial underlying price of the first asset. This figure illustrates the values of the gamma of the FSRPO with respect to $S_1(0)$ under different values of $S_1(0)$ and $S_2(0)$. We consider the guaranteed minimum strike price K to be 90, 100, and 110, the forward-start date t to be 0.25, 0.5, and 0.75, and $r = 0.05$, $\rho_{12} = -0.5$, $\sigma_1 = \sigma_2 = 0.3$, and $T = 1$. The values of gamma are positive since the option value is shown to be a convex function with respect to $S_1(0)$ in Fig. 4

gamma, the set of the discontinuity points should be theoretically the union of the point sets for the discontinuities of the delta with respect to either $S_1(0)$ or $S_2(0)$, i.e., $\{(S_1(0), S_2(0)) : K \leq S_1(0) = S_2(0), S_2(0) \leq S_1(0) = K, \text{ or } S_1(0) \leq S_2(0) = K\}$. The results of the jumps of delta and gamma are consistent with the findings in Cheng and Zhang (2000) and Liao and Wang (2003).

5 Conclusion

This paper examines a new type of path-dependent rainbow options–forward-start rainbow options. The analytic pricing formula for forward-start rainbow options is first proposed, and it can be a general formula for several existing options, including options on the maximum or minimum of multiple assets, single-asset reset options, and plain vanilla options. In addition, the delta, gamma, and cross gamma for forward-start rainbow options are derived analytically. Equipped with these analytic formulae, numerical experiments are conducted to examine the characteristics of the option value
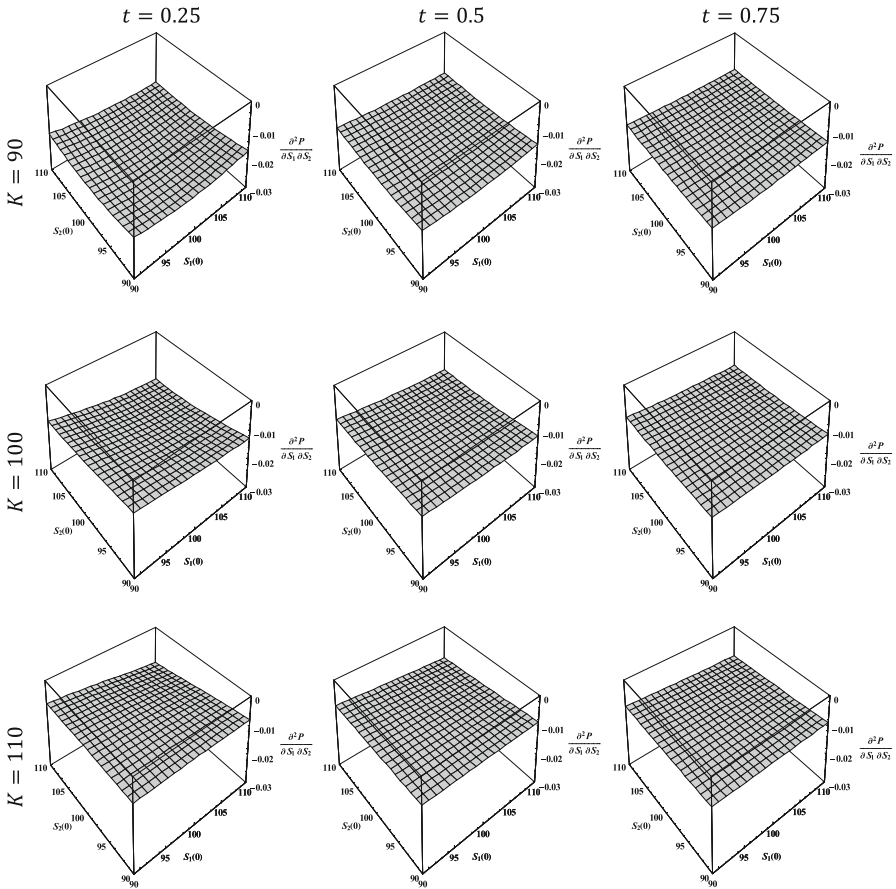
Fig. 8 Values of cross gamma. This figure plots the values of the cross gamma of the FSRPO with respect to $S_1(0)$ and $S_2(0)$. Given each combination of the guaranteed minimum strike price K (chosen from 90, 100, 110) and the forward-start date t (chosen from 0.25, 0.5, and 0.75), the values of the other parameters are $r = 0.05$, $\rho_{12} = -0.5$, $\sigma_1 = \sigma_2 = 0.3$, and $T = 1$. Note that the cross gamma of the FSRPO is negative. This negative cross gamma indicates an additional dimension of risk imposing on the holders of the FSRPOs

and Greek letters of forward-start rainbow options. It is the first time to find that the price behavior of forward-start rainbow options is significantly different from that of single-asset forward-start/reset options.

Furthermore, another contribution of this paper is the proposal of a novel martingale pricing technique that can be applied to pricing not only the forward-start rainbow options but also other options dependent on multiple assets and time points, such as discrete-sampling lookback rainbow options. Due to the wide applicability of this novel technique, this paper broaches a new direction in designing new option contracts with payoffs that depend on both multiple underlying assets and path-dependent features.
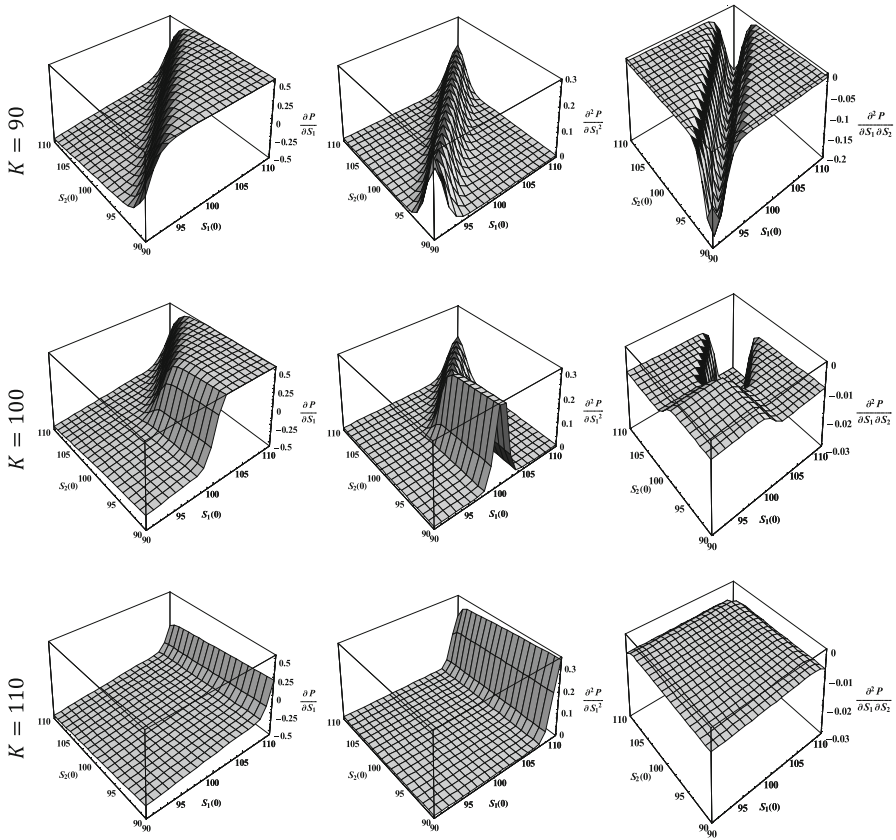
Fig. 9 Jumps of delta, gamma, and cross gamma near the forward-start date. This figure illustrates the possible jumps for the delta, gamma, and cross gamma of the FSRPO when the forward-start date t approaches zero. More specifically, we consider $t = 0.001$ to generate all diagrams in this figure. The other parameters are $r = 0.05$, $\rho_{12} = -0.5$, $\sigma_1 = \sigma_2 = 0.3$, $T = 1$, and $K = 90$, 100, and 110. It is clear that the discontinuities of these Greek letters appear when t approaches zero, so the surfaces of delta, gamma, and cross gamma with respect to $S_1(0)$ and $S_2(0)$ are different from the smooth ones in Figs. 6, 7 and 8. The results of the possible jumps of the delta and gamma are consistent with the findings in Cheng and Zhang (2000) and Liao and Wang (2003)

6 Appendix 1: Proof of Theorem 2.1

The martingale pricing method relies heavily on the Cameron–Martin–Girsanov theorem, which is the key to evaluating the conditional expectation of the price processes with the multivariate standard normal cumulative distribution function (CDF). Therefore, the proof starts with the review of the general n-dimensional Cameron-Martin-Girsanov theorem based on the description in Baxter and Rennie (2000), with a slight change in notations.

