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Determinants of S&P 500 index option returns

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Abstract We analyze common factors that affect returns on S&P 500 index options and find that 93% of the variation in option returns can be explained by three factors, which respectively account for 87%, 4%, and 2% of the variation in option returns. Furthermore, we test diffusion option pricing models by using mean–variance spanning properties implied in the models. The spanning tests reject one-factor diffusion models, as well as the hypothesis that the underlying asset and an equally weighted option index span options. Our results fail to reject that the underlying asset and an at-the-money option can span out-of-the-money options, but does reject that they span in-the-money options.

Keywords Option returns · Factor analysis · Option implied volatility · Equally weighted option index · Mean–variance spanning

JEL Classification G10 · G12 · G13

1 Introduction

Since the seminal work of Black and Scholes (1973) and Merton (1973), there have been extensive studies to extend one-factor option pricing models. Examples of

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multifactor pricing models and related tests include, among others, Bakshi et al. (1997, 2000), Bates (1996, 2000), Buraschi and Jackwerth (2001), Chernov and Ghysels (2000), Cochrane and Saa-Requejo (2000), Heston (1993), Hull and White (1987), Scott (1987, 1997), Stein and Stein (1991), and Wiggins (1987). Recent work of Bakshi et al. (2003), Bondarenko (2003) and Jones (2006) focuses on understanding the nature of option returns. However, several important questions remain unanswered. They include: How many factors are sufficient to describe most of the variation in option returns? What is the contribution of each factor and what are good proxies for these factors? In this paper, we empirically determine and analyze the common factors that affect option returns. Exploring the number of common factors and the nature of these factors provides insights into how important each factor is, how many state variables should be selected in the option modeling framework, and how to construct hedging portfolios.

Our empirical investigation focuses on option returns and is based on daily prices of S&P 500 index options from 1988 to 1994. We begin our analysis with the construction of representative option returns with constant moneyness and maturity. These options allow us to use both cross-sectional and time-series information in option prices, and to separate the effect of moneyness and maturity on option returns from other systematic factors. We identify the number of common factors using factor analysis. To interpret the unobservable common factors obtained through factor analysis, we follow Knez et al. (1994) and adopt a particular factor rotation scheme. Under this scheme, rotated factors best mimic the variables that can be justified based on normative grounds, such as the underlying asset return, the option index return, and the volatility. Further, we directly examine what fraction of the variation in option returns can be explained by observable proxies for the factors.

In factor analysis, we examine three-factor models and find that such factors explain an average of 93% of the total variation in option returns. The first, second, and third factors, respectively account for 87%, 4%, and 2% of the total variation in option returns. In particular, the first factor explains more than 90% of the variation in the at-the-money and the in-the-money option returns, while the second factor explains 9% of the variation in the out-of-the-money option returns. Thus, the contribution of the second factor to the out-of-the money options is expected to be larger. We interpret the first factor as the underlying factor and the second factor as the volatility factor.

Since the true factors are not observable, we construct proxies for the three factors and examine what fraction of option returns can be explained by these proxies. A natural candidate variable for the first factor is the underlying S&P 500 index returns. For the second factor, we consider two alternative proxies: one is the equally weighted option index (a "market" portfolio for options), and the other is the option-implied volatility. Results from the regression analysis indicate that call (put) option returns are positively (negatively) associated with S&P 500 returns, positively related to the equally weighted option index returns, and positively associated with the implied volatility. When the second factor is proxied by the option index, it offers significantly incremental explanatory power for option returns, especially for the out-of-the-money options. Further, our results indicate that the equally weighted option index is a better proxy for the second factor than the option implied volatility.

Hedging effectiveness is an important yardstick for evaluating the performance of an option pricing model. The essential element of any hedging test is whether selected hedging instruments (for example, the underlying asset and the risk-free bonds) can span the space of target option returns. While most hedging tests require the specification of a parametric option pricing model, spanning tests are model independent, as long as one knows how many factors, and what factors, are in the modeling framework. The rejection of the spanning tests implies that selected hedging instruments cannot effectively hedge the target options, which makes it unnecessary to test each parametric option model in the same class.

We perform mean–variance spanning tests based on the framework of Hansen and Jagannathan (1991), Huberman and Kandel (1987) and Kan and Zhou (2001). We use the underlying asset and an equally weighted option index (or an at-the-money option) as benchmark assets to span out-of-the-money, at-the-money or in-the-money option returns, both individually and jointly. Our evidence rejects the null hypothesis that the underlying asset and an equally weighted option index span option returns. We find that the underlying asset and the at-the-money option can span the out-of-the-money option returns, but not the in-the-money option returns. Our results indicate that one, or more than one, option instrument(s) should be used in conjunction with the underlying asset in order to devise hedging strategies when the hedging target is an out-of-the money or in-the-money option.

This paper is organized as follows: Section 2 provides a description of the S&P 500 index options data along with the procedures we use to construct returns of constant moneyness and maturity options. Section 3 discusses our factor analysis, the rotation method, and the related empirical results. Section 4 presents results of the regression analysis from one-, two-, and three-factor models, using observable variables as proxies for factors. In Sect. 5, we conduct mean–variance spanning tests for index option returns. Section 6 examines the robustness of our results. Concluding remarks are offered in Sect. 7.

2 Data

2.1 Data description

We use the prices of S&P 500 index options in our empirical investigation for several reasons. The most important reason is that these options are the most actively traded European-style contracts. Many empirical studies focus on S&P 500 index options and test alternative option pricing models [see, e.g., Bakshi et al. (1997), Bates (1991), Dumas et al. (1998), Rubinstein (1994)]. Further, many contracts with different strike prices and maturities are traded, and their prices are actively quoted. On a typical day, the strike price of these option contracts ranges from 15% out-of-money to 15% in-the-money, and the days-to-expiration ranges from less than 30 days to 1 year. Since there are many different contracts, it is possible to construct prices of options with constant moneyness and maturity.

The initial sample is obtained from the Berkeley Option Database and consists of the intra-daily transaction prices and bid-ask quotes for S&P 500 index options traded on the CBOE. Our sample period extends from June 1, 1988 through May 31, 1994. To eliminate the potential impact of the option bid-ask spread, our analysis uses the mid-point of the bid-ask quotes as a proxy for the market value of the option contract. We note that for each of the bid-ask quotes for S&P 500 index options, the recorded index value is the index level at the moment the option price is recorded, not the daily closing index level. We apply two standard filters to our option sample: (1) Observations with obvious recording errors are excluded; (2) Option prices that are time-stamped later than 3:00 pm Central Time are dropped. The second filter ensures that the spot price and the option price are synchronized. The final sample comprises 92,637 calls and 83,020 puts prices placed during 1,506 trading days of the sample period.

In addition to the option data, we obtain daily S&P 500 index levels from the Center for Research in Security Prices (CRSP). The daily dividend distributions for the S&P 500 index are obtained from the S&P 500 Information Bulletin. Daily Treasury-bill rates with various maturities are from Datastream International. To obtain the yield corresponding to the maturity of a given option contract, we use the two Treasury bills that straddle the option's expiration date.

2.2 Options with constant moneyness and maturity

To separate the effect of moneyness and maturity on option returns from other systematic factors, we must have returns of option contracts with constant moneyness and time-to-expiration. However, we cannot directly observe options with constant moneyness and maturity because the moneyness of an option contract constantly changes (as the underlying index changes), and because the option's time-to-expiration always decreases as time goes by. In this subsection, we discuss how to construct returns of options with constant moneyness and maturity.

During our sample period, there are an average of 100 S&P 500 index option contracts available on each day (49 calls and 51 puts, respectively). These contracts are different in their strike prices (e.g., moneyness) and days-to-expiration. From these contracts, we obtain six representative option prices for calls and another six for puts using interpolations.[1] Each of the resulting option contracts is characterized by a particular combination of a moneyness and a maturity. These options are listed as below:

- CSTOTM: short-term out-of-the-money calls
- CSTATM: short-term at-the-money calls
- CSTITM: short-term in-the-money calls
- CLTOTM: long-term out-of-the-money calls
- CLTATM: long-term at-the-money calls
- CLTITM: long-term in-the-money calls
- PSTOTM: short-term out-of-the-money puts
- PSTATM: short-term at-the-money puts
- PSTITM: short-term in-the-money puts
- PLTOTM: long-term out-of-the-money puts

[1] Bates (1991) interpolates option prices across strikes. Buraschi and Jackwerth (2001) use interpolated Black-Scholes implied volatilities to obtain corresponding option prices.

- PLTATM: long-term at-the-money puts
- PLTITM: long-term in-the-money puts

We define short-term options as 30-day options and long-term options as 180-day options.[2] The out-of-the-money (OTM), at-the-money (ATM), and in-the-money (ITM) calls (or puts) are those with $\frac{S}{K}$ $\left(\text{or } \frac{K}{S}\right)$ close to 0.95, 1.00, and 1.05, respectively, where S is the spot price and K the strike price.

We use short-term OTM calls (e.g., $\tau = 30$ days and $\frac{S}{K} \approx 0.95$) as an example to illustrate how to obtain prices of options with constant moneyness and maturity. Let K and τ be the strike price and days-to-expiration, and $C(t, \tau, K)$ denote the observed price at t. For convenience, we assume there are no 30-day options available on day t, i.e., a contract's days-to-expiration is either less than or greater than 30 days. We adopt the following procedure:

1. For a given day t, find all available strike prices for S&P 500 call options. For each strike price K, identify the contract whose days-to-expiration is less than, but the closest to, 30 days. Next, find another contract whose days-to-expiration is greater than, but the closest, to 30 days. Denote these two call prices by $C(t, \tau_1, K)$ with $\tau_1 < 30$ and $C(t, \tau_2, K)$ with $\tau_2 > 30$, respectively. Use the two prices and linear interpolation to obtain the price of a 30-day call option, $C(t, 30, K)$.
2. For day $t-1$, repeat Step 1 and obtain the 30-day call option price $C(t-1, 30, K)$ for each strike price K.
3. Among all available strike prices on day t, find the one in which $\frac{S}{K}$ is the closest to 0.95. Let K^* denote the resulting strike price. Calculate the 30-day out-of-the money call-option return on day t as $R_t = log\frac{C(t,30,K^*)}{C(t-1,30,K^*)}$.
4. Repeat Steps 1 through 3 for each day to obtain a return time series for call options that have a constant moneyness $\left(\frac{S}{K} \approx 0.95\right)$ and a constant maturity (30 days).

Returns on other call and put options can be constructed in a similar manner.

Table 1 presents descriptive statistics for the 12 representative daily option returns and the underlying S&P 500 index. It reveals several systematic patterns. First, call-option returns are persistently positive, and put-option returns are persistently negative, regardless of the moneyness or the maturity of the contract. Returns on short-term options are generally higher (lower) than their long-term counterparts for calls (puts), and returns on OTM calls (puts) are higher (lower) than those on ITM calls (puts). For example, daily option return averages 1.16% (0.46%) and ranges from −10.38% to 12.34% (from −4.02% to 4.71%) for short-term (long-term) ATM calls. Second, for a fixed maturity, the return volatility decreases monotonically from OTM to ATM and then to ITM calls (or puts). For instance, for short-term calls, CSTOTM, CSTATM, and CSTITM, the standard deviations of daily returns are 29.76%, 21.47%, and 10.18%, respectively. For a given moneyness, short-term options are more volatile than their long-term counterparts. Finally, option returns are more volatile than the underlying

[2] We choose 30 and 180 days based on several reasons. First, the corresponding option contracts should have distinct characteristics in terms of days-to-expiration. Second, option contracts with days-to-expiration surrounding the two chosen values are available on a typical day. To examine the robustness of our results, we use alternative definitions of short- or long-term options (e.g., we define long-term options as 240-day options) and find the results are qualitatively similar.