Cameron–Martin–Girsanov Theorem *Let* $\vec{W}^Q(z) = (W_1^Q(z), \ldots, W_n^Q(z))$ *be a vector of n-dimensional independent Q-Brownian motions up to time T adapted with the natural filtration $\mathcal{F}$. Suppose that $\vec{v}(z) = (v_1(z), \ldots, v_n(z))$ is an $\mathcal{F}$-previsible n-vector process that satisfies the condition $E^Q[\exp(\frac{1}{2}\int_0^T |\vec{v}^2(\tau)| d\tau)] < \infty$. Then there is a new measure R such that*

1. *Q is equivalent to R up to time T,*

2. $\frac{dR}{dQ} = \exp\left(\sum_{j=1}^n \int_0^T v_j(\tau) dW_j^Q(\tau) - \frac{1}{2}\int_0^T |\vec{v}(\tau)|^2 d\tau\right)$, *and*

3. $W_j^R(z) = W_j^Q(z) - \int_0^z v_j(\tau) d\tau$, *and* $\vec{W}^R(z) = (W_1^R(z), \ldots, W_n^R(z))$ *is a vector of n-dimensional R-Brownian motion up to time T.*

Based upon the general *n*-dimensional Cameron–Martin–Girsanov theorem, the most important finding in this paper is stated in Theorem 2.1.

Theorem 2.1 *Let $W_j^Q(z)$ and $W_i^Q(z)$ be two correlated Q-Brownian motions, and ρ_{ji} is their correlation coefficient. By defining the Radon–Nikodym derivative as $dR_{is}/dQ = \exp(\sigma_i W_i^Q(s) - \sigma_i^2 s/2)$, we can obtain that*

$$W_j^{R_{is}}(z) = W_j^Q(z) - \rho_{ji}\sigma_i \min(s, z)$$

is an R_{is}-Brownian motion for any time points s and z.

Before proving Theorem 2.1, we state a useful identity equation. Suppose $W_j^P(z)$ and $W_i^P(z)$ are correlated Brownian motions with a correlation coefficient ρ_{ji} under any probability measure *P*. Once we define $\widetilde{W}_j^P(z) \equiv (W_j^P(z) - \rho_{ji}W_i^P(z))/\sqrt{1 - \rho_{ji}^2}$, then $\widetilde{W}_j^P(z)$ is an independent *P*-Brownian motion with respect to $W_i^P(z)$. Rearranging the above equation, we can obtain

$$W_j^P(z) = \rho_{ji}W_i^P(z) + \sqrt{1 - \rho_{ji}^2}\,\widetilde{W}_j^P(z), \tag{6.1}$$

an equation that is helpful in the following proof.

Proof Suppose $W_i^Q(z)$ and $\widetilde{W}_j^Q(z)$ are two independent *Q*-Brownian motions satisfying Eq. (6.1), that is, $W_j^Q(z) = \rho_{ji}W_i^Q(z) + \sqrt{1 - \rho_{ji}^2}\,\widetilde{W}_j^Q(z)$. By considering $\vec{v}(z) = (v_i(z), v_j(z)) = (\sigma_i \cdot I(z \le s), 0)$ in the Cameron–Martin–Girsanov theorem, the Radon–Nikodym derivative can be derived as

$$\frac{dR_{is}}{dQ} = \exp(\sum_{u=i,j}\int_0^T v_u(\tau) dW_u^Q(\tau) - \frac{1}{2}\int_0^T |\vec{v}(\tau)|^2 d\tau)$$

$$= \exp(\sigma_i W_i^Q(s) - \frac{1}{2}\sigma_i^2 s),$$

and for any time point z,

$$W_i^{R_{is}}(z) = W_i^Q(z) - \int_0^z v_i(\tau)d\tau = W_i^Q(z) - \int_0^z \sigma_i \cdot I(\tau \leq s)d\tau$$

$$= W_i^Q(z) - \sigma_i \min(s, z),$$

$$\widetilde{W}_j^{R_{is}}(z) = \widetilde{W}_j^Q(z) - \int_0^z v_j(\tau)d\tau = \widetilde{W}_j^Q(z).$$

Finally, we have the following equations and the proof is complete:

$$W_j^{R_{is}}(z) = \rho_{ji} W_i^{R_{is}}(z) + \sqrt{1 - \rho_{ji}^2} \widetilde{W}_j^{R_{is}}(z) \qquad \text{(by (6.1))}$$

$$= \rho_{ji}(W_i^Q(z) - \sigma_i \min(s, z)) + \sqrt{1 - \rho_{ji}^2} \widetilde{W}_j^Q(z)$$

$$= \rho_{ji} W_i^Q(z) + \sqrt{1 - \rho_{ji}^2} \widetilde{W}_j^Q(z) - \rho_{ji}\sigma_i \min(s, z)$$

$$= W_j^Q(z) - \rho_{ji}\sigma_i \min(s, z). \qquad \text{(by (6.1))}.$$

$$\square$$

After proving Theorem 2.1, we would like to illustrate the specialty of Theorem 2.1 regarding the time dimension by evaluating $E^Q[S_a(s) \cdot I(S_b(u) \geq S_c(v))]$ (given $u \leq s \leq v$ for example), which is a problem arising exclusively when pricing FSROs or other path-dependent rainbow options. To evaluate this expectation, we first rewrite it with $S_a(s) = S_a(0) \exp((r - \sigma_a^2/2)s + \sigma_a W_a^Q(s))$, i.e.,

$$E^Q[S_a(s) \cdot I(S_b(u) \geq S_c(v))]$$
$$= S_a(0)e^{rs} E^Q \left[\exp(\sigma_a W_a^Q(s) - \sigma_a^2 s/2) \cdot I(S_b(u) \geq S_c(v)) \right]. \qquad (6.2)$$

By defining the Radon–Nikodym derivative as $dR_{as}/dQ = \exp(\sigma_a W_a^Q(s) - \sigma_a^2 s/2)$ and applying Theorem 2.1, we can obtain

$$W_b^{R_{as}}(u) = W_b^Q(u) - \rho_{ba}\sigma_a \min(s, u) = W_b^Q(u) - \rho_{ba}\sigma_a u, \qquad (6.3)$$

$$W_c^{R_{as}}(v) = W_c^Q(v) - \rho_{ca}\sigma_a \min(s, v) = W_c^Q(v) - \rho_{ca}\sigma_a s. \qquad (6.4)$$

As a result, Eq. (6.2) can be expressed as

$$S_a(0)e^{rs} E^{R_{as}}[I(S_b(u) \geq S_c(v))]$$
$$= S_a(0)e^{rs} Pr^{R_{as}}(S_b(u) \geq S_c(v))$$
$$= S_a(0)e^{rs} Pr^{R_{as}}(\ln S_b(u) \geq \ln S_c(v))$$
$$= S_a(0)e^{rs} Pr^{R_{as}}(\ln S_b(0) + (r - \sigma_b^2/2)u + \sigma_b(W_b^{R_{as}}(u) + \rho_{ba}\sigma_a u)$$
$$\geq \ln S_c(0) + (r - \sigma_c^2/2)v + \sigma_c(W_c^{R_{as}}(v) + \rho_{ca}\sigma_a s))$$