Table 1 Sample statistics for returns on S&P 500 index options and on equally weighted option indexes

Return series	Mean	SD	25% Quartile	Median	75% Quartile
Calls					
CSTOTM	1.085	29.762	−15.708	1.040	16.983
CSTATM	1.165	21.479	−10.383	0.725	12.345
CSTITM	0.385	10.181	−4.157	0.335	5.525
CLTOTM	0.655	12.897	−6.503	0.435	7.307
CLTATM	0.467	8.054	−4.022	0.265	4.715
CLTITM	0.278	5.374	−2.572	0.215	3.105
Puts					
PSTOTM	−1.121	23.842	−15.290	−2.170	11.797
PSTATM	−1.176	22.649	−14.110	−1.435	11.138
PSTITM	−0.589	13.306	−7.712	−0.590	6.263
PLTOTM	−0.366	9.923	−6.198	−0.300	4.547
PLTATM	−0.339	9.075	−5.488	−0.420	4.390
PLTITM	−0.271	6.524	−3.743	−0.370	2.943
Option indexes					
EW	0.014	2.950	−1.734	−0.167	1.549
EW short-term	−0.042	4.716	−2.695	−0.344	2.414
EW long-term	0.071	2.139	−1.120	0.019	1.163
EW OTM	0.063	6.488	−3.783	−0.159	3.571
EW ATM	0.029	2.875	−1.643	−0.178	1.422
EW ITM	−0.050	1.856	−0.831	−0.051	0.729
Underlying asset					
S&P 500 index	0.035	0.805	−0.355	0.033	0.460

This table presents summary statistics for returns on S&P 500 index options and on option indexes. Results are reported for 12 option-return time series (six for calls and six for puts) with different moneyness-maturity categories: (1) CSTOTM (short-term out-of-the-money calls), (2) CSTATM (short-term at-the-money calls), (3) CSTITM (short-term in-the-money calls), (4) CLTOTM (long-term out-of-the money calls), (5) CLTATM (long-term at-the-money calls), (6) CLTITM (long-term in-the-money calls), (7) PSTOTM (short-term out-of-the-money puts), (8) PSTATM (short-term at-the-money puts), (9) PSTITM (short-term in-the-money puts), (10) PLTOTM (long-term out-of-the-money puts), (11) PLTATM (long-term at-the-money puts), and (12) PLTITM (long-term in-the-money puts)

These return series are used to construct six equally weighted option indexes. They are equally weighted "market" (EW), equally weighted short-term (EW short-term), equally weighted long-term (EW long-term), equally weighted OTM (EW OTM), equally weighted ATM (EW ATM), and equally weighted ITM (EW ITM) option indexes. Return is defined as logarithmic price change (in %)

The sample period extends from June 1, 1988 through May 31, 1994 for a total of 1,506 daily observations

asset return by a large margin. The standard deviation of daily return on S&P 500 index is 0.80%, while the standard deviation of each daily option return is above 5%.

We construct six equally weighted option return indexes from the above 12 option return series.[3] These option indexes are:

- EW: equally weighted "market" option index
- EW short-term: equally weighted short-term option index
- EW long-term: equally weighted long-term option index
- EW OTM: equally weighted OTM option index
- EW ATM: equally weighted ATM option index
- EW ITM: equally weighted ITM option index

[3] In a later section, we will elaborate on how these option indexes relate to common factors.

As shown in Table 1, the average return on the equally weighted "market" option index is close to zero, but the option index's standard deviation is more than three times as large as that of the underlying index (2.95% compared to 0.80%). Between the short- and long-term option indexes, the former's volatility is much larger. Among the OTM, ATM, and ITM option indexes, the return volatility decreases as the moneyness increases.

Table 2 reports the correlation matrix for the 12 individual option-return time series. There is a considerably high correlation between any two option returns. For a given pair of call (or put) option returns, the correlation is positive, and ranges from 0.77 to 0.96 (or from 0.83 to 0.94). Notice that the correlation between a call option return and a put option return is always negative, ranging from −0.92 to −0.69. This evidence indicates that prices of call options generally move in the same direction, and they move in the opposite direction to those of put options. However, these correlations are significantly different from those predicted by the one-dimensional diffusion option models (1.0 between two call, or two put, option returns and −1.0 between a call and a put option returns).

To get a sense of how strongly option returns are related to the underlying asset and other factors, Table 3 reports correlations among option returns and the underlying S&P 500 index returns, as well as among option returns and option index returns. It is noted that the correlation between the S&P 500 returns and short-term call returns increases as the option's moneyness increases from the OTM to the ATM, then to the ITM. The correlations are 0.78, 0.89 and 0.90, respectively, for short-term OTM, ATM, and ITM calls. We observe a similar pattern for put options, except that the correlation between the underlying and put-option returns is negative. The high correlation between the underlying and option returns suggests that the underlying asset return is the primary factor affecting option returns. However, its impact depends on the option's moneyness.

The correlation between a given option and the equally weighted "market" option index (EW) is substantially lower than that between the same option and the S&P 500 index. For example, correlations between CSTOTM and EW and between CSTOTM and the S&P 500 index are 0.36 and 0.78, respectively. We also observe that the correlation between EW and option returns decreases as the option's moneyness increases from OTM to ITM. This is true for both short- and long-term options, and for both calls and puts. Finally, the correlation between any option and an option index is positive. Consequently, if any of the option index is used as a proxy for a systematic factor, its impact on call- and put-option returns is unidirectional, and its impact is larger on the OTM options than on the ITM options.[4]

Our results suggest that the correlation between each of the six option indexes and the S&P 500 index is close to zero. For instance, the correlations between the EW, EW short-term, or EW long-term option index and the S&P 500 index are −0.01, −0.01 and 0.00, respectively.[5] The moderate correlations between individual option returns and option indexes and the low correlations between option indexes and the S&P 500

[4] Recall that call-option return is positively, put-option return negatively, associated with the underlying return.

[5] For brevity, these results are not reported in the table.

Table 2 Correlations of S&P 500 index option returns

Return series	CSTOTM	CSTATM	CSTITM	CLTOTM	CLTATM	CLTITM	PSTOTM	PSTATM	PSTITM	PLTOTM	PLTATM	PLTITM
CSTOTM	1.00											
CSTATM	0.88	1.00										
CSTITM	0.79	0.89	1.00									
CLTOTM	0.80	0.84	0.77	1.00								
CLTATM	0.85	0.91	0.84	0.88	1.00							
CLTITM	0.84	0.90	0.85	0.86	0.96	1.00						
PSTOTM	−0.70	−0.81	−0.79	−0.72	−0.83	−0.85	1.00					
PSTATM	−0.80	−0.91	−0.89	−0.79	−0.89	−0.90	0.93	1.00				
PSTITM	−0.82	−0.92	−0.89	−0.81	−0.90	−0.91	0.87	0.94	1.00			
PLTOTM	−0.72	−0.78	−0.75	−0.69	−0.81	−0.83	0.83	0.84	0.83	1.00		
PLTATM	−0.78	−0.87	−0.86	−0.78	−0.88	−0.90	0.87	0.92	0.92	0.90	1.00	
PLTITM	−0.78	−0.87	−0.86	−0.79	−0.89	−0.91	0.85	0.90	0.92	0.86	0.94	1.00

Reported below are correlations among 12 option returns time series: (1) CSTOTM (short-term out-of-the-money calls), (2) CSTATM (short-term at-the-money calls), (3) CSTITM (short-term in-the-money calls), (4) CLTOTM (long-term out-of-the money calls), (5) CLTATM (long-term at-the-money calls), (6) CLTITM (long-term in-the-money calls), (7) PSTOTM (short-term out-of-the-money puts), (8) PSTATM (short-term at-the-money puts), (9) PSTITM (short-term in-the-money puts), (10) PLTOTM (long-term out-of-the-money puts), (11) PLTATM (long-term at-the-money puts), and (12) PLTITM (long-term in-the-money puts)

The sample period extends from June 1, 1988 through May 31, 1994, for a total of 1,506 daily observations

Table 3 Correlations among S&P 500 index returns, S&P 500 index option returns and option index returns

Return series	CSTOTM	CSTATM	CSTITM	CLTOTM	CLTATM	CLTITM	PSTOTM	PSTATM	PSTITM	PLTOTM	PLTATM	PLTITM
S&P 500 index	0.78	0.89	0.90	0.77	0.87	0.88	−0.81	−0.89	−0.92	−0.80	−0.89	−0.90
EW	0.36	0.17	0.06	0.26	0.11	0.06	0.30	0.15	0.05	0.21	0.14	0.10
EW short-term	0.37	0.15	0.06	0.13	0.05	0.01	0.34	0.17	0.06	0.14	0.10	0.07
EW long-term	0.18	0.12	0.03	0.42	0.20	0.13	0.08	0.03	0.01	0.28	0.16	0.10

Reported below are correlations between a pair of option return and the S&P 500 index return (or an equally weighted option index return)

The option-return time series includes (1) CSTOTM (short-term out-of-the-money calls), (2) CSTATM (short-term at-the-money calls), (3) CSTITM (short-term in-the-money calls), (4) CLTOTM (long-term out-of-the money calls), (5) CLTATM (long-term at-the-money calls), (6) CLTITM (long-term in-the-money calls), (7) PSTOTM (short-term out-of-the-money puts), (8) PSTATM (short-term at-the-money puts), (9) PSTITM (short-term in-the-money puts), (10) PLTOTM (long-term out-of-the-money puts), (11) PLTATM (long-term at-the-money puts), and (12) PLTITM (long-term in-the-money puts). The option indexes include equally weighted "market" (EW), equally weighted short-term (EW Short-term), and equally weighted long-term (EW Long-term) option indexes

The sample period extends from June 1, 1988 through May 31, 1994, for a total of 1,506 daily observations index suggest that option indexes could relate to some additional orthogonal factors that affect option returns.

3 Factor analysis of option returns

In this section, we use factor analysis to empirically determine the number of common factors underlying option returns. This section starts with a brief review of the linear factor model and the estimation procedure, followed by the estimation results.

3.1 The linear factor model

We assume that the option returns follow a linear k-factor structure:

$$r_i = \mu_i + \sum_{j=1}^{k} b_{ij} f_j + \varepsilon_i, \qquad i = 1, \ldots, p, \tag{1}$$

where $\mu_i = E[r_i]$ denotes the expected return for option i, and $\{f_1, f_2, \ldots\ldots, f_k\}$ represents the k unobservable common factors. The coefficient b_{ij} represents the loading of the ith option on the jth factor. The residual term ε_i is the idiosyncratic risk component. Rewriting Eq. 1 in matrix notation, we have:

$$R - \mu = Bf + \varepsilon, \tag{2}$$

where R, μ and ε are $(p \times 1)$-vectors, $B = (b_{ij})$ is a $(p \times k)$-matrix, and f is a $(k \times 1)$-vector.

As is standard in the literature, we assume that:

$$\begin{array}{ll} E[f] = 0, & Cov(f) = E[ff'] = I, \\ E[\varepsilon] = 0, & Cov(\varepsilon) = E[\varepsilon\varepsilon'] = \Psi, \\ Cov(\varepsilon, f) = E[\varepsilon f'] = 0, & \end{array} \tag{3}$$

where I is a $(k \times k)$ identity matrix and $\Psi = \text{diag}(\psi_1, \psi_2, \ldots, \psi_p)$, which is a $(p \times p)$ diagonal matrix. Equations 2 and 3 define an orthogonal factor model. In this model, the variance-covariance matrix of R, denoted by $\Sigma = (\sigma_{ij})$, has the following structure:

$$\Sigma = BB' + \Psi. \tag{4}$$

Notice that Eqs. (2–4) remain unchanged under the transformation:

$$B^* = BG \quad \text{and} \quad f^* = G'f \quad \text{with} \quad GG' = I. \tag{5}$$

As a result, factor loadings B are unique only up to an orthogonal transformation G.

3.2 Estimation of factor loadings

The two methods that are commonly used to estimate the loadings matrix B are the principal component and the maximum likelihood methods. The former is simple and computationally more efficient. The latter has desirable asymptotic properties, but requires the assumption that option returns are normally distributed. To maintain minimum assumptions about option returns, we use the principal component method to estimate the matrix B.

Let the option return covariance matrix Σ have eigenvalue-eigenvector pairs $(\lambda_i, e_i)_{1\le i\le p}$, with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p \ge 0$. Using the principal component method, the loadings matrix B for a k-factor ($k < p$) model is given by:

$$B = (b_1, b_2, \ldots, b_k) \;=\; \left(\sqrt{\lambda_1}e_1, \sqrt{\lambda_2}e_2, \ldots, \sqrt{\lambda_k}e_k\right),$$

where b_i, $1 \le i \le k$, is a $(p \times 1)$-vector that represents the loadings on the ith factor. It follows that the idiosyncratic covariance matrix is given by:

$$\Psi = \mathrm{diag}(\psi_1, \psi_2, \ldots, \psi_p) \;\text{ with } \psi_i = \sigma_{ii} - \sum_{j=1}^{p} b_{ij}^2.$$

We obtain the principal component solution of the loadings matrix B and idiosyncratic covariance matrix Ψ by replacing Σ by the sample covariance matrix $S = (s_{ij})$. Let $\hat{B}$ and $\hat{\Psi}$ be the estimates of B and Ψ, respectively. We have:

$$\hat{B} = \left(\sqrt{\hat{\lambda}_1}\hat{e}_1, \sqrt{\hat{\lambda}_2}\hat{e}_2, \ldots, \sqrt{\hat{\lambda}_k}\hat{e}_k\right),$$

$$\hat{\Psi} = \mathrm{diag}(\hat{\psi}_1, \hat{\psi}_2, \ldots, \hat{\psi}_p) \;\text{ with } \hat{\psi}_i = s_{ii} - \sum_{j=1}^{p} \hat{b}_{ij}^2,$$

where $(\hat{\lambda}_i, \hat{e}_i)_{1\le i\le p}$ are eigenvalue-eigenvector pairs of S with $\hat{\lambda}_1 \ge \hat{\lambda}_2 \ge \cdots \ge \hat{\lambda}_p \ge 0$.