The last equation is derived by replacing $W_b^Q(u)$ and $W_c^Q(v)$ according to Eqs. (6.3) and (6.4), respectively. The straightforward algebraic calculation leads to

$$
S_a(0)e^{rs}Pr^{Ras}\left(\frac{\sigma_c W_c^{Ras}(v) - \sigma_b W_b^{Ras}(u)}{\sqrt{\mathrm{var}(\sigma_c W_c^{Ras}(v) - \sigma_b W_b^{Ras}(u))}}\right.
$$

$$
\leq \left(\frac{\ln(S_b(0)/S_c(0)) + r(u-v) + \sigma_c^2 v/2 - \sigma_b^2 u/2 + \rho_{ba}\sigma_b\sigma_a u - \rho_{ca}\sigma_c\sigma_a s}{\sqrt{\mathrm{var}(\sigma_c W_c^{Ras}(v) - \sigma_b W_b^{Ras}(u))}}\right)
$$

$$
\equiv S_a(0)e^{rs}Pr^{Ras}(Z_{bu,cv}^{Ras} \leq d_{bu,cv}^{Ras})
$$

$$
= S_a(0)e^{rs}N(d_{bu,cv}^{Ras}),
$$

where $\mathrm{var}(\sigma_c W_c^{Ras}(v) - \sigma_b W_b^{Ras}(u))$ equals $\sigma_c^2 v - 2\rho_{cb}\sigma_c\sigma_b \min(v,u) + \sigma_b^2 u$ by definition. In addition, since $Z_{bu,cv}^{Ras} \equiv \frac{\sigma_c W_c^{Ras}(v) - \sigma_b W_b^{Ras}(u)}{\sqrt{\mathrm{var}(\sigma_c W_c^{Ras}(v) - \sigma_b W_b^{Ras}(u))}}$ follows the standard normal distribution, the last equation in the above can be derived with defining $N(\cdot)$ as the cumulative distribution function of the standard normal distribution.

It should be noted that when we transform $W_b^Q(u)$ to $W_b^{Ras}(u)$ in Eq. (6.3), the adjustment term is $\rho_{ba}\sigma_a u$, which is proportional to u as expected. On the other hand, when we transform $W_c^Q(v)$ to $W_c^{Ras}(v)$ in Eq. (6.4), although both $W_c^Q(v)$ and $W_c^{Ras}(v)$ are Brownian motions evolving up to the time point v, the adjustment term is $\rho_{ca}\sigma_a s$ rather than $\rho_{ca}\sigma_a v$ according to Theorem 2.1. In fact, if we apply, without any modification, the commonly used change-of-probability-measure techniques for option pricing, e.g., those in Ouwehand and West (2006) and Liao and Wang (2003), this special characteristic associated with the time dimension cannot be captured such that we cannot derive the correct formula corresponding to $E^Q[S_a(s) \cdot I(S_b(u) \geq S_c(v))]$.

7 Appendix 2: Derivations of the pricing formula

Theorem 2.1 makes it straightforward to derive the pricing formula for the FSRPO. Recall that in Eq. (2.5) the price today is

$$
P(0) = e^{-rT}E^Q[P(T)|\mathcal{F}_0]
$$

$$
= \sum_{m=1}^{n} Ke^{-rT}E^Q\Big[I\left(\{K \geq S_i(t)\}_{1\leq i \leq n},\right.
$$

$$
\left.\{S_j(T) \geq S_m(T)\}_{1\leq j \neq m \leq n}, K \geq S_m(T)\right)|\mathcal{F}_0\Big]
$$

$$
- \sum_{m=1}^{n} e^{-rT}E^Q\Big[S_m(T) \cdot I\left(\{K \geq S_i(t)\}_{1\leq i \leq n},\right.
$$

$$
\left.\{S_j(T) \geq S_m(T)\}_{1\leq j \neq m \leq n}, K \geq S_m(T)\right)|\mathcal{F}_0\Big]
$$

$$+ \sum_{1 \leq m, M \leq n} e^{-rT} E^Q \left[S_M(t) \cdot I \left(\{ S_M(t) \geq S_{it} \}_{1 \leq i \leq n}, \right. \right.$$

$$\left. \{ S_j(T) \geq S_m(T) \}_{1 \leq j \neq m \leq n}, S_M(t) \geq S_m(T) \right) | \mathcal{F}_0 \right]$$

$$- \sum_{1 \leq m, M \leq n} e^{-rT} E^Q \left[S_m(T) \cdot I \left(\{ S_M(t) \geq S_{it} \}_{1 \leq i \leq n}, \right. \right.$$

$$\left. \{ S_j(T) \geq S_m(T) \}_{1 \leq j \neq m \leq n}, S_M(t) \geq S_m(T) \right) | \mathcal{F}_0 \right], \tag{7.1}$$

where $S_{it} \equiv \begin{cases} K & i = M \\ S_i(t) & i \neq M \end{cases}$.

We evaluate the option value in Eq. (7.1) term by term in the following paragraphs.

7.1 The first summation in Eq. (7.1)

The expectation inside the first summation in Eq. (7.1) can be evaluated directly in the probability measure Q:

$$E^Q \left[I \left(\{ K \geq S_i(t) \}_{1 \leq i \leq n}, \{ S_j(T) \geq S_m(T) \}_{1 \leq j \neq m \leq n}, K \geq S_m(T) \right) | \mathcal{F}_0 \right]$$

$$= Pr^Q \left(\{ K \geq S_i(t) \}_{1 \leq i \leq n}, \{ S_j(T) \geq S_m(T) \}_{1 \leq j \neq m \leq n}, K \geq S_m(T) | \mathcal{F}_0 \right)$$

$$= Pr^Q \left(\left\{ d_{K,it}^Q \geq Z_{K,it}^Q \right\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^Q \geq Z_{jT,mT}^Q \right\}_{1 \leq j \neq m \leq n}, d_{K,mT}^Q \geq Z_{K,mT}^Q | \mathcal{F}_0 \right)$$

$$= N_{2n} \left(\left\{ d_{K,it}^Q \right\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^Q \right\}_{1 \leq j \neq m \leq n}, d_{K,mT}^Q; R^{K,m} \right),$$

where

$$d_{K,it}^Q = \frac{\ln(K/S_i(0)) - (r - \sigma_i^2/2)t}{\sigma_i \sqrt{t}}, \qquad Z_{K,it}^Q = \frac{W_i^Q(t)}{\sqrt{t}},$$

$$d_{jT,mT}^Q = \frac{\ln(S_j(0)/S_m(0)) + (\sigma_m^2 - \sigma_j^2)T/2}{\sigma_{jm} \sqrt{T}}, \quad Z_{jT,mT}^Q = \frac{\sigma_m W_m^Q(T) - \sigma_j W_j^Q(T)}{\sigma_{jm} \sqrt{T}},$$

$$d_{K,mT}^Q = \frac{\ln(K/S_m(0)) - (r - \sigma_m^2/2)T}{\sigma_m \sqrt{T}}, \qquad Z_{K,mT}^Q = \frac{W_m^Q(T)}{\sqrt{T}}.$$

In the above equations, σ_{ab} is defined as $\sqrt{\sigma_a^2 - 2\rho_{ab}\sigma_a\sigma_b + \sigma_b^2}$, and the notation $d_{as,b\tau}^P$ is referred to as the parameter of the multivariate standard normal CDF under the event $\{ S_a(s) \geq S_b(\tau) \}$ with respect to any probability measure P. The notation $d_{K,b\tau}^P$ is defined similarly but corresponding to the event $\{ K \geq S_b(\tau) \}$.

For the correlation matrix $R^{K,m}$, we first concatenate $\{ Z_{K,it}^Q \}_{1 \leq i \leq n}$, $\{ Z_{jT,mT}^Q \}_{1 \leq j \neq m \leq n}$, and $Z_{K,mT}^Q$ to form $\{ Z_p^{K,m} \}_{1 \leq p \leq 2n}$, i.e., $\{ Z_p^{K,m} \}_{1 \leq p \leq 2n} \equiv \{ \{ Z_{K,it}^Q \}_{1 \leq i \leq n}, \{ Z_{jT,mT}^Q \}_{1 \leq j \neq m \leq n}, Z_{K,mT}^Q \}$. Similarly, $\{ Z_q^{K,m} \}_{1 \leq q \leq 2n} \equiv \{ \{ Z_{K,kt}^Q \}_{1 \leq k \leq n}, \{ Z_{lT,mT}^Q \}_{1 \leq l \neq m \leq n}, Z_{K,mT}^Q \}$. In addition, we also define $\rho_{p,q}^{K,m} \equiv \text{corr}(Z_p^{K,m}, Z_q^{K,m})$,

and $R^{K,m} \equiv (\rho_{p,q}^{K,m})_{1 \le p,q \le 2n}$ is a $2n \times 2n$ correlation coefficient matrix as follows:

$$
R^{K,m} = \begin{pmatrix} (I)_{n \times n} & (II)_{n \times (n-1)} & (III)_{n \times 1} \\ (II)_{(n-1) \times n}' & (IV)_{(n-1) \times (n-1)} & (V)_{(n-1) \times 1} \\ (III)_{1 \times n}' & (V)_{1 \times (n-1)}' & 1 \end{pmatrix}_{2n \times 2n} ,
$$

where

$$
(I)_{n \times n} = \left(\mathrm{corr}(Z_{K,it}^Q, Z_{K,kt}^Q) \right)_{\substack{1 \le i \le n \\ 1 \le k \le n}} = \left(\mathrm{corr}\left(\frac{W_i^Q(t)}{\sqrt{t}}, \frac{W_k^Q(t)}{\sqrt{t}} \right) \right)_{\substack{1 \le i \le n \\ 1 \le k \le n}}
$$

$$
= (\rho_{ik})_{\substack{1 \le i \le n \\ 1 \le k \le n}},
$$

and other parts listed below can be computed similarly:

$$
(II)_{n \times (n-1)} = \left(\mathrm{corr}(Z_{K,it}^Q, Z_{lT,mT}^Q) \right)_{\substack{1 \le i \le n \\ 1 \le l \ne m \le n}} = \left(\frac{\rho_{im}\sigma_m - \rho_{il}\sigma_l}{\sigma_{lm}} \sqrt{\frac{t}{T}} \right)_{\substack{1 \le i \le n \\ 1 \le l \ne m \le n}} ,
$$

$$
(III)_{n \times 1} = \left(\mathrm{corr}(Z_{K,it}^Q, Z_{K,mT}^Q) \right)_{1 \le i \le n} = \left(\rho_{im} \sqrt{\frac{t}{T}} \right)_{1 \le i \le n} ,
$$

$$
(IV)_{(n-1) \times (n-1)} = \left(\mathrm{corr}(Z_{jT,mT}^Q, Z_{lT,mT}^Q) \right)_{\substack{1 \le l \ne m \le n \\ 1 \le l \ne m \le n}}
$$

$$
= \left(\frac{\sigma_m^2 - \rho_{lm}\sigma_m\sigma_l - \rho_{jm}\sigma_j\sigma_m + \rho_{jl}\sigma_j\sigma_l}{\sigma_{jm}\sigma_{lm}} \right)_{\substack{1 \le j \ne m \le n \\ 1 \le l \ne m \le n}} ,
$$

$$
(V)_{(n-1) \times 1} = \left(\mathrm{corr}(Z_{jT,mT}^Q, Z_{K,mT}^Q) \right)_{1 \le l \ne m \le n} = \left(\frac{\sigma_m - \rho_{jm}\sigma_j}{\sigma_{jm}} \right)_{1 \le j \ne m \le n} .
$$

7.2 The second summation in Eq. (7.1)

Under the risk-neutral measure Q, the underlying price of asset m at time T is

$$
S_m(T) = S_m(0) \exp \left((r - \sigma_m^2/2)T + \sigma_m W_m^Q(T) \right).
$$

It is convenient to introduce the probability measure R_{mT} by setting the corresponding Radon–Nikodym derivative to

$$
\frac{dR_{mT}}{dQ} = \exp \left(\sigma_m W_m^Q(T) - \sigma_m^2 T/2 \right).
$$

With Theorem 2.1, $W_j^{R_{mT}}(z) = W_j^Q(z) - \rho_{jm}\sigma_m \min(z, T) = W_j^Q(z) - \rho_{jm}\sigma_m z$ is a standard Brownian motion under the probability measure R_{mT}, where j is the

index for the underlying asset. Then we can rewrite the expectation inside the second summation in Eq. (7.1) evaluated with respect to the R_{mT} measure as follows:

$$E^Q \left[S_m(T) \cdot I \left(\{K \geq S_i(t)\}_{1 \leq i \leq n}, \{S_j(T) \geq S_m(T)\}_{1 \leq j \neq m \leq n}, K \geq S_m(T) \right) | \mathcal{F}_0 \right]$$

$$= S_m(0) e^{rT} E^{R_{mT}} \left[I \left(\{K \geq S_i(t)\}_{1 \leq i \leq n}, \{S_j(T) \geq S_m(T)\}_{1 \leq j \neq m \leq n}, \right. \right.$$

$$\left. K \geq S_m(T)) | \mathcal{F}_0 \right]$$

$$= S_m(0) e^{rT} Pr^{R_{mT}} \left(\left\{ d_{K,it}^{R_{mT}} \geq Z_{K,it}^{R_{mT}} \right\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^{R_{mT}} \geq Z_{jT,mT}^{R_{mT}} \right\}_{1 \leq j \neq m \leq n}, \right.$$

$$\left. d_{K,mT}^{R_{mT}} \geq Z_{K,mT}^{R_{mT}} | \mathcal{F}_0 \right)$$

$$= S_m(0) e^{rT} N_{2n} \left(\left\{ d_{K,it}^{R_{mT}} \right\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^{R_{mT}} \right\}_{1 \leq j \neq m \leq n}, d_{K,mT}^{R_{mT}}; R^{K,m} \right),$$

where

$$d_{K,it}^{R_{mT}} = \frac{\ln(K/S_i(0)) - (r - (\sigma_{im}^2 - \sigma_m^2)/2)t}{\sigma_i \sqrt{t}}, \quad Z_{K,it}^{R_{mT}} = \frac{W_i^{R_{mT}}(t)}{\sqrt{t}},$$

$$d_{jT,mT}^{R_{mT}} = \frac{\ln(S_j(0)/S_m(0)) - \sigma_{jm}^2 T/2}{\sigma_{jm} \sqrt{T}}, \quad Z_{jT,mT}^{R_{mT}} = \frac{\sigma_m W_m^{R_{mT}}(T) - \sigma_j W_j^{R_{mT}}(T)}{\sigma_{jm} \sqrt{T}},$$

$$d_{K,mT}^{R_{mT}} = \frac{\ln(K/S_m(0)) - (r + \sigma_m^2/2)T}{\sigma_m \sqrt{T}}, \quad Z_{K,mT}^{R_{mT}} = \frac{W_m^{R_{mT}}(T)}{\sqrt{T}}.$$

It is worth noting that $Z_{\bullet,\bullet}^{R_{mT}}$ are essentially the same as $Z_{\bullet,\bullet}^Q$ in the first summation of Eq. (7.1) except under a different probability measure. Hence for the expectation inside the second summation, the correlation matrix of the multivariate standard normal CDF is the same as $R^{K,m}$ in the first summation in Eq. (7.1).