It is noted that the contribution of each factor to the variance of option i (or the total variance of p options) can be written as a function of eigenvalues and eigenvectors of Σ. For example, the contribution to option i's variance σ_{ii} from the jth factor is $b_{ij}^2 = \lambda_j e_{ij}$, where e_{ij} is the ith element of eigenvector e_j. The contribution to option i's variance σ_{ii} from all k common factors is $\sum_{j=1}^{k} b_{ij}^2 = \sum_{j=1}^{k} \lambda_j e_{ij}$. The contribution to the total variance of p options, $\mathrm{tr}(\Sigma)$, from the jth factor is $\sum_{i=1}^{p} b_{ij}^2 = \sum_{i=1}^{p} \lambda_j e_{ij} = \lambda_j$ (eigenvectors have unity length), and the contribution to the total variance of p assets from all k common factors is $\sum_{j=1}^{k}\sum_{i=1}^{p} b_{ij}^2 = \sum_{j=1}^{k} \lambda_j$.

3.3 Estimation of factors

We can estimate the k unobservable common factors by using the weighted least squares, the cross-sectional regression, or by constructing mimicking portfolios, which are perfectly correlated with the k factors (see Grinblatt and Titman 1987, Huberman et al. 1987, Lehmann and Modest 1988).

Given the estimated loadings matrix $\hat{B}$, we can treat R and B in Eq. 2 as dependent and independent variables and estimate the factors f by using a cross-sectional regression. This yields:

$$f = (B'\Psi^{-1}B)^{-1}B'\Psi^{-1}(R-\mu).$$

Notice that estimates of f are obtained by minimizing the sum of weighted squared errors $[(R-\mu)-Bf]'\Psi^{-1}[(R-\mu)-Bf]$, and therefore, they are the weighted least squares estimator of f. It follows that the estimated factor values at t are given by:

$$\hat{f}_t = \left[(\hat{B}'\hat{\Psi}^{-1}\hat{B})^{-1}\hat{B}'\hat{\Psi}^{-1}\right]\left(R_t - \bar{R}\right), \tag{6}$$

where $\bar{R}$ is the sample mean of R. Using Eq. 6 for each t, $t = 1, \ldots, T$, yields a time series of the factor scores. Equation 6 implies that the factor scores are a linear function of option returns R_t. This indicates that option-return portfolios can be constructed from the p option returns, r_i, $i = 1, \ldots, p$, to mimic the factors. In fact, the matrix in the brackets in Eq. 6 represents the set of k-mimicking portfolios. Once we obtain mimicking portfolios, we can use their returns as estimates of the factors.

When we estimate $\hat{B}$ and $\hat{\Psi}$ by using the principal component solution, we follow a standard practice to assume that the residual variances ψ_{ii}, $i = 1, \ldots, p$, are almost equal. It follows from Eq. 6 that:

$$\hat{f}_t = (\hat{B}'\hat{B})^{-1}\hat{B}'(R_t - \bar{R}) = \left(\frac{\hat{e}_1}{\sqrt{\hat{\lambda}_1}}, \frac{\hat{e}_2}{\sqrt{\hat{\lambda}_2}}, \ldots, \frac{\hat{e}_k}{\sqrt{\hat{\lambda}_k}}\right)'(R_t - \bar{R}).$$

We use the above equation to obtain estimates of factor values.

3.4 Factor rotations

Both factor loadings and factors are unique only up to an orthogonal transformation (rotation). Since we examine option returns in this article, a guide to the rotation is that the most important factor affecting option returns should be the underlying asset, and that the second most important, the volatility factor. There is abundant evidence showing that return volatility changes stochastically over time, and it has important implications for pricing options [see e.g., French et al. (1987), Madan et al. (1998), Melino and Turnbull (1990), Schwert (1989, 1990), Scott (1987, 1997), and Wiggins (1987)].

To facilitate our interpretation of the factors, we rotate the estimated loadings matrix $\hat{B}$ in such a way that the first factor implied by the rotated loadings is the closest to the underlying S&P 500 index returns (e.g., the distance between the two vectors is minimized), and the second factor is the closest to the change in volatility.[6] This requires the construction of an orthonormal transformation matrix G to achieve the desired rotation. Following Knez et al. (1994), we use the k-mimicking portfolios obtained earlier to construct G. Specifically, we adopt a three-step procedure to obtain the matrix G and use the three-factor ($k = 3$) model for the point of discussion.

First, we regress S&P 500 index returns on returns of the $k(= 3)$ mimicking portfolios and use the regression coefficients as the first column of G:

$$g_1 = (\hat{F}'\hat{F})^{-1}\hat{F}'R^{\text{S\&P 500}}, \tag{7}$$

where $\hat{F} = (\hat{f}_1, \hat{f}_2, \hat{f}_3)$ is a $(T \times 3)$ matrix that represents initially estimated factor values, and $R^{\text{S\&P 500}}$ is the $(T \times 1)$ vector of the S&P 500 returns. Next, we regress the second observable factor, say a measure of volatility, on the three mimicking portfolios, with the constraint that the vector of the regression coefficients is orthogonal to g_1. The resulting regression coefficients are used as the second column of G. As shown in Appendix A:

$$g_2 = g_2^0 - \alpha_0(\hat{F}'\hat{F})^{-1}g_1, \tag{8}$$

with

$$g_2^0 = (\hat{F}'\hat{F})^{-1}\hat{F}'R_2, \qquad \text{and} \qquad \alpha_0 = \frac{g_1' g_2^0}{g_1'(\hat{F}'\hat{F})^{-1}g_1}, \tag{9}$$

where R_2 is the $(T \times 1)$ vector of the second observable factor, and g_2^0 is the (3×1) vector of the unconstrained regression coefficients.

Finally, we regress the residual from the previous regression against the three mimicking portfolios, with the constraint that the vector of the regression coefficients is orthogonal to both g_1 and g_2. The regression coefficients, g_3, are taken as the last column of G, and are given by:

$$g_3 = g_3^0 - (\hat{F}'\hat{F})^{-1}(\alpha_1 g_1 + \alpha_2 g_2), \tag{10}$$

with

$$g_3^0 = (\hat{F}'\hat{F})^{-1}\hat{F}'e^*, \qquad \alpha_1 = \frac{c_1 a_2 - c_2 a_1}{a_0 a_2 - a_1^2}, \qquad \alpha_2 = \frac{c_2 a_0 - c_1 a_1}{a_0 a_2 - a_1^2}. \tag{11}$$

[6] Our subsequent empirical analysis indicates that a three-factor model can explain about 93% of the variation in option returns. Therefore, our discussion is based on the three-factor model. Implicit in our discussion is the assumption that the underlying asset is the most important, and that volatility is the next most important factor. The rotation method discussed here is applicable when other variables are used as proxies for common factors. We also redo the factor rotation by using the equally weighted option index as the second observable factor and find the results are qualitatively similar.

(See Appendix A.) Note that e^* is the $(T \times 1)$ vector of residuals from the second regression and:

$$\begin{aligned} a_0 &= g_1'(\hat{F}'\hat{F})^{-1}g_1, \quad & a_1 &= g_1'(\hat{F}'\hat{F})^{-1}g_2, \quad & a_2 &= g_2'(\hat{F}'\hat{F})^{-2}g_2, \\ c_1 &= g_1'g_3^0, & c_2 &= g_2'g_3^0. \end{aligned}$$

With the orthogonal transformation matrix $G = (g_1, g_2, g_3)$, we use Eq. 5 to obtain a new loadings matrix and new factor scores by rotating the initial loadings matrix $\hat{B}$ and initial factors $\hat{f}$.

3.5 Estimation results

To present estimation results from the three-factor ($k = 3$) model, we begin with the results based on the original factor loadings reported in Table 4. First, according to the original factor loadings matrix, all six call-option returns have a large positive sensitivity, and all six put-option returns have a large negative sensitivity, to the first factor. The estimated loading monotonically decreases (increases) from OTM to ITM calls (puts). This is true for both short- and long-term options. For example, the estimated loadings are 26.10 and 9.32 for short-term OTM and ITM calls, and −8.79 and −6.21 for long-term OTM and ITM puts. Thus, the OTM option returns are the most sensitive to the first factor. Second, all call- and put-option returns have a *positive* loading on the second factor, indicating that the impact of the second factor on both call and put option returns is in the same direction. This contrasts with signs of loadings on the first factor, which are all positive for calls and all negative for puts. The estimated loading on the second factor is the largest for the OTM options, and the smallest (close to zero) for the ITM options. Finally, the loadings on the third factor are relatively small.

The rotated factor loadings on the first and second factors are similar to the original ones, in both sign and magnitude. For example, all six call (put) option returns have positive (negative) loadings on the rotated first factor. Since we choose the rotation so that the rotated first factor is consistent with returns on S&P 500 index, this factor can be interpreted as representing the return of the underlying asset. Next, all 12 option returns have positive loadings on the rotated second factor. Recall that the rotation is such that the new loadings are consistent with the second factor being a proxy for volatility. Thus, the rotated second factor can be interpreted to be a volatility proxy.[7] The loadings on the third factor change noticeably after the rotation. However, the rotated loadings on the third factor do not seem to have a systematic pattern. The interpretation of this factor remains open.

Table 5 reports proportions of the variance explained by the three original (and rotated) factors for each of the 12 option returns. This decomposition shows the relative importance of each factor. In addition, this table presents summary results for options in a given maturity category (e.g., short- or long-term options) and a given moneyness category (e.g., OTM, ATM, or ITM options). On average, the three factors

[7] There might be alternative interpretations of the second factor. As demonstrated in the next section, the equally weighted option index also has a positive impact on both call and put option returns.

Table 4 Factor loadings from the principal component method

Return series	Original factor loadings			Rotated factor loadings		
	F_1	F_2	F_3	F_1	F_2	F_3
Calls						
CSTOTM	26.10	9.62	−0.54	25.67	9.56	4.81
CSTATM	20.48	3.17	−2.56	20.55	3.63	0.88
CSTITM	9.32	0.37	−3.12	9.64	0.96	−1.70
CLTOTM	11.26	4.29	2.67	10.68	3.73	4.93
CLTATM	7.72	1.01	0.86	7.51	0.85	2.08
CLTITM	5.17	0.33	0.53	5.04	0.24	1.29
Puts						
PSTOTM	−21.55	6.42	0.42	−21.58	6.19	−1.31
PSTATM	−21.82	2.81	2.61	−22.04	2.23	0.10
PSTITM	−12.85	0.38	1.40	−12.93	0.08	−0.30
PLTOTM	−8.79	2.90	−2.30	−8.48	3.27	−2.89
PLTATM	−8.67	1.58	−0.33	−8.59	1.60	−1.20
PLTITM	−6.21	0.77	−0.12	−6.15	0.77	−0.82

This table presents estimates of the original and rotated factor loadings from the principal component method

Results are reported for the 12 option-return time series: (1) CSTOTM (short-term out-of-the money calls), (2) CSTATM (short-term at-the-money calls), (3) CSTITM (short-term in-the-money calls), (4) CLTOTM (long-term out-of-the money calls), (5) CLTATM (long-term at-the-money calls), (6) CLTITM (long-term in-the-money calls), (7) PSTOTM (short-term out-of-the money puts), (8) PSTATM (short-term at-the-money puts), (9) PSTITM (short-term in-the-money puts), (10) PLTOTM (long-term out-of-the money puts), (11) PLTATM (long-term at-the-money puts), and (12) PLTITM (long-term in-the-money puts)

The sample period extends from June 1, 1988 through May 31, 1994 for a total of 1,506 daily observations

explain 92.82% of the variation in all 12 option returns. The first, second, and third factors, explain 86.73%, 3.97%, and 2.11%, respectively, of the variation. The results for short- and long-term options are similar. For example, the three factors explain, on average, 92.41% (93.23%) of the variation in all six short-term (long-term) option returns. Among the three moneyness categories, the first factor accounts for a significantly larger percentage of the variation in the ATM option returns (91.75%) and the ITM option returns (90.13%) than it does in the OTM option returns (78.31%). In contrast, the second factor is significantly more important for the OTM options (9.33%) than it is for the ATM (2.09%) and the ITM options (0.50%).

For each of the 12 option returns, the first factor accounts for the largest percentage variation in returns, ranging from 76.22% for long-term OTM calls (CLTOTM), to 93.33% for short-term ITM puts (PSTITM). The second factor explains the second largest percentage variation in option returns (except CSTITM, CLTITM, and PSTITM), especially for the OTM options.

Judging by the fraction of the total variance explained by each of the factors, we conclude that the first factor dominates all other factors, and that it is more important for the ATM and ITM options than for the OTM options. And yet, the contribution of the second factor toward the variation in option returns is significant, especially for the OTM options. The fact that results from rotated factor loadings are qualitatively similar (see Table 5) further substantiates our earlier argument that the first and second factors can be interpreted as the "underlying" factor and the "volatility" factor.

Table 5 Factor analysis of S&P 500 index option returns

Return series	Original factor loadings			Rotated factor loadings			Total (%)
	F_1 (%)	F_2 (%)	F_3 (%)	F_1 (%)	F_2 (%)	F_3 (%)	
Calls							
CSTOTM	76.89	10.44	0.03	74.36	10.39	2.61	87.37
CSTATM	90.94	2.17	1.42	91.51	2.85	0.17	94.54
CSTITM	83.89	0.13	9.39	89.72	0.90	2.78	93.41
CLTOTM	76.22	11.06	4.31	68.59	8.35	14.64	91.60
CLTATM	92.01	1.60	1.16	86.95	1.11	6.70	94.77
CLTITM	92.75	0.39	1.00	88.18	0.19	5.76	94.15
Puts							
PSTOTM	81.65	7.26	0.03	81.90	6.74	0.30	88.94
PSTATM	92.77	1.54	1.33	94.67	0.96	0.00	95.64
PSTITM	93.33	0.08	1.11	94.48	0.00	0.05	94.53
PLTOTM	78.46	8.55	5.38	73.03	10.84	8.51	92.40
PLTATM	91.27	3.05	0.13	89.57	3.12	1.76	94.46
PLTITM	90.55	1.40	0.03	89.00	1.40	1.58	92.00
Average							
All	86.73	3.97	2.11	85.16	3.91	3.74	92.82
Short-term	86.58	3.60	2.22	87.77	3.64	0.98	92.41
Long-term	86.88	4.34	2.00	82.56	4.17	6.49	93.23
OTM	78.30	9.33	2.44	74.47	9.08	6.52	90.08
ATM	91.75	2.09	1.01	90.68	2.01	2.16	94.86
ITM	90.13	0.50	2.88	90.35	0.62	2.54	93.52

Reported below are proportions of the total variance explained by factors 1, 2 or 3, respectively
Results are reported for the 12 option-return time series: (1) CSTOTM (short-term out-of-the money calls), (2) CSTATM (short-term at-the-money calls), (3) CSTITM (short-term in-the-money calls), (4) CLTOTM (long-term out-of-the money calls), (5) CLTATM (long-term at-the-money calls), (6) CLTITM (long-term in-the-money calls), (7) PSTOTM (short-term out-of-the money puts), (8) PSTATM (short-term at-the-money puts), (9) PSTITM (short-term in-the-money puts), (10) PLTOTM (long-term out-of-the money puts), (11) PLTATM (long-term at-the-money puts), and (12) PLTITM (long-term in-the-money puts)
The sample period extends from June 1, 1988 through May 31, 1994 for a total of 1,506 daily observations
OTM, ATM, and ITM denote out-of-the money, at-the-money, and in-the-money options, respectively

3.6 Correlations between factors and their proxies

Since option contracts are derivative securities, we consider the underlying asset return as a proxy for the first factor. However, the choice of a proxy for the second factor is not straightforward. One candidate is a measure of the volatility of the underlying asset return. Since the volatility is not observable, we use the option's implied volatility as a proxy for the second factor. Specifically, we use the Black-Scholes model to back out the implied volatility. The Black-Scholes model implied volatility on day t is estimated by using all available call and put option (closing) prices on that day, and by minimizing the sum of squared pricing errors.[8] Bates (1991), Dumas et al. (1998), Longstaff (1995), and Whaley (1982) have used a similar procedure.

[8] Appendix B provides details on how to use daily cross-sectional option prices to estimate implied volatility from the Black-Scholes model. The results based on the implied volatility from more complicated models (e.g., the stochastic volatility models) will be discussed in Sect. 6.

An alternative proxy for the second factor is the return on the equally weighted option index that is constructed from both call and put options in different moneyness-maturity categories. The rationale for choosing this index is as follows: Components of the index are pairs of call and put options with similar strike prices and maturities (e.g., short-term OTM calls and short-term ITM puts). Therefore, the first-order effect of the underlying asset on calls and puts cancels itself out. As a result, the second factor becomes dominant.

This point can be illustrated by the following example. Consider an equally weighted option portfolio consisting of an S&P 500 call option and an otherwise identical put option. For simplicity, assume that option returns are driven by two factors, the underlying index and a volatility factor. Returns on this portfolio over a short period of time can be decomposed as:

$$\frac{\Delta C}{C} + \frac{\Delta P}{P} = (\cdot)\Delta t + (\Omega_s^c + \Omega_s^p)\frac{\Delta S}{S} + (\Omega_v^c + \Omega_v^p)\frac{\Delta v}{v},$$

where the superscripts c and p denote the call and put options. Variables Ω_s^c and Ω_v^c are the call option's elasticities with respect to the underlying asset and the volatility respectively, $\Omega_s^c \equiv \frac{S}{C}\frac{\partial C}{\partial S}$, and $\Omega_v^c \equiv \frac{v}{C}\frac{\partial C}{\partial v}$. The put option's elasticities are defined similarly. In general, Ω_s^c and Ω_s^p have opposite signs, and the call and put prices move in the opposite directions with the underlying asset. In contrast, Ω_v^c and Ω_v^p have the same sign as the volatility changes. As a result, we expect the coefficient on the change in the underlying ($\Delta S/S$) to be small relative to the coefficient on the change in volatility ($\Delta v/v$).

To investigate whether the equally weighted option index return (EW) is a good proxy for the second factor, we regress it on the three factor scores obtained earlier. Panel A of Table 6 reports the regression result. The estimated coefficients on the first and the third factors are both small. However, the coefficient on the second factor is large, positive, and significant, and the adjusted R^2 is as high as 90.6%. Panel B of Table 6 displays proportions of the variance in EW explained by each of the three factors. We note that the first and third factors together explain less than 1% of the total variation, but the second factor explains more than 90% of the total variation in the equally weighted option return. Two additional option index returns, "EW short-term" and "EW long-term," are also used in the regression. The results (reported in Table 6) are qualitatively similar. Finally, as shown in Table 6, the regression results based on the rotated factors are similar to those based on the original factors. Taken together, these results indicate that the equally weighted option index return is strongly associated with the second factor, and is essentially uncorrelated with the other two factors.

We also examine whether a factor proxy is good by looking at its correlation with the factor itself. Table 7 presents a correlation matrix for each of the three factors (both the original and the rotated) and their proxies. The proxy for the first factor is the S&P 500 index return. The proxy for the second factor includes three measures of the change in Black-Scholes implied volatility (e.g., all-options-, short-term-options-, and long-term-options-based volatility), and three option index returns (e.g., EW, EW short-term and EW long-term). We begin our discussion with the original factors.

Table 6 Regression analysis of equally weighted option return indexes

Return series	Original factors				Rotated factors				Adj. R^2 (%)
	β_0	β_1	β_2	β_3	β_0	β_1	β_2	β_3	
Panel A: Regression results									
EW	0.01	0.01	2.80	−0.03	0.01	−0.07	2.09	0.10	90.60
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.03)	(0.02)	(0.00)	
EW short-term	−0.04	−0.05	3.79	−0.29	−0.04	−0.15	2.86	0.09	65.22
	(0.07)	(0.07)	(0.07)	(0.07)	(0.07)	(0.09)	(0.05)	(0.01)	
EW long-term	0.07	0.08	1.81	0.22	0.07	0.00	1.32	0.12	73.44
	(0.03)	(0.03)	(0.03)	(0.03)	(0.04)	(0.03)	(0.02)	(0.01)	

Return series	Original factors loadings			Rotated factors loadings			Total (%)
	F_1 (%)	F_2 (%)	F_3 (%)	F_1 (%)	F_2 (%)	F_3 (%)	
Panel B: Proportion of the total variance explained by each factor							
EW	0.00	90.59	0.01	0.03	87.83	2.74	90.60
EW short-term	0.01	64.82	0.39	0.06	64.40	0.76	65.22
EW long-term	0.14	72.24	1.06	0.00	66.47	6.97	73.44

This table reports results from the following regression:

$$R_t = \beta_0 + \beta_1 \hat{f}_{1t} + \beta_2 \hat{f}_{2t} + \beta_3 \hat{f}_{3t} + \epsilon_t,$$

where R_t is an equally-weighted option-index return, and $\hat{f}_{1t}$, $\hat{f}_{2t}$ and $\hat{f}_{3t}$ denote estimates of the three common factors. Three equally weighted option return indexes are used as the dependent variable separately. They are equally weighted "market" (EW), equally weighted short-term (EW short-term), and equally weighted long-term (EW long-term) option indexes, respectively

The sample period extends from June 1, 1988 through May 31, 1994 for a total of 1,506 daily observations. Panel A reports coefficient estimates and corresponding standard errors. The standard errors (in parentheses) are White's (1980) heteroskedasticity consistent estimator

Panel B reports proportions of the total variance explained by each of the common factors

First, factor 1 has a large positive correlation (0.92) with the S&P 500 index return, and a large negative correlation (−0.56) with all-options-based volatility change. We note that the correlation between the S&P 500 index return and all-options-based volatility change is also −0.56. Thus, interpreting the first factor as the underlying asset return is consistent with the observed correlation pattern. Second, the correlation between factor 2 and the underlying asset is close to zero. However, the second factor has a large, positive correlation with the volatility factor. This result supports the view that the second factor can be interpreted as the change in volatility. Among the three volatility measures, the long-term volatility measure is weakly associated with the third factor. Third, the equally weighted option index is uncorrelated with the first factor, but strongly correlated with the second factor. In fact, the option index EW and the second factor have a correlation of 0.95. For both short- and long-term option indexes, correlations with the second factor are also high. This confirms that the equally weighted option index is a good proxy for the second factor. Finally, the equally weighted long-term option index is weakly correlated with the third factor. It shows a correlation of 0.10 (and a correlation of 0.26 with the rotated third factor). This suggests that the option index can be used as a proxy for the third factor. The results based on rotated factors yield similar patterns.

Table 7 Correlations of changes in implied volatility and factors

	S&P 500 index	BS implied volatility			Original factors			Rotated factors		
		All	Short	Long	F_1	F_2	F_3	F_1	F_2	F_3
S&P 500 index		−0.56	−0.51	−0.44	0.92	−0.02	−0.12	0.93	0.00	0.00
BS Implied volatility										
All			0.71	0.84	−0.56	0.61	−0.03	−0.56	0.60	0.00
Short				0.45	−0.52	0.52	−0.01	−0.53	0.51	0.00
Long					−0.40	0.46	0.06	−0.42	0.44	0.09
Original factors										
Factor 1								0.99	0.00	0.13
Factor 2								−0.02	0.98	0.18
Factor 3								−0.13	−0.18	0.97
Option indexes										
EW	−0.01	0.55	0.56	0.38	0.01	0.95	−0.01	−0.01	0.94	0.16
EW short-term	−0.01	0.42	0.55	0.24	−0.01	0.81	−0.06	−0.02	0.80	0.09
EW long-term	0.00	0.58	0.32	0.52	0.03	0.85	0.10	0.00	0.82	0.26

Reported below is the correlation matrix among changes in the Black-Scholes implied volatility, the original factors, the rotated factors, and the option return indexes. For each day, the Black-Scholes implied volatility is obtained by minimizing sum of squared pricing errors. The results under “All” are obtained by using all call and put options on a given day. Those under “Short” and “Long” are obtained by using all call/put options with days to expiration less than 45 days, or more than 180 days, respectively

The estimates of the original and rotated factors are from the factor analysis. “EW”, “EW short-term” and “EW long-term”, respectively, represent equally weighted option return index, equally weighted short-term option return index, and equally weighted long-term option return index

The sample period extends from June 1, 1988 through May 31, 1994 for a total of 1,506 daily observations

4 Regression analysis

The preceding section has demonstrated that the first, second, and third factors explain 87%, 4%, and 2% of the total variation in option returns. Further, we have found that the underlying S&P 500 index return is a good proxy for the first factor, and that both the equally weighted option index and the volatility are good proxies for the second factor. From a practical perspective, the true factors are not directly observable, and one is bounded to use proxies of factors and to explain option returns. We ask what fraction of the variation in option returns can be explained by these proxies. The answer to this question guides us on how to choose, among many observable proxies, the right set of observable variables for our option modeling framework. In this section, we use a regression analysis to study the association between option returns and proxies of factors. Specifically, we consider the one-factor, two-factor, and three-factor models.