7.3 The third summation in Eq. (7.1)

Following the same technique, we compute the expectation inside the third summation in Eq. (7.1). The underlying price of asset M at time t under the risk-neutral measure Q is

$$S_M(t) = S_M(0) \exp \left((r - \sigma_M^2/2)t + \sigma_M W_M^Q(t) \right).$$

Let R_{Mt} be the equivalent probability measure defined by

$$\frac{dR_{Mt}}{dQ} = \exp \left(\sigma_M W_M^Q(t) - \sigma_M^2 t/2 \right).$$

According to Theorem 2.1, $W_j^{R_{Mt}}(z) = W_j^Q(z) - \rho_{jM} \sigma_M \min(z, t)$ is a standard Brownian motion under the probability measure R_{Mt}, where j is the index for the underlying asset. Note that if the examined time point z is before t, $W_j^{R_{Mt}}(z) =$

$W_j^Q(z) - \rho_{jM}\sigma_M z$; if the examined time point z is after t, $W_j^{R_{Mt}}(z) = W_j^Q(z) - \rho_{jM}\sigma_M t$ even though both $W_j^{R_{Mt}}(z)$ and $W_j^Q(z)$ evolve up to z. The expectation inside the third summation in Eq. (7.1) evaluated with respect to the R_{Mt} measure is as follows:

$$
E^Q \left[S_M(t) \cdot I \left(\{S_M(t) \geq S_{it}\}_{1 \leq i \leq n}, \{S_j(T) \geq S_m(T)\}_{1 \leq j \neq m \leq n}, \right. \right.
$$
$$
\left. \left. S_M(t) \geq S_m(T)) \,|\mathcal{F}_0 \right] \right.
$$
$$
= S_M(0)e^{rt} \, Pr^{R_{Mt}} \left(\left\{ d_{it}^{R_{Mt}} \geq Z_{it}^{R_{Mt}} \right\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^{R_{Mt}} \geq Z_{jT,mT}^{R_{Mt}} \right\}_{1 \leq j \neq m \leq n}, \right.
$$
$$
\left. d_{Mt,mT}^{R_{Mt}} \geq Z_{Mt,mT}^{R_{Mt}} |\mathcal{F}_0 \right)
$$
$$
= S_M(0)e^{rt} \, N_{2n} \left(\left\{ d_{it}^{R_{Mt}} \right\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^{R_{Mt}} \right\}_{1 \leq j \neq m \leq n}, d_{Mt,mT}^{R_{Mt}}; \, R^{M,m} \right),
$$

where

$$
d_{it}^{R_{Mt}} = \begin{cases} d_{Mt,K}^{R_{Mt}} = \frac{\ln(S_M(0)/K)+(r+\sigma_M^2/2)t}{\sigma_M \sqrt{t}} & i = M \\ d_{Mt,it}^{R_{Mt}} = \frac{\ln(S_M(0)/S_i(0))+\sigma_{iM}^2 t/2}{\sigma_{iM} \sqrt{t}} & i \neq M \end{cases},
$$
$$
Z_{it}^{R_{Mt}} = \begin{cases} Z_{Mt,K}^{R_{Mt}} = -\frac{W_M^{R_{Mt}}(t)}{\sqrt{t}} & i = M \\ Z_{Mt,it}^{R_{Mt}} = \frac{\sigma_i W_i^{R_{Mt}}(t)-\sigma_M W_M^{R_{Mt}}(t)}{\sigma_{iM} \sqrt{t}}, & i \neq M \end{cases},
$$
$$
d_{jT,mT}^{R_{Mt}} = \frac{\ln(S_j(0)/S_m(0))+(\sigma_{Mt,mT}^2 - \sigma_{Mt,jT}^2)/2}{\sigma_{jm} \sqrt{T}}, \quad Z_{jT,mT}^{R_{Mt}} = \frac{\sigma_m W_m^{R_{Mt}}(T)-\sigma_j W_j^{R_{Mt}}(T)}{\sigma_{jm} \sqrt{T}},
$$
$$
d_{Mt,mT}^{R_{Mt}} = \frac{\ln(S_M(0)/S_m(0))+r(t-T)+\sigma_{Mt,mT}^2/2}{\sigma_{Mt,mT}}, \quad Z_{Mt,mT}^{R_{Mt}} = \frac{\sigma_m W_m^{R_{Mt}}(T)-\sigma_M W_M^{R_{Mt}}(t)}{\sigma_{Mt,mT}},
$$

and the notation $\sigma_{as,b\tau}$ is defined as $\sqrt{\sigma_a^2 s - 2\rho_{ab}\sigma_a\sigma_b \min(s,\tau) + \sigma_b^2 \tau}$.

Next, we define $\{Z_p^{M,m}\}_{1 \leq p \leq 2n} \equiv \{\{Z_{it}^{R_{Mt}}\}_{1 \leq i \leq n}, \{Z_{jT,mT}^{R_{Mt}}\}_{1 \leq j \neq m \leq n}, Z_{Mt,mT}^{R_{Mt}}\}$, $\{Z_q^{K,m}\}_{1 \leq q \leq 2n} \equiv \{\{Z_{kt}^{R_{Mt}}\}_{1 \leq i \leq n}, \{Z_{lT,mT}^{R_{Mt}}\}_{1 \leq j \neq m \leq n}, Z_{Mt,mT}^{R_{Mt}}\}$, $\rho_{p,q}^{M,m} \equiv$ corr$(Z_p^{M,m}, Z_q^{M,m})$, and $R^{M,m} \equiv (\rho_{p,q}^{M,m})_{1 \leq p,q \leq 2n}$ as a $2n \times 2n$ correlation coefficient matrix. For simplicity, we abuse the notation slightly and set $\sigma_i = 0$ when $i = M$. Consequently, as $i = M$, $\sigma_{iM} = \sqrt{\sigma_i^2 - 2\rho_{iM}\sigma_i\sigma_M + \sigma_M^2} = \sqrt{\sigma_M^2} = \sigma_M$. Then the correlation coefficient matrix $R^{M,m}$ is presented as follows:

$$
R^{M,m} = \begin{pmatrix} (VI)_{n \times n} & (VII)_{n \times (n-1)} & (VIII)_{n \times 1} \\ (VII)'_{(n-1) \times n} & (IX)_{(n-1) \times (n-1)} & (X)_{(n-1) \times 1} \\ (VIII)'_{1 \times n} & (X)'_{1 \times (n-1)} & 1 \end{pmatrix}_{2n \times 2n},
$$

where

$$(VI)_{n \times n} = \left(\text{corr}(Z_{it}^{R_{Mt}}, Z_{kt}^{R_{Mt}}) \right)_{\substack{1 \leq i \leq n \\ 1 \leq k \leq n}}$$

$$= \left(\frac{\rho_{ik}\sigma_i\sigma_k - \rho_{kM}\sigma_k\sigma_M - \rho_{iM}\sigma_i\sigma_M + \sigma_M^2}{\sigma_{iM} \ \sigma_{kM}} \right)_{\substack{1 \leq i \leq n \\ 1 \leq k \leq n}},$$

$$(VII)_{n \times (n-1)} = \left(\text{corr}(Z_{it}^{R_{Mt}}, Z_{lT,mT}^{R_{Mt}}) \right)_{\substack{1 \leq i \leq n \\ 1 \leq l \neq m \leq n}}$$

$$= \left(\frac{\rho_{im}\sigma_i\sigma_m - \rho_{il}\sigma_i\sigma_l - \rho_{mM}\sigma_m\sigma_M + \rho_{lM}\sigma_l\sigma_M}{\sigma_{iM} \ \sigma_{lm}} \sqrt{\frac{t}{T}} \right)_{\substack{1 \leq i \leq n \\ 1 \leq l \neq m \leq n}},$$

$$(VIII)_{n \times 1} = \left(\text{corr}(Z_{it}^{R_{Mt}}, Z_{Mt,mT}^{R_{Mt}}) \right)_{1 \leq i \leq n}$$

$$= \left(\frac{\rho_{im}\sigma_i\sigma_m - \rho_{iM}\sigma_i\sigma_M - \rho_{mM}\sigma_m\sigma_M + \sigma_M^2}{\sigma_{iM} \ \sigma_{Mt,mT}} \sqrt{t} \right)_{1 \leq i \leq n},$$

$$(IX)_{(n-1) \times (n-1)} = \left(\text{corr}(Z_{jT,mT}^{R_{Mt}}, Z_{lT,mT}^{R_{Mt}}) \right)_{\substack{1 \leq l \neq m \leq n \\ 1 \leq l \neq m \leq n}}$$

$$= \left(\frac{\sigma_m^2 - \rho_{jm}\sigma_j\sigma_m - \rho_{lm}\sigma_l\sigma_m + \rho_{jl}\sigma_j\sigma_l}{\sigma_{jm} \ \sigma_{lm}} \right)_{\substack{1 \leq j \neq m \leq n \\ 1 \leq l \neq m \leq n}},$$

$$(X)_{(n-1) \times 1} = \left(\text{corr}(Z_{jT,mT}^{R_{Mt}}, Z_{Mt,mT}^{R_{Mt}}) \right)_{1 \leq l \neq m \leq n}$$

$$= \left(\frac{\sigma_m^2 T - \rho_{mM}\sigma_m\sigma_M t - \rho_{jm}\sigma_j\sigma_m T + \rho_{jM}\sigma_j\sigma_M t}{\sigma_{jm}\sqrt{T} \ \sigma_{Mt,mT}} \right)_{1 \leq j \neq m \leq n}.$$