4.1 The one-factor model

The one-factor model we investigate is specified as:

$$R_{it} = \alpha_{i0} + \alpha_{i1} R_t^{\text{S\&P 500}} + \epsilon_{it}, \tag{12}$$

where R_{it} is one of the following option returns at t: CSTOTM, CSTATM, CSTITM, CLTOTM, CLTATM, CLTITM, PSTOTM, PSTATM, PSTITM, PLTOTM, PLTATM, or PLTITM. $R_t^{\text{S\&P 500}}$ is the return on the S&P 500 index. This specification is consistent with the discretized version of the one-factor diffusion option pricing model. To appreciate this point, let S_t denote the level of the underlying S&P 500 index at t, and τ and K be the time-to-expiration and strike price of a European call option, and $C_t = C(S_t, K, \tau)$ be the value of the call option. Assume that the index level evolves according to the one-dimensional diffusion process:

$$dS_t = \mu_s(\cdot)\, S_t dt + \sigma_s(S_t, t)\, S_t dz_s,$$

where μ_s and σ_s are the drift and volatility functions, respectively, and z_s is a standard Brownian motion under a real probability measure.

Using Ito's lemma and the partial differential equation (PDE) of the pricing function C, we have:

$$\frac{dC_t}{C_t} = r_t(1 - \Omega_s^c)dt + \Omega_s^c \frac{dS_t}{S_t},$$

where $\Omega_s^c = \frac{S}{C}\frac{\partial C}{\partial S}$ denotes the elasticity of the call option with respect to S, and r_t is the risk-free rate. Rewriting the above equation in terms of logarithmic return yields:

$$d(\ln C_t) = \left(r_t + \frac{1}{2}\sigma_s^2 \Omega_s^c\right)\left(1 - \Omega_s^c\right) dt + \Omega_s^c\, d(\ln S_t). \tag{13}$$

The regression model given in Eq. 12 can be justified under two conditions: (1) The first term on the RHS of Eq. 13, namely, the time-decay component during dt, is constant; and (2) the variation in Ω_s^c is small. The dependent variables in Eq. 12 are daily option returns with a constant moneyness and maturity. Given that dt is small, the time-decay component is close to zero, and the first condition is expected to hold. The second condition requires the option elasticity to be a function of the time-to-expiration and the moneyness (or the ratio of the underlying price S to the strike K) only, and not sensitive to the interest rate and the underlying return volatility. Elasticity Ω_s^c is a function of τ if the underlying process is stationary. One sufficient condition for Ω_s^c to depend on S and K only through the ratio S/K is that the price function of a call (or put) option is homogeneous of degree one in S and K.[9] It is known that the Black-Scholes option pricing model has such a homogeneity property. Merton (1973) and Ingersoll (1987) provide sufficient conditions for the option price to be first-degree homogeneous in S and K. We assume that the interest rate is constant and the variation in Ω_s^c due to r_t is small. Daily variations in Ω_s^c due to volatility are expected to be small. However, there could be large variations over a long horizon. Nonetheless, results from our regressions are still useful, qualitatively.

[9] From the homogeneity property, $C(aS, aK) = a\ C(S, K)\ \forall\ a > 0$, we have $\Omega_s^c(aS, aK) = \frac{aS}{C(aS,aK)}\frac{\partial C(aS,aK)}{\partial(aS)} = \Omega_s^c(S, K), \forall\, a > 0$. Setting $a = 1/K$ leads to $\Omega_s^c(S, K) = \Omega_s^c(S/K, 1)$.

In Table 8 we report estimated regression coefficients and corresponding standard errors, which are White's (1980) heteroskedasticity consistent estimator. The estimation is done for each of the 12 option returns. The estimated constant is small and insignificant from zero. This indicates that time decay over one day is indeed small. However, there is a strong positive (negative) association between a call (put) option return and S&P 500 index return. All 12 estimated coefficients of α_1 are significant at a 5% level. For a fixed maturity, the coefficient of α_1 monotonically decreases from OTM to ITM call options. Between a pair of short- and long-term call options (e.g., CSTOTM and CLTOTM), the coefficient for short-term options is larger. These conclusions also hold for put options, except that the coefficient of α_1 is negative. Thus, returns on short-term and out-of-the money options are far more sensitive to the underlying asset than are those on the long-term and in-the-money options.

Although all option returns are significantly related to the underlying asset return, the explanatory power varies substantially across option's moneyness or maturity. Overall, the average adjusted R^2 is 74.7% for the 12 regressions. Across moneyness categories, the average adjusted R^2 is 63.4% for OTM options, 79.3% for ATM options, and 81.3% for ITM options.[10] Clearly, the underlying asset return explains a significant percentage of the variation in the option return. This is particularly true for the ITM options. Yet, about 20–40% of the variation in option returns cannot be attributed to changes in the underlying value. Therefore, it is necessary to include additional factors to explain variations in option returns that are not otherwise explained by the S&P 500 index return.

4.2 Multifactor regressions based on option return indexes

Since the one-factor model does not satisfactorily explain variations in option returns, we examine multifactor regression models. Our objective is to investigate incremental explanatory power, over the underlying S&P 500 index, that the second and the third factors offer.

We begin with a two-factor regression model, with the second factor being the return on the equally weighted option index. The estimated regression model is:

$$R_{it} = \beta_{i0} + \beta_{i1} R_t^{\text{S\&P 500}} + \beta_{i2} R_t^{EW} + \epsilon_{it}, \tag{14}$$

where R_{it} is one of the 12 option returns, $R_t^{\text{S\&P 500}}$ is the return on the S&P 500 index, and R_t^{EW} is the return on the equally weighted option index.

Similar to the one-factor model given in Eqs. 12 and 14 can be considered as a discretized version of a two-factor diffusion option pricing model. To see this point, let S_t denote the underlying S&P 500 index and y_t the value of the second state variable at t. Assume that the dynamics of S_t and y_t are given by:

[10] Recall that the rotated first factor explains 85.17% of the total variation in option returns. One reason why the R^2 from the principal component analysis is higher than the regression R^2 is that we obtain the principal component solution by maximizing the variance explained by factors.

Table 8 Regression analysis of S&P 500 index option returns: One-, two- and three-factor models with S&P 500 index return and option index return as factors

Return series	100×		Adj. R^2	100×			Adj. R^2	100×				Adj. R^2
	α_0	α_1		β_0	β_1	β_2		γ_0	γ_1	γ_2	γ_3	
CSTOTM	0.07	29.06	61.8	0.01	29.26	3.82	76.2	0.12	29.26	2.35	0.57	77.0
	(0.47)	(0.58)		(0.37)	(0.46)	(0.12)		(0.36)	(0.46)	(0.08)	(0.18)	
CSTATM	0.33	23.85	80.0	0.31	23.92	1.32	83.3	0.30	23.92	0.61	0.82	83.3
	(0.24)	(0.31)		(0.22)	(0.28)	(0.08)		(0.22)	(0.28)	(0.05)	(0.11)	
CSTITM	−0.01	11.35	80.6	−0.01	11.36	0.28	81.2	−0.01	11.36	0.17	0.05	81.2
	(0.11)	(0.14)		(0.11)	(0.14)	(0.04)		(0.11)	(0.14)	(0.03)	(0.06)	
CLTOTM	0.22	12.48	60.7	0.20	12.53	1.19	68.0	0.03	12.53	−0.08	2.68	79.4
	(0.20)	(0.26)		(0.18)	(0.23)	(0.06)		(0.15)	(0.18)	(0.04)	(0.08)	
CLTATM	0.16	8.74	76.4	0.15	8.76	0.35	78.0	0.10	8.75	−0.04	0.83	80.9
	(0.10)	(0.12)		(0.10)	(0.12)	(0.03)		(0.09)	(0.11)	(0.02)	(0.05)	
CLTITM	0.07	5.90	78.1	0.07	5.90	0.14	78.6	0.04	5.90	−0.04	0.39	80.2
	(0.06)	(0.08)		(0.06)	(0.08)	(0.02)		(0.06)	(0.08)	(0.01)	(0.03)	
PSTOTM	−0.27	−24.25	67.1	−0.31	−24.13	2.38	75.8	−0.15	−24.12	1.80	−0.68	78.5
	(0.35)	(0.43)		(0.30)	(0.38)	(0.10)		(0.28)	(0.35)	(0.07)	(0.14)	
PSTATM	−0.29	−25.26	80.7	−0.31	−25.21	1.06	82.6	−0.23	−25.20	0.86	−0.46	83.4
	(0.26)	(0.32)		(0.24)	(0.30)	(0.08)		(0.24)	(0.30)	(0.05)	(0.12)	
PSTITM	−0.05	−15.23	85.0	−0.06	−15.22	0.17	85.1	−0.03	−15.22	0.21	−0.29	85.5
	(0.13)	(0.17)		(0.14)	(0.16)	(0.04)		(0.13)	(0.16)	(0.03)	(0.06)	
PLTOTM	−0.02	−9.86	64.1	−0.03	−9.83	0.68	68.2	−0.10	−9.83	0.05	1.22	71.7
	(0.15)	(0.20)		(0.14)	(0.18)	(0.05)		(0.13)	(0.16)	(0.03)	(0.07)	
PLTATM	0.01	−10.07	79.9	0.00	−10.05	0.40	81.6	−0.02	−10.05	0.07	0.60	82.4
	(0.10)	(0.13)		(0.10)	(0.12)	(0.03)		(0.09)	(0.12)	(0.02)	(0.05)	
PLTITM	−0.01	−7.31	81.5	−0.02	−7.30	0.19	82.3	−0.03	−7.30	0.04	0.27	82.5
	(0.07)	(0.09)		(0.07)	(0.09)	(0.02)		(0.07)	(0.08)	(0.02)	(0.04)	

Table 8 continued

This table reports results from the following three regressions:

Regression 1: $R_{it} = \alpha_{i0} + \alpha_{i1}\, R_t^{\text{S\&P 500}} + \epsilon_{it},$

Regression 2: $R_{it} = \beta_{i0} + \beta_{i1}\, R_t^{\text{S\&P 500}} + \beta_{i2}\, R_t^{\text{EW}} + \epsilon_{it},$

Regression 3: $R_{it} = \gamma_{i0} + \gamma_{i1}\, R_t^{\text{S\&P 500}} + \gamma_{i2}\, R_t^{\text{EWshort-term}} + \gamma_{i3}\, R_t^{\text{EWlong-term}} + \epsilon_{it}, \quad t = 1, \ldots, T,$

where R_{it} is one of the following option return series: (1) CSTOTM (short-term out-of-the money calls), (2) CSTATM (short-term at-the-money calls), (3) CSTITM (short-term in-the-money calls), (4) CLTOTM (long-term out-of-the money calls), (5) CLTATM (long-term at-the-money calls), (6) CLTITM (long-term in-the-money calls), (7) PSTOTM (short-term out-of-the money puts), (8) PSTATM (short-term at-the-money puts), (9) PSTITM (short-term in-the-money puts), (10) PLTOTM (long-term out-of-the money puts), (11) PLTATM (long-term at-the-money puts), and (12) PLTITM (long-term in-the-money puts). $R_t^{\text{S\&P 500}}$ is the S&P 500 index return. R_t^{EW}, $R_t^{\text{EWshort-term}}$, and $R_t^{\text{EWlong-term}}$, respectively, represent returns on equally weighted "market", equally weighted short-term, and equally weighted long-term option indexes. Return is defined as logarithmic price change

The sample period extends from June 1, 1988 through May 31, 1994, for a total of 1,506 daily observations. The standard errors (reported in parentheses) are White's (1980) heteroskedasticity consistent estimator

$$dS_t = \mu_s(\cdot)\, S_t dt + \sigma_s(S_t, t)\, S_t dz_s,$$
$$dy_t = \mu_y(\cdot)\, y_t dt + \sigma_y(y_t, t)\, y_t dz_y,$$

where z_s and z_y are two standard Brownian motions with a correlation coefficient of ρ_{sy}. Further assume that the value of a call option written on the S&P 500 index is given by $C = C(S_t, y_t, K, \tau)$. Using Ito's lemma and the PDE of the option price, we have:

$$\frac{dC_t}{C_t} = [r_t(1 - \Omega_s^c) - (\mu_y - \sigma_y\lambda_y)\Omega_y^c]dt + \Omega_s^c \frac{dS_t}{S_t} + \Omega_y^c \frac{dy_t}{y_t}, \tag{15}$$

where $\Omega_y^c = \frac{y}{C}\frac{\partial C}{\partial y}$ is the elasticity of the option price with respect to y. Parameter λ_y represents the market price of risk associated with the second factor. Note that if the factor represented by y is a traded asset, then $(\mu_y - \sigma_y\lambda_y) = r_t$. Rewriting Eq. 15, we have:

$$d(\ln C_t) = \left[\left(r_t - \frac{\sigma_c^2}{2}\right) - \Omega_s^c\left(r_t - \frac{\sigma_s^2}{2}\right) - \Omega_y^c\left(\mu_y - \sigma_y\lambda_y - \frac{\sigma_y^2}{2}\right)\right]dt$$
$$+ \Omega_s^c d(\ln S_t) + \Omega_y^c d(\ln y_t), \tag{16}$$

where $\sigma_c^2 = (\Omega_s^c)^2\sigma_s^2 + (\Omega_y^c)^2\sigma_y^2 + 2\Omega_s^c\Omega_y^c\sigma_s\sigma_y\rho_{sy}$. Equation 16 provides a basis for the regression model in Eq. 14.