7.4 The fourth summation in Eq. (7.1)

The probability measure R_{mT} used for the expectation inside the fourth summation in Eq. (7.1) is the same as that in the second summation of Eq. (7.1). In addition, comparing with the derivation details of the third summation, it can be easily shown that $Z_{\bullet,\bullet}^{R_{mT}}$ are in essence the same as $Z_{\bullet,\bullet}^{R_{Mt}}$ except under a different probability measure. Hence the correlation matrix for the multivariate standard normal CDF inside the fourth summation will be $R^{M,m}$. Consequently, the expectation inside the fourth summation in Eq. (7.1) is evaluated as follows:

$$E^Q \left[S_m(T) \cdot I \left(\{S_M(t) \geq S_{it}\}_{1 \leq i \leq n}, \{S_j(T) \geq S_m(T)\}_{1 \leq j \neq m \leq n}, \right. \right.$$

$$\left. \left. S_M(t) \geq S_m(T)) | \mathcal{F}_0 \right] \right.$$

$$= S_m(0)e^{rT} \ Pr^{R_{mT}} \left(\left\{ d_{it}^{R_{mT}} \geq Z_{it}^{R_{mT}} \right\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^{R_{mT}} \geq Z_{jT,mT}^{R_{mT}} \right\}_{1 \leq j \neq m \leq n}, \right.$$

$$d_{Mt,mT}^{R_{mT}} \geq Z_{Mt,mT}^{R_{mT}} | \mathcal{F}_0 \bigg)$$

$$= S_m(0)e^{rT} \ N_{2n} \left(\left\{ d_{it}^{R_{mT}} \right\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^{R_{mT}} \right\}_{1 \leq j \neq m \leq n}, d_{Mt,mT}^{R_{mT}}; \ R^{M,m} \right),$$

where

$$d_{it}^{R_{mT}} = \begin{cases} d_{Mt,K}^{R_{mT}} = \dfrac{\ln(S_M(0)/K)+(r-(\sigma_{Mm}^2-\sigma_m^2)/2)t}{\sigma_M \sqrt{t}} & i = M \\[2mm] d_{Mt,it}^{R_{mT}} = \dfrac{\ln(S_M(0)/S_i(0))+(\sigma_{im}^2-\sigma_{Mm}^2)t/2}{\sigma_{iM} \sqrt{t}} & i \neq M \end{cases},$$

$$Z_{it}^{R_{mT}} = \begin{cases} Z_{Mt,K}^{R_{mT}} = -\dfrac{W_M^{R_{mT}}(t)}{\sqrt{t}} & i = M \\[2mm] Z_{Mt,it}^{R_{mT}} = \dfrac{\sigma_i W_i^{R_{mT}}(t)-\sigma_M W_M^{R_{mT}}(t)}{\sigma_{iM} \sqrt{t}} & i \neq M \end{cases},$$

$$d_{jT,mT}^{R_{mT}} = \frac{\ln(S_j(0)/S_m(0))-\sigma_{jm}^2 T/2}{\sigma_{jm} \sqrt{T}}, \qquad Z_{jT,mT}^{R_{mT}} = \frac{\sigma_m W_m^{R_{mT}}(T)-\sigma_j W_j^{R_{mT}}(T)}{\sigma_{jm} \sqrt{T}},$$

$$d_{Mt,mT}^{R_{mT}} = \frac{\ln(S_M(0)/S_m(0))+r(t-T)-\sigma_{Mt,mT}^2/2}{\sigma_{Mt,mT}}, \qquad Z_{Mt,mT}^{R_{mT}} = \frac{\sigma_m W_m^{R_{mT}}(T)-\sigma_M W_M^{R_{mT}}(t)}{\sigma_{Mt,mT}}.$$

As a result, the pricing formula of the forward-start put option can be summarized as follows.

$$\sum_{m=1}^{n} Ke^{-rT} N_{2n} \left(\left\{ d_{K,it}^{Q} \right\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^{Q} \right\}_{1 \leq j \neq m \leq n}, d_{K,mT}^{Q}; \ R^{K,m} \right)$$

$$- \sum_{m=1}^{n} S_m(0) \ N_{2n} \left(\left\{ d_{K,it}^{R_{mT}} \right\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^{R_{mT}} \right\}_{1 \leq j \neq m \leq n}, d_{K,mT}^{R_{mT}}; \ R^{K,m} \right)$$

$$+ \sum_{1 \leq m, M \leq n} S_M(0)e^{-r(T-t)} \ N_{2n} \left(\{d_{it}^{R_{Mt}}\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^{R_{Mt}} \right\}_{1 \leq j \neq m \leq n}, d_{Mt,mT}^{R_{Mt}}; \ R^{M,m} \right)$$

$$- \sum_{1 \leq m, M \leq n} S_m(0) \ N_{2n} \left(\left\{ d_{it}^{R_{mT}} \right\}_{1 \leq i \leq n}, \left\{ d_{jT,mT}^{R_{mT}} \right\}_{1 \leq j \neq m \leq n}, d_{Mt,mT}^{R_{mT}}; \ R^{M,m} \right).$$

8 Appendix 3: Delta and gamma

The delta of an option can be directly derived by differentiating the pricing formula with respect to the initial price of the underlying asset. However, here we employ an alternative method that uses the linear homogeneity of the option pricing formula to derive the formula for delta. For any linearly homogeneous function, that is, $f(\lambda x, \lambda y) = \lambda f(x, y)$, the Euler's rule implies

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = f(x, y).$$

To simplify the expression, we use S_i as shorthand for $S_i(0)$ hereafter. Since the general pricing formula (2.6) is linearly homogeneous, that is,

$$P(\lambda S_1, \lambda S_2, \ldots, \lambda S_n, \lambda K) = \lambda P(S_1, S_2, \ldots, S_n, K),$$

by the Euler's rule the equation can be written as

$$P(S_1, S_2, \ldots, S_n, K) = \sum_{a=1}^{n} S_a \frac{\partial P}{\partial S_a} + K \frac{\partial P}{\partial K} = S_1 \frac{\partial P}{\partial S_1} + \cdots + S_n \frac{\partial P}{\partial S_n} + K \frac{\partial P}{\partial K}.$$

To obtain the delta with respect to the a-th asset, $\frac{\partial P}{\partial S_a}$, we need to collect all the terms that contain S_a. Recall that the pricing formula of $P(0)$ in Eq. (2.6) is

$$
\begin{aligned}
&\sum_{m=1}^{n} K e^{-rT} N_{2n} \left(\left\{ d_{K,it}^{Q} \right\}_{1 \le i \le n}, \left\{ d_{jT,mT}^{Q} \right\}_{1 \le j \ne m \le n}, d_{K,mT}^{Q}; R^{K,m} \right) \\
&- \sum_{m=1}^{n} S_m N_{2n} \left(\left\{ d_{K,it}^{R_{mT}} \right\}_{1 \le i \le n}, \{ d_{jT,mT}^{R_{mT}} \}_{1 \le j \ne m \le n}, d_{K,mT}^{R_{mT}}; R^{K,m} \right) \\
&+ \sum_{1 \le m, M \le n} S_M e^{-r(T-t)} N_{2n} \left(\left\{ d_{it}^{R_{Mt}} \right\}_{1 \le i \le n}, \{ d_{jT,mT}^{R_{Mt}} \}_{1 \le j \ne m \le n}, d_{Mt,mT}^{R_{Mt}}; R^{M,m} \right) \\
&- \sum_{1 \le m, M \le n} S_m N_{2n} \left(\left\{ d_{it}^{R_{mT}} \right\}_{1 \le i \le n}, \left\{ d_{jT,mT}^{R_{mT}} \right\}_{1 \le j \ne m \le n}, d_{Mt,mT}^{R_{mT}}; R^{M,m} \right).
\end{aligned}
\tag{8.1}
$$

Inside the above formula, S_a appears in the terms of the second and fourth summations when m is equal to a and in the third summation when M is equal to a. The term that contains S_a in the second summation of the formula (8.1) is

$$-S_a N_{2n} \left(\left\{ d_{K,it}^{R_{aT}} \right\}_{1 \le i \le n}, \left\{ d_{jT,aT}^{R_{aT}} \right\}_{1 \le j \ne a \le n}, d_{K,aT}^{R_{aT}}; R^{K,a} \right).$$

Similarly, the terms with S_a in the third summation of the formula (8.1) can be grouped as follows.

$$\sum_{a'=1}^{n} S_a e^{-r(T-t)} N_{2n} \left(\left\{ d_{it}^{R_{at}} \right\}_{1 \le i \le n}, \left\{ d_{jT,a'T}^{R_{at}} \right\}_{1 \le j \ne a' \le n}, d_{at,a'T}^{R_{at}}; R^{a,a'} \right),$$

where a' is merely an index of the underlying assets. Finally, we consider the terms that contain S_a in the fourth summation of the formula (8.1) and the corresponding sum of these terms is

$$- \sum_{a'=1}^{n} S_a N_{2n} \left(\left\{ d_{it}^{R_{aT}} \right\}_{1 \le i \le n}, \left\{ d_{jT,aT}^{R_{aT}} \right\}_{1 \le j \ne a \le n}, d_{a't,aT}^{R_{aT}}; R^{a',a} \right).$$

After rearranging the terms in the formula (8.1) with respect to each S_a and the strike price K, we can obtain the formula for delta with respect to the a-th asset as follows:

$$
\begin{aligned}
\frac{\partial P}{\partial S_a} &= -N_{2n}\left(\left\{d_{K,it}^{R_aT}\right\}_{1\leq i\leq n}, \left\{d_{jT,aT}^{R_aT}\right\}_{1\leq j\neq a\leq n}, d_{K,aT}^{R_aT}; \ R^{K,a}\right) \\
&+ \sum_{a'=1}^{n} e^{-r(T-t)} N_{2n}\left(\left\{d_{it}^{R_at}\right\}_{1\leq i\leq n}, \left\{d_{jT,a'T}^{R_at}\right\}_{1\leq j\neq a'\leq n}, d_{at,a'T}^{R_at}; \ R^{a,a'}\right) \\
&- \sum_{a'=1}^{n} N_{2n}\left(\left\{d_{it}^{R_aT}\right\}_{1\leq i\leq n}, \left\{d_{jT,aT}^{R_aT}\right\}_{1\leq j\neq a\leq n}, d_{a't,aT}^{R_aT}; \ R^{a',a}\right).
\end{aligned}
$$

Based on the formula for delta, we can proceed to derive the formulae for gamma with respect to the a-th asset and cross gamma with respect to the a-th and b-th assets, $\frac{\partial^2 P}{\partial S_a^2}$ and $\frac{\partial^2 P}{\partial S_a \partial S_b}$. Consider the derivation of $\frac{\partial^2 P}{\partial S_a^2}$ first:

$$
\begin{aligned}
\frac{\partial^2 P}{\partial S_a^2} &= \frac{\partial}{\partial S_a}\left(\frac{\partial P}{\partial S_a}\right) \\
&= -\frac{\partial}{\partial S_a} N_{2n}\left(\left\{d_{K,it}^{R_aT}\right\}_{1\leq i\leq n}, \left\{d_{jT,aT}^{R_aT}\right\}_{1\leq j\neq a\leq n}, d_{K,aT}^{R_aT}; \ R^{K,a}\right) \\
&+ \sum_{a'=1}^{n} e^{-r(T-t)} \frac{\partial}{\partial S_a} N_{2n}\left(\left\{d_{it}^{R_at}\right\}_{1\leq i\leq n}, \left\{d_{jT,a'T}^{R_at}\right\}_{1\leq j\neq a'\leq n}, d_{at,a'T}^{R_at}; \ R^{a,a'}\right) \\
&- \sum_{a'=1}^{n} \frac{\partial}{\partial S_a} N_{2n}\left(\left\{d_{it}^{R_aT}\right\}_{1\leq i\leq n}, \left\{d_{jT,aT}^{R_aT}\right\}_{1\leq j\neq a\leq n}, d_{a't,aT}^{R_aT}; \ R^{a',a}\right).
\end{aligned}
$$

We obtain the derivative of the first term in the above equation by the chain rule.

$$
\begin{aligned}
&\frac{\partial}{\partial S_a} N_{2n}\left(\left\{d_{K,it}^{R_aT}\right\}_{1\leq i\leq n}, \left\{d_{jT,aT}^{R_aT}\right\}_{1\leq j\neq a\leq n}, d_{K,aT}^{R_aT}; R^{K,a}\right) \\
&= \sum_{1\leq i\leq n} \frac{\partial N_{2n}}{\partial d_{K,it}^{R_aT}} \frac{\partial d_{K,it}^{R_aT}}{\partial S_a} + \sum_{1\leq j\neq a\leq n} \frac{\partial N_{2n}}{\partial d_{jT,aT}^{R_aT}} \frac{\partial d_{jT,aT}^{R_aT}}{\partial S_a} + \frac{\partial N_{2n}}{\partial d_{K,aT}^{R_aT}} \frac{\partial d_{K,aT}^{R_aT}}{\partial S_a}.
\end{aligned}
$$

The key element in deriving the formula for gamma is the calculation of the derivatives of the multivariate standard normal CDFs. The method adopted in this paper is as follows. Let $\{d_u\}_{1\leq u\leq 2n} \equiv \{\{d_{K,it}^{R_aT}\}_{1\leq i\leq n}, \{d_{jT,aT}^{R_aT}\}_{1\leq j\neq a\leq n}, d_{K,aT}^{R_aT}\}$, and $\rho_{u,v}^{K,a}$ be the correlation coefficient element of $R^{K,a}$. Taking $\partial N_{2n}/\partial d_{K,aT}^{R_aT}$ as an example, we rewrite the above multivariate standard normal CDF according to the results in Curnow and Dunnett (1962).

$$N_{2n}\left(\left\{d_{K,it}^{R_aT}\right\}_{1\leq i\leq n},\left\{d_{jT,aT}^{R_aT}\right\}_{1\leq j\neq a\leq n},d_{K,aT}^{R_aT};R^{K,a}\right)$$

$$=N_{2n}\left(\{d_u\}_{1\leq u\leq 2n};R^{K,a}\right)$$

$$=\int_{-\infty}^{d_{2n}}N_{2n-1}\left(\{\widehat{d}_u(x)\}_{1\leq u<2n};\left(\widehat{\rho}_{u,v}^{K,a}\right)_{\substack{1\leq u<2n\\1\leq v<2n}}\right)\phi(x)\,dx.$$

where

$$d_{2n}\equiv d_{K,aT}^{R_aT},\quad\widehat{d}_u(x)=\frac{d_u-\rho_{u,2n}^{K,a}x}{\sqrt{1-\left(\rho_{u,2n}^{K,a}\right)^2}},\quad\text{and}\quad\widehat{\rho}_{u,v}^{K,a}=\frac{\rho_{u,v}^{K,a}-\rho_{u,2n}^{K,a}\rho_{v,2n}^{K,a}}{\sqrt{1-\left(\rho_{u,2n}^{K,a}\right)^2}\sqrt{1-\left(\rho_{v,2n}^{K,a}\right)^2}},$$

and $\phi(x)$ is the standard normal probability density function. Hence, $\partial N_{2n}/\partial d_{2n}=\partial N_{2n}/\partial d_{K,aT}^{R_aT}$ can be calculated as

$$\frac{\partial N_{2n}}{\partial d_{2n}}=N_{2n-1}\left(\{\widehat{d}_u(x)\}_{1\leq u<2n};\left(\widehat{\rho}_{u,v}^{K,a}\right)_{\substack{1\leq u<2n\\1\leq v<2n}}\right)\phi(x)\Bigg|_{x=d_{2n}=d_{K,aT}^{R_aT}}.$$

After similarly calculating other terms, we obtain the formula for the FSRPO's gamma. This paper only briefly describes the method to derive the formulae of gamma and cross gamma, and the detailed formulae are omitted for brevity but available upon request.