Table 8 reports estimation results of the two-factor model for each of the 12 option returns. The estimated coefficients for the S&P 500 index return are close to those from the one-factor model; they are positive for calls and negative for puts. More important, the coefficients of the equally weighted option index are positive and significant at 5% for all call and put option returns. When we compare estimates of β_1 and β_2, we find that the estimated β_1 is much larger than the estimated β_2. This is an indication that option returns are more sensitive to the underlying return than to the second factor. The estimated coefficient for the equally weighted option index monotonically decreases from the OTM to the ITM options. Thus, the OTM options bear a stronger relation to the second factor than to the ITM options. For example, for a point change in the EW, the corresponding changes in CSTOTM, CSTATM, and CSTITM are 3.82, 1.32, and 0.28 points, respectively. Further, the regression results show that including the second factor increases the adjusted R^2 in general. Yet, the most noticeable increase in the R^2 is for the OTM options (e.g., for CSTOTM, CLTOTM, PSTOTM, and PLTOTM). For example, the adjusted R^2 is 61.8% for the one-factor model, and 76.2% for the two-factor model, for CSTOTM. Therefore, the second factor is more important in explaining variations in the OTM option returns than in the ATM and ITM option returns.

Finally, we estimate a three-factor regression model that includes the underlying asset and the two equally weighted option indexes as independent variables. The model is specified as:

$$R_{it} = \gamma_{i0} + \gamma_{i1}R_t^{\text{S\&P 500}} + \gamma_{i2}R_t^{\text{EWshort-term}} + \gamma_{i3}R_t^{\text{EWlong-term}} + \epsilon_{it}, \tag{17}$$

where $R_t^{\text{EWshort-term}}$ and $R_t^{\text{EWlong-term}}$ denote equally weighted short- and long-term option return indexes, respectively. The regression results are displayed in Table 8. For coefficient estimates of γ_0 and γ_1, they are very close to those from the two-factor model. The coefficient estimates of γ_2 are positive and significant at 5%, but the estimates of γ_3 are only weakly significant (or insignificant) for each of short-term option returns.

For long-term option returns, the opposite is true. Replacing the equally weighted option index by two maturity-specific option indexes does not lead to a significant increase in the adjusted R^2, except for long-term OTM options (CLTOTM and PLTOTM). Thus, the overall contribution of the proxy of the third factor toward the variation in option returns is not significant.

4.3 Multifactor regressions based on implied volatility

In the preceding regression analysis, we used the equally weighted option index as a proxy for the second factor. In this subsection, we use an alternative proxy, the option implied volatility, for the second factor in the regression. The estimated two- and three-factor models are the same as those in Eqs. 14 and 17, except that we replace the option return index with the logarithmic change in the implied volatility. These models are:

$$R_{it} = \beta_{i0} + \beta_{i1} R_t^{\text{S\&P 500}} + \beta_{i2} log\left(\frac{\sigma_t^{\text{All}}}{\sigma_{t-1}^{\text{All}}}\right) + \epsilon_{it},$$

and

$$R_{it} = \gamma_{i0} + \gamma_{i1} R_t^{\text{S\&P 500}} + \gamma_{i2} log\left(\frac{\sigma_t^{\text{Short-term}}}{\sigma_{t-1}^{\text{Short-term}}}\right) + \gamma_{i3} log\left(\frac{\sigma_t^{\text{Long-term}}}{\sigma_{t-1}^{\text{Long-term}}}\right) + \epsilon_{it},$$

where σ_t^{All} is the volatility on day t implied by the Black-Scholes model using all call and put options, $\sigma_t^{\text{short-term}}$ is the volatility implied by the BS model using all short-term call/put options (with days-to-expiration less than 45 days), and $\sigma_t^{\text{Long-term}}$ is the volatility implied by the BS model using long-term call/put options (with days-to-expiration more than 180 days).

Table 9 presents the estimation results. The estimated coefficients for the S&P 500 return are similar to those from the two-factor model reported in Table 8. The coefficients of the change in the implied volatility (β_2) are positive and significant at 5% for all option returns except for CSTITM and CLTITM, indicating that the larger the change in the volatility, the higher the option return. The sensitivity of option returns to the change in volatility is largest for the OTM options, followed by the ATM options, and then the ITM options. This result holds true for both short- and long-term options, and for both calls and puts. Between alternative specifications of the two-factor models, the adjusted R^2 is generally higher for the model with the option index as the proxy for the second factor (see Table 8). Taking short-term call option returns as an

Table 9 Regression analysis of S&P 500 index option returns: Two- and three-factor models with S&P 500 index return and implied volatility as factors

Return series	100×			Adj. R^2	100×				Adj. R^2
	β_0	β_1	β_2		γ_0	γ_1	γ_2	γ_3	
CSTOTM	0.07	32.72	2.41	63.9	0.00	33.91	0.86	1.44	65.4
	(0.45)	(0.69)	(0.25)		(0.45)	(0.68)	(0.09)	(0.24)	
CSTATM	0.31	25.26	0.93	80.6	0.30	25.60	0.32	0.46	80.9
	(0.24)	(0.36)	(0.09)		(0.24)	(0.36)	(0.05)	(0.13)	
CSTITM	−0.01	11.41	0.04	80.6	−0.01	11.44	0.03	0.01	80.6
	(0.11)	(0.17)	(0.06)		(0.11)	(0.17)	(0.02)	(0.06)	
CLTOTM	0.19	14.30	1.20	63.5	0.19	14.19	0.10	1.12	64.0
	(0.20)	(0.30)	(0.11)		(0.20)	(0.30)	(0.04)	(0.10)	
CLTATM	0.15	9.28	0.36	77.0	0.16	9.23	−0.06	0.62	78.4
	(0.10)	(0.14)	(0.06)		(0.10)	(0.14)	(0.02)	(0.05)	
CLTITM	0.07	5.94	0.03	78.1	0.07	6.00	−0.05	0.26	78.9
	(0.06)	(0.09)	(0.04)		(0.06)	(0.09)	(0.02)	(0.03)	
PSTOTM	−0.33	−20.08	2.75	71.4	−0.38	−19.50	1.36	−0.19	74.5
	(0.32)	(0.50)	(0.18)		(0.31)	(0.46)	(0.06)	(0.17)	
PSTATM	−0.32	−23.22	1.34	81.8	−0.36	−22.58	0.87	−0.42	83.9
	(0.24)	(0.37)	(0.14)		(0.23)	(0.35)	(0.05)	(0.12)	
PSTITM	−0.06	−14.80	0.28	85.1	−0.07	−15.00	0.11	−0.13	85.1
	(0.13)	(0.19)	(0.07)		(0.13)	(0.20)	(0.03)	(0.07)	
PLTOTM	−0.06	−7.23	1.74	74.1	−0.05	−7.81	0.32	0.74	70.0
	(0.13)	(0.20)	(0.07)		(0.14)	(0.21)	(0.03)	(0.08)	
PLTATM	−0.01	−8.41	1.10	86.6	−0.01	−8.80	0.20	0.47	82.6
	(0.09)	(0.14)	(0.05)		(0.09)	(0.14)	(0.02)	(0.05)	
PLTITM	−0.02	−6.53	0.52	83.5	−0.03	−6.75	0.08	0.21	82.5
	(0.06)	(0.10)	(0.03)		(0.07)	(0.10)	(0.02)	(0.04)	

This table reports results from the following two regressions:

$$\text{Regression 1:}\quad R_{it} = \beta_{i0} + \beta_{i1}\, R_t^{\text{S\&P 500}} + \beta_{i2}\, \log\left(\frac{\sigma_t^{\text{All}}}{\sigma_{t-1}^{\text{All}}}\right) + \epsilon_{it},$$

$$\text{Regression 2:}\quad R_{it} = \gamma_{i0} + \gamma_{i1}\, R_t^{\text{S\&P 500}} + \gamma_{i2}\, \log\left(\frac{\sigma_t^{\text{short-term}}}{\sigma_{t-1}^{\text{short-term}}}\right) + \gamma_{i3}\, \log\left(\frac{\sigma_t^{\text{long-term}}}{\sigma_{t-1}^{\text{long-term}}}\right) + \epsilon_{it},$$

$$t = 1, \ldots, T,$$

where R_{it} is one of the following option return series: (1) CSTOTM (short-term out-of-the money calls), (2) CSTATM (short-term at-the-money calls), (3) CSTITM (short-term in-the-money calls), (4) CLTOTM (long-term out-of-the money calls), (5) CLTATM (long-term at-the-money calls), (6) CLTITM (long-term in-the-money calls), (7) PSTOTM (short-term out-of-the money puts), (8) PSTATM (short-term at-the-money puts), (9) PSTITM (short-term in-the-money puts), (10) PLTOTM (long-term out-of-the money puts), (11) PLTATM (long-term at-the-money puts), and (12) PLTITM (long-term in-the-money puts)

$R_t^{\text{S\&P 500}}$ is the S&P 500 index return. Return is defined as logarithmic price change. σ_t^{All} is the volatility on day t implied by the Black-Scholes (BS) model using all call and put options, $\sigma_t^{\text{short-term}}$ is the volatility on day t implied by the BS model using all short-term call/put options (with days-to-expiration less than 45 days), and $\sigma_t^{\text{long-term}}$ is the volatility on day t implied by the BS model using long-term call/put options (with days-to-expiration more than 180 days)

The sample period extends from June 1, 1988 through May 31, 1994, for a total of 1,506 daily observations

The standard errors (reported in parentheses) are White's (1980) heteroskedasticity consistent estimator example, the adjusted R^2s are 76.2%, 83.3%, and 81.2% when the EW is used as the second factor. These statistics are 63.9%, 80.6%, and 80.6%, respectively, when the implied volatility is used as a proxy for the second factor. It appears that the equally weighted option index is a better proxy for the second factor than is the option-implied volatility.

Next we examine a three-factor model and see if maturity-specific volatilities can explain option returns better than can a single-volatility factor. Since the short- and long-term implied volatilities are different in their levels and variabilities (e.g., the variation of volatility), using maturity-specific volatilities might improve estimation results. Table 9 presents results for the three-factor model. Among the 12 regressions, nine pairs of coefficients for short- and long-term volatility are significant at 5%. Further, for a given short-term (long-term) option, the t-statistic associated with the short-term (long-term) volatility is more significant than that associated with the long-term (short-term) volatility. Thus, the short-term volatility plays a dominant role in explaining short-term option returns. However, the long-term volatility also contributes significantly to the variation in short-term option returns, and vice versa.[11] Finally, the adjusted R^2s from the three-factor model are close to those of the two-factor model with a single-volatility factor. This result suggests that performance difference between the two-factor model (with one volatility factor) and the three-factor model (with two volatility factors) is small. The two-factor model is sufficient to explain most of the variation in option returns.

5 Mean–variance spanning tests

In the preceding analysis we investigated the common factors affecting option returns and the characteristics of these factors. Another approach to examining this issue is to use mean–variance spanning tests within the framework of Hansen and Jagannathan (1991). While factor analysis purports to identify the factors and the option return-generating process, spanning tests examine whether a certain set of (basis) assets can span or, equivalently, replicate payoffs of options. Motivated by the equivalence between the Hansen and Jagannathan tests and the Huberman and Kandel (1987) regression-based tests for spanning (see Bekaert and Urias (1996); Ferson (1995)), we reformalize our tests within the framework of Huberman and Kandel. The likelihood ratio test is then used to test the Huberman and Kandel spanning restrictions (HK-restrictions).

5.1 Restrictions of the test

Let R_{t+1} represent an $(n \times 1)$-vector of asset returns at time $t+1$, m_{t+1} be the stochastic discount factor or pricing kernel, ℓ be an $(n \times 1)$ unit vector, and $\mathcal{F}_t$ denote the information available at time t. By a standard result, the Euler equation gives a general pricing restriction: $E[m_{t+1}R_{t+1}|\mathcal{F}_t] = \ell_n$. Applying the law of iterated expectations yields:

[11] The estimated coefficient of γ_2 is negative, but small in magnitude, when we use CLTATM and CLTITM as dependent variables. This can be partially explained by the correlation between the short- and long-term volatility factors (see Table 7).

$$E[m_{t+1}R_{t+1}] = \ell_n, \tag{18}$$

which forms the basis of mean–variance spanning tests for option returns. Equation 18 places restrictions on stochastic discount factors that should be consistent with the given set of asset returns R_{t+1}. Hansen and Jagannathan (1991) show that among these discount factors, the linear projection of m_{t+1} onto R_{t+1} and a constant has the minimum variance. Thus we consider: $m_{t+1} = E[m_{t+1}] + [R_{t+1} - E(R_{t+1})]'\bar{\beta}$, where $\bar{\beta}$ is an $n \times 1$ vector of parameters to be determined.

Following Bekaert and Urias (1996) and DeSantis (1995), we consider partitions:

$$R_{t+1} = \begin{bmatrix} R^a_{t+1} \\ \\ R^b_{t+1} \end{bmatrix}, \quad \bar{\beta} = \begin{bmatrix} \bar{\beta}^a \\ \\ \bar{\beta}^b \end{bmatrix},$$

where R^a_{t+1} and $\bar{\beta}_a$ are $n_a \times 1$ vectors and R^b_{t+1} and $\bar{\beta}_b$ are $n_b \times 1$ vectors. For convenience, we refer to the assets with returns R^a_{t+1} as test assets and the assets with returns R^b_{t+1} as benchmark assets. If the pricing kernel m_{t+1} depends only on R^b_{t+1}, then benchmark assets span returns on both benchmark and test assets. To test whether returns R^b_{t+1} span R_{t+1}, we can examine the Hansen and Jagannathan pricing restrictions (HJ-restrictions):

$$\ell_n = E[(E[m_{t+1}] + [R_{t+1} - E(R_{t+1})]'\bar{\beta})R_{t+1}], \qquad \bar{\beta}^a = 0_{n_a}.$$

Since these restrictions hold for any given arbitrary value of $E[m_{t+1}]$, we can choose two different values for $E[m_{t+1}]$ and then test the HJ-restrictions.

Bekaert and Urias (1996) and Ferson (1995) show that the HJ-restrictions are equivalent to the HK-restrictions. Specifically, the HK-restrictions are given by:

$$\begin{aligned} R^a_t &= a + BR^b_t + \varepsilon_t \\ E[\varepsilon_t] = 0_{n_a}, &\qquad E[\varepsilon_{it}R^b_t] \; = \; 0_{n_b}, \quad i = 1, \ldots, n_a \qquad (19) \\ a = 0_{n_a}, &\qquad B\ell_{n_b} = \ell_{n_a}, \qquad (20) \end{aligned}$$

where a represents an $n_a \times 1$ vector of parameters, B is an $n_a \times n_b$ matrix of parameters, and ε_t is an $n_a \times 1$ vector of error term. The HK-restrictions imply that, if returns R^b_t span R^a_t, then, up to an orthogonal (zero-mean) error factor, a test asset can be replicated by a portfolio of the benchmark assets. Notice that this interpretation is also reflected in (15), in which an option return is spanned by the underlying asset and the equally weighted option index. Our subsequent spanning tests focus on the HK-restrictions.

5.2 Test statistics

Here, we describe how to use the generalized method of moments (GMM) (Hansen 1982) to test the HK-restrictions. Let $\beta = \text{vec}[(a, B)']$ represent the $n_a(n_b + 1) \times 1$ vector of parameters. The moment conditions corresponding to Eq. 19 are:

$$g_T(\beta) = \frac{1}{T}\sum_{t=1}^{T} g_t(\beta) \equiv \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} \varepsilon_t \\ \varepsilon_t \otimes R_t^b \end{bmatrix},$$

where T denotes the number of observations in each option-return time series and the symbol $\otimes$ represents the Kronecker product. The GMM estimator of β is given by:

$$\hat{\beta} = \text{argmin}_{\beta}\{J_T(\beta) \equiv g_T(\beta)' \, W_T \, g_T(\beta)\}.$$

W_T is a positive semi-definite matrix and satisfies $\text{plim} W_T = W_0$, where W_0 is a positive definite matrix. For any such W_T, the GMM estimator is a consistent estimator of β under some regularity conditions. Hansen (1982) shows that the (asymptotically) efficient GMM estimator is obtained by choosing the weighting matrix $W_0^* = \Omega_0^{-1}$, where $\Omega_0 = \sum_{\ell=-\infty}^{\infty} E\left[g_t(\beta)g_{t-\ell}(\beta)'\right]$. The asymptotic covariance matrix of the efficient GMM estimator is given by $\text{avar}(\hat{\beta}^*) = [G_0'\Omega_0^{-1}G_0]^{-1}$, where $G_0 = E\left[\partial g_t(\beta)/\partial\beta\right]$.

To conduct the spanning tests using the GMM, we substitute the HK-restrictions into moment conditions, estimate the restricted system, and then test the overidentified restrictions. This test is a likelihood ratio test. Since there are $n_a(n_b+1)$ moment conditions and $n_a(n_b-1)$ unknown parameters, the system is overidentified. As a result, the GMM estimator depends on the weighting matrix W_T, and we use an iteration over W_T to improve the small sample properties of the GMM estimator (see Ferson and Foerster 1994). A likelihood ratio statistic can be constructed to test the HK-restrictions. Let $\hat{\beta}_r$ denote the restricted estimator of β. Under the null hypothesis that the test assets can be spanned by the benchmark assets, we have:

$$TJ(\hat{\beta}_r) = T g_T(\hat{\beta}_r)'\Omega_T^{-1} g_T(\hat{\beta}_r) \overset{a}{\sim} \chi^2(2n_a).$$

5.3 Spanning results

In conducting the mean–variance spanning tests, we choose three sets of benchmark assets:

(1) $R_t^{\text{S\&P 500}}$ and R_t^{EW};
(2) $R_t^{\text{S\&P 500}}$, $R_t^{\text{EW short-term}}$, and $R_t^{\text{EW long-term}}$;
(3) $R_t^{\text{S\&P 500}}$, $R_t^{\text{EW OTM}}$, $R_t^{\text{EW ATM}}$, and $R_t^{\text{EW ITM}}$.

We first test if any set of benchmark assets can span any of the 12 option returns. Table 10 reports the likelihood ratio statistic and corresponding p-values for the mean–variance spanning test. Regardless of which one of the three sets of benchmark assets we use, we strongly reject mean–variance spanning for each of option returns (all p-values are less than 1%).[12] Put differently, a given option return, say short-term OTM call-option return, cannot be replicated (spanned) by the underlying S&P 500 index and the equally weighted option index, nor can it be replicated by the underlying, the short- and long-term option indexes. Further, using the underlying and three

[12] The Newey and West (1987) weighting matrix with lag 5 is used throughout our GMM tests.

Table 10 Mean–variance spanning tests using returns on S&P 500 index and option indexes as benchmark assets

Test asset	Benchmark assets: $R^{\text{S\&P 500}}$, R^{EW}	$R^{\text{S\&P 500}}$, $R^{\text{EWshort-term}}$, $R^{\text{EWlong-term}}$	$R^{\text{S\&P 500}}$, R^{EWOTM}, R^{EWATM}, R^{EWITM}
CSTOTM	108.79	171.15	12.16
	(<0.01)	(<0.01)	(<0.01)
CSTATM	91.18	111.09	16.45
	(<0.01)	(<0.01)	(<0.01)
CSTITM	57.89	71.95	13.67
	(<0.01)	(<0.01)	(<0.01)
CLTOTM	97.68	124.55	18.38
	(<0.01)	(<0.01)	(<0.01)
CLTATM	100.04	141.25	16.36
	(<0.01)	(<0.01)	(<0.01)
CLTITM	90.61	139.77	15.55
	(<0.01)	(<0.01)	(<0.01)
PSTOTM	102.91	160.93	14.16
	(<0.01)	(<0.01)	(<0.01)
PSTATM	99.63	110.45	16.98
	(<0.01)	(<0.01)	(<0.01)
PSTITM	88.46	110.41	13.60
	(<0.01)	(<0.01)	(<0.01)
PLTOTM	98.80	127.22	12.74
	(<0.01)	(<0.01)	(<0.01)
PLTATM	96.81	129.82	13.99
	(<0.01)	(<0.01)	(<0.01)
PLTITM	92.12	129.90	13.77
	(<0.01)	(<0.01)	(<0.01)

This table reports likelihood ratio statistics and p-values (in parentheses) for the mean–variance spanning tests. The null hypothesis is that returns on benchmark assets span the return on the test asset.
We estimate the following regression models and test the null hypothesis using Generalized Method of Moments (GMM):

Regression 1: $R_{it} = \beta_{i0} + \beta_{i1} R_t^{\text{S\&P 500}} + \beta_{i2} R_t^{\text{EW}} + \epsilon_{it}$,
$H_0 : \beta_0 = 0, \quad \beta_1 + \beta_2 = 1$

Regression 2: $R_{it} = \beta_{i0} + \beta_{i1} R_t^{\text{S\&P 500}} + \beta_{i2} R_t^{\text{EWshort-term}} + \beta_{i3} R_t^{\text{EWlong-term}} + \epsilon_{it}$,
$H_0 : \beta_0 = 0, \quad \beta_1 + \beta_2 + \beta_3 = 1$

Regression 3: $R_{it} = \beta_{i0} + \beta_{i1} R_t^{\text{S\&P 500}} + \beta_{i2} R_t^{\text{EWOTM}} + \beta_{i3} R_t^{\text{EWATM}} + \beta_{i4} R_t^{\text{EWITM}} + \epsilon_{it}$,
$H_0 : \beta_0 = 0, \quad \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$

where R_{it} 0is the return on a test asset. $R_t^{\text{S\&P 500}}$ is the S&P 500 index return, R_t^{EW}, $R_t^{\text{EWshort-term}}$, $R_t^{\text{EWlong-term}}$, $R_t^{\text{EW OTM}}$, $R_t^{\text{EW ATM}}$, and $R_t^{\text{EW ITM}}$, respectively, represent returns on equally weighted "market", equally weighted short-term, long-term, OTM, ATM, and ITM option indexes
The sample period extends from June 1, 1988 through May 31, 1994, for a total of 1,506 daily observations

moneyness-based option indexes (e.g., $R_t^{\text{EW OTM}}$, $R_t^{\text{EW ATM}}$, and $R_t^{\text{EW ITM}}$), we still cannot span option returns, although the rejection of the mean–variance spanning is the weakest in this case.

We then implement the spanning tests for a given set of test assets, and test whether benchmark assets can span a set of test assets jointly. We use the following three sets of test assets:

(1) all call-option returns (CSTOTM, CSTATM, CSTITM, CLTOTM, CLTATM and CLTITM);
(2) all put-option returns (PSTOTM, PSTATM, PSTITM, PLTOTM, PLTATM and PLTITM);
(3) all 12 call- and put-option returns.

Our likelihood ratio tests reject the null that a given set of benchmark assets span a set of test assets jointly. For all tests, the p-values are less than 1%. For the sake of brevity, detailed results are not reported here.

Finally, we test if returns of an out-of-the money (or in-the-money) option can be spanned by the underlying asset and an at-the-money option. For example, we see if $R_t^{\text{S\&P 500}}$ and CSTATM can span CSTOTM, and $R_t^{\text{S\&P 500}}$ and CLTATM can span CLTOTM. Intuitively, we can use any option contract, say short-term ITM option, as the benchmark asset to span returns of other options. We choose the ATM option as a benchmark asset because it is usually the most liquid and has low transaction costs, and it is more likely to be used as a hedging instrument. The results of the spanning tests are presented in Table 11. When we use the underlying asset and the ATM option to span the OTM-option returns (e.g., CSTOTM, CLTOTM, PSTOTM, or PLTOTM), we fail to reject the spanning for each of the OTM-option returns. Furthermore, among the four regressions, the estimated coefficients for the underlying S&P 500 index are insignificant in three cases, but the coefficients for the ATM option return are close to one. Between the underlying and the ATM option, the latter is more important in spanning the OTM-option returns.[13] In contrast, the likelihood ratio tests strongly reject the null that the underlying and the ATM option span returns of ITM options (e.g., CSTITM, CLTITM, PSTITM, or PLTITM). The regression coefficients for the underlying asset return are large (positive for calls and negative for puts) and far from one. The coefficients for the ATM option are small, although significantly different from zero. Taken together, the results from Tables 10 and 11 suggest that the underlying asset is not sufficient to hedge option contracts. Additional instruments, such as the ATM options, should be used to devise efficient hedging strategies.

6 Robustness of empirical results

Using the entire sample period data, we have concluded that almost all of the variation in option returns can be explained by three factors. Further, the mean–variance spanning tests have rejected the one-factor diffusion models and the hypothesis that the underlying asset and an equally weighted option index span options. However, we wish demonstrate that these conclusions still hold when alternative test designs and different sample periods are used. Below, we briefly report results from four controlled experiments.

[13] Cochrane and Saa-Requejo (2000) show that for a given option, other options with different strikes are better hedging instruments than is the underlying asset.

Table 11 Mean–variance spanning tests using returns on S&P 500 index and at-the-money options as benchmark assets

Test asset	Benchmark assets	LR-statistic (p-value)
CSTOTM	$R^{\text{S\&P 500}}$, CSTATM	1.32 (0.52)
CSTITM	$R^{\text{S\&P 500}}$, CSTATM	26.22 (0.00)
CLTOTM	$R^{\text{S\&P 500}}$, CLTATM	1.64 (0.44)
CLTITM	$R^{\text{S\&P 500}}$, CLTATM	9.52 (0.01)
PSTOTM	$R^{\text{S\&P 500}}$, PSTATM	1.89 (0.39)
PSTITM	$R^{\text{S\&P 500}}$, PSTATM	67.34 (0.00)
PLTOTM	$R^{\text{S\&P 500}}$, PLTATM	0.27 (0.87)
PLTITM	$R^{\text{S\&P 500}}$, PLTATM	48.78 (0.00)

This table reports likelihood ratio statistics and p-values (in parentheses) for the mean–variance spanning tests

The null hypothesis is that returns on benchmark assets span the return on the test asset

We estimate the following regression model and test the null hypothesis using Generalized Method of Moments (GMM):

$$R_{it} = \beta_{i0} + \beta_{i1} R_t^{\text{S\&P 500}} + \beta_{i2} R_t^{\text{ATM}} + \epsilon_{it},$$
$$H_0 : \beta_0 = 0, \quad \beta_1 + \beta_2 = 1,$$

where R_{it} is the return on a test asset. $R_t^{\text{S\&P 500}}$ is the S&P 500 index return, and R_t^{ATM} is the return on a chosen ATM option

The sample period extends from June 1, 1988 through May 31, 1994, for a total of 1,506 daily observations

We partition the sample into two subsamples, June 1988 to May 1991, and June 1991 to May 1994. For both subsamples, we re-implement each of the tests, and find that the results are similar to those for the full sample. For example, the first, second, and third factors explain, on average, 88%, 5%, and 2% of the variation in option returns during the first subsample, and 86%, 4%, and 2% during the second subsample.

Using moneyness-specific option indexes (e.g., $R^{\text{EW OTM}}$, $R^{\text{EW ATM}}$, and $R^{\text{EW ITM}}$) as proxies for factors, and repeating the regression analysis in Table 5, we find that the overall fitting results are similar to those reported in Table 5. In addition, we use alternative definitions of option returns in the factor analysis and regression analysis, such as the percentage returns (which are more skewed than the logarithmic return), the standardized returns, and the de-meaned returns. Again, the results are qualitatively similar to those using the logarithmic returns.

In Sect. 5.3 we use the underlying asset and the equally weighted option index (or an ATM option) as benchmark assets to span option returns. We also test the null hypothesis that the underlying asset and the risk-free bond span option returns. The results, again, strongly reject the null hypothesis.

Note that we use the Black-Scholes model implied volatility as a proxy for the second factor in factor analysis and regression analysis. Admittedly, the BS model is mis-specified and its implied volatility is not the best measurement of volatility. Extant studies have found that the BS implied volatility obtained from the ATM options is similar to that from other option models e.g., the stochastic volatility models. For this reason, we use the ATM option volatility implied by the Black-Scholes model to perform factor analysis and regression analysis. The results remain the same as those from all-options-based implied volatility. To check whether our results are robust with respect to which model we use to obtain the implied volatility, we implement the stochastic volatility (SV) model of Heston (1993), infer the implied volatility from this model, and then redo the regression analysis.[14] We find that, although the regression coefficients for the volatility factor are smaller, the adjusted R^2s and the overall explanatory power of factor models are similar to those reported in Table 9. Take the two-factor model as an example. Using the BS model implied volatility, the coefficient estimate β_2 for the second factor is 0.93 and the adjusted R^2 is 80.6% for CSTATM (see Table 9). With the SV model implied volatility, the coefficient estimate is 0.25 and the R^2 is 80.2%. To summarize, our empirical results are robust to different sample periods and test designs.

7 Conclusion

This paper has studied the common factors that affect option returns. Using prices of S&P 500 index options, we construct daily returns on the index option with constant moneyness and maturity. The results from factor analysis indicate that three factors can explain, on average, 93% of the total variation in option returns. The first and second factors represent the underlying security factor and the volatility factor, and account for 87% and 4% of the total variation in option returns, respectively. The third factor accounts for an average of 2% of the total variation in option returns. Therefore, an option pricing model with two-state variables is sufficient to describe almost all of the variation in option returns.

We have also examined what fraction of the variation in option returns can be explained by proxies for the factors. The underlying asset return is used as the first factor. We construct two alternative proxies for the second factor: the equally weighted option index and the option implied volatility. When proxied by the option index, the second factor provides significantly incremental explanatory power for option returns, especially for the out-of-the-money option returns. Our results also suggest that the equally weighted option index is a better proxy for the second factor than is the option implied volatility. Further, compared to two-factor models based on the underlying asset and a single volatility factor, three-factor regression models based on the underlying asset and two maturity-specific volatility factors (one from short-term and the other from long-term options) yield little improvement in explaining option returns.

The question of how many assets are sufficient to span the space of option returns interests both academics and practitioners, because it is the foundation of all hedging strategies. The mean–variance spanning tests, which we base on the framework of Hansen and Jagannathan (1991) and Huberman and Kandel (1987), show that the underlying asset and the option index cannot span returns on the OTM, ATM, or ITM options, either individually or jointly. The underlying asset and the ATM option successfully span the OTM option returns, but they fail to span the ITM option returns. These results highlight the importance of using one or more options with different characteristics as hedging instruments when hedging targets are options.

[14] Appendix C presents the option price formula under the SV model and parameter estimation procedure.

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Appendix A: Construction of the orthonormal matrix *G*

We derive the orthonormal transformation matrix G used in Sect. 3.4 to rotate the initial factors. The first column of G, g_1, is the OLS regression coefficients given by Eq. 7. The second column of G solves:

$$\min_g \ (R_2 - \hat{F}g)'(R_2 - \hat{F}g)$$
$$\text{s.t.} \ \ g'g_1 = 0$$

Introduce the Lagrangian multiplier $2\alpha_0$. The first-order conditions become:

$$\frac{\partial L}{\partial g} = 2\hat{F}'(R_2 - \hat{F}g) + 2\alpha_0 g_1 \ = \ 0, \tag{21}$$

$$\frac{\partial L}{\partial \alpha_0} = 2g'g_1 = 0. \tag{22}$$

Solving (21) for g yields:

$$g = (\hat{F}'\hat{F})^{-1}\hat{F}'R_2 - \alpha_0(\hat{F}'\hat{F})^{-1}g_1 = g_2^0 - \alpha_0(\hat{F}'\hat{F})^{-1}g_1, \tag{23}$$

where g_2^0 is defined in (9). Multiplying g_1' on both sides of Eq. 23, we have:

$$\alpha_0 = \frac{g_1'g_2^0}{g_1'(\hat{F}'\hat{F})^{-1}g_1}.$$

The third column of G solves:

$$\min_g \ (e^* - \hat{F}g)'(e^* - \hat{F}g)$$
$$\text{s.t.} \ \ g'g_1 = 0 \ \text{ and } \ g'g_2 = 0$$

Introduce the Lagrangian multipliers $2\alpha_1$ and $2\alpha_2$. The first-order conditions become:

$$\frac{\partial L}{\partial g} = 2\hat{F}'(e^* - \hat{F}g) + 2\alpha_1 g_1 + 2\alpha_2 g_2 \ = \ 0, \tag{24}$$

$$\frac{\partial L}{\partial \alpha_1} = 2g'g_1 = 0, \tag{25}$$

$$\frac{\partial L}{\partial \alpha_2} = 2g'g_2 = 0. \tag{26}$$

Solving (24) for g yields:

$$g = (\hat{F}'\hat{F})^{-1}\hat{F}'e^* - (\hat{F}'\hat{F})^{-1}(\alpha_1 g_1 + \alpha_2 g_2) = g_3^0 - (\hat{F}'\hat{F})^{-1}(\alpha_1 g_1 + \alpha_2 g_2), \quad (27)$$

where g_3^0 is defined in Eq. 11. Multiplying g_1' on both sides of Eq. 27 and using (25) yields:

$$0 = g_1' g = g_1' g_3^0 - g_1'(\hat{F}'\hat{F})^{-1}(\alpha_1 g_1 + \alpha_2 g_2).$$

Repeating the same procedure with g_2' leads to:

$$0 = g_2' g = g_2' g_3^0 - g_2'(\hat{F}'\hat{F})^{-1}(\alpha_1 g_1 + \alpha_2 g_2).$$

The above two equations define a system of linear equations of α_1 and α_2. One can verify that the solutions for α_1 and α_2 are given by Eq. 11. This completes the proof. □

Appendix B: Estimation of the implied volatility from the Black-Scholes model

Using a set of observed option prices (both calls and puts) on day t, we estimate the implied variance $V(t)(=\sigma_t^2)$ by minimizing the sum of squared pricing errors. Let N be the number of observed option prices, τ_n and K_n be the time-to-expiration and the strike price of the nth option, $\hat{O}_n(t, \tau_n, K_n)$ and $O_n(t, \tau_n, K_n; V(t))$ be respectively the observed and the Black-Scholes model price of the n-th option. We find $V(t)$ by solving:

$$SSE(t) \equiv \min_{V(t)} \sum_{n=1}^{N} \mid \hat{O}_n(t, \tau_n, K_n) - O_n(t, \tau_n, K_n; V(t)) \mid^2,$$

where $SSE(t)$ represents a goodness-of-fit statistic of day t's option prices by the Black-Scholes model. Repeat this procedure for each day of the sample to produce a time series of the BS model implied volatility.

Appendix C: Estimation of the implied volatility from the Stochastic Volatility Model

In the stochastic-volatility model of Heston (1993), the underlying price and its return variance $V(t)$ follow (under the equivalent martingale measure) the respective processes below:

$$\frac{dS(t)}{S(t)} = r\,dt + \sqrt{V(t)}\,dz_s(t),$$
$$dV(t) = [\theta_v - \kappa_v V(t)]\,dt + \sigma_v \sqrt{V(t)} dz_v(t),$$

where the structural parameter, κ_v, reflects the speed at which $V(t)$ approaches its long-run mean $\frac{\theta_v}{\kappa_v}$, σ_v is its variation coefficient, and z_s and z_v are two standard Brownian motions with a correlation of ρ. Heston (1993) shows that the European call-pricing formula is:

$$C(t,\tau) = S(t)\,\Pi_1(t,\tau) - K\,e^{-r\tau}\,\Pi_2(t,\tau), \tag{28}$$

where the probabilities, $\Pi_j\,(t,\tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi}}{i\phi}\,f_j(t,\tau;\phi