9 Appendix 4: Formulae of discrete-sampling lookback options

This appendix demonstrates the generality of our main theorem by applying it to discrete-sampling lookback rainbow options. In order to describe the formulae more clearly, the notation is slightly different from that in Sect. 2. Let $S_i(t_j)$ denote the price of the i-th stock at time t_j for $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, and $0=t_0<t_1<\cdots<t_n=T$. The return dynamics of S_i for $i=1,2,\ldots,m$ follow the same stochastic differential equations as Eq. (2.2). In addition, M denotes the historical highest price up to now. Except for above differences, the rest of the notation, such as r, σ_i, K, T, and ρ_{ij}, is the same.

Consider the discrete-sampling lookback rainbow option that is a call option on the maximum among all the prices of every risky asset at every monitoring time during the option's life. Let $C_{\max}(T)$ be the payoff of this option at maturity T with the historical highest price M until t_0 and constant strike price K:

$$C_{\max}(T)=\left(\max_{k,l}(M,S_k(t_l))-K\right)^+=\left(\max_{k,l}(M,S_k(t_l))-K\right)\cdot I_A,$$

where

$$A_{ij} = \left\{ S_i(t_j) = \max_{k,l}(M, S_k(t_l)) \geq K \right\},$$

$$A_M = \left\{ M = \max_{k,l}(M, S_k(t_l)) \geq K \right\},$$

$$A = \bigcup_{i,j} \left\{ S_i(t_j) = \max_{k,l}(M, S_k(t_l)) \geq K \right\} \cup \left\{ M = \max_{k,l}(M, S_k(t_l)) \geq K \right\}$$

$$= \left(\bigcup_{i,j} A_{ij} \right) \cup A_M.$$

By the risk-neutral valuation, the option value today can be expressed as follows:

$$
\begin{aligned}
C_{\max}(0) &= e^{-rT} E^Q[C_{\max}(T)] \\
&= e^{-rT} E^Q[(\max_{k,l}(M, S_k(t_l)) - K) \cdot I_A] \\
&= e^{-rT} E^Q[(\max_{k,l}(M, S_k(t_l)) \cdot I_A] - K e^{-rT} E^Q[I_A] \\
&= e^{-rT} \sum_{i,j} E^Q[(S_i(t_j) \cdot I_{A_{ij}})] + M e^{-rT} E^Q[I_{A_M}] - K e^{-rT} E^Q[I_A].
\end{aligned}
$$

$$(9.1)$$

As $M \geq K$, $I_A = 1$, the equation (9.1) can be expressed as

$$
\begin{aligned}
C_{\max}(0) &= e^{-rT} \sum_{i,j} E^Q[(S_i(t_j) \cdot I_{A_{ij}})] + M e^{-rT} E^Q[I_{A_M}] - K e^{-rT} \\
&= \left\{ \sum_{i,j} S_i(0) e^{-r(T-t_j)} N_{mn} \left[\left(D1_{kl}^{ij} \right)_{mn}; \left(F1_{kl,pq}^{ij} \right)_{mn \times mn} \right] \right\} \\
&\quad + M e^{-rT} N_{mn} \left[(d1_{ij})_{mn}; (f1_{ij,kl})_{mn \times mn} \right] - K e^{-rT},
\end{aligned}
$$

$$(9.2)$$

where

$$d1_{ij} = -\frac{\ln(S_i(0)/M) + (r - \frac{1}{2}\sigma_i^2)t_j}{\sigma_i \sqrt{t_j}},$$

$$f1_{ij,kl} = \frac{\rho_{ik} \min(t_j, t_l)}{\sqrt{t_j \cdot t_l}},$$

$$D1_{kl}^{ij} = \begin{cases} \dfrac{\ln(S_i(0)/M) + (r + \frac{1}{2}\sigma_i^2)t_j}{\sigma_i \sqrt{t_j}} & \text{for } k = i \text{ and } l = j \\ \dfrac{\ln(S_i(0)/S_k(0)) + r(t_j - t_l) + \frac{1}{2}\Sigma_{ijkl}^2}{\Sigma_{ijkl}} & \text{otherwise} \end{cases},$$

$$F1_{kl,pq}^{ij} = \begin{cases} 1 & \text{for } kl = pq = ij \\ \dfrac{\sigma_i t_j - \rho_{ip}\sigma_p \min(t_j,t_q)}{\sqrt{t_j}\,\Sigma_{ijpq}} & \text{for } kl = ij,\ pq \neq ij \\ \dfrac{\sigma_i t_j - \rho_{ik}\sigma_k \min(t_j,t_l)}{\sqrt{t_j}\,\Sigma_{ijkl}} & \text{for } kl \neq ij,\ pq = ij \\ \dfrac{H}{\Sigma_{ijkl}\Sigma_{ijpq}} & \text{for } kl \neq ij,\ pq \neq ij \end{cases},$$

and

$$\Sigma_{ijkl} = \sqrt{\sigma_i^2 t_j - 2\rho_{ik}\sigma_i\sigma_k \min(t_j,t_l) + \sigma_k^2 t_l},$$

and

$$H = \sigma_i^2 t_j - \rho_{ik}\sigma_i\sigma_k \min(t_j,t_l) - \rho_{ip}\sigma_i\sigma_p \min(t_j,t_q) + \rho_{kp}\sigma_k\sigma_p \min(t_l,t_q).$$

In the other case of $M < K$, $I_{A_M} = 0$, the formula is similar to the case of $M \geq K$, and Eq. (9.1) can be expressed as

$$C_{\max}(0) = e^{-rT}\sum_{i,j} E^Q\left[(S_i(t_j) \cdot I_{A_{ij}})\right] - Ke^{-rT}E^Q[I_A]$$

$$= \left\{ \sum_{i,j} S_i(0)e^{-r(T-t_j)} N_{mn}\left[(D2_{kl}^{ij})_{mn};\,(F2_{kl,pq}^{ij})_{mn \times mn}\right] \right\}$$

$$- Ke^{-rT}\left(1 - N_{mn}\left[(d2_{ij})_{mn};\,(f2_{ij,kl})_{mn \times mn}\right]\right), \qquad (9.3)$$

where

$$d2_{ij} = -\frac{\ln(S_i(0)/K) + (r - \frac{1}{2}\sigma_i^2)t_j}{\sigma_i\sqrt{t_j}},$$

$$f2_{ij,kl} = f1_{ij,kl},$$

$$D2_{kl}^{ij} = \begin{cases} \dfrac{\ln(S_i(0)/K)+(r+\frac{1}{2}\sigma_i^2)t_j}{\sigma_i\sqrt{t_j}} & \text{for } k=i \text{ and } l=j \\ D1_{kl}^{ij} & \text{otherwise} \end{cases},$$

$$F2_{kl,pq}^{ij} = F1_{kl,pq}^{ij}.$$

This appendix shows that the pricing technique of Theorem 2.1 can be employed to price different types of discrete-sampling path-dependent rainbow options. Therefore, the technique proposed in this paper can broach many possible directions in the design of new contracts with both the path dependent and multiple asset features.

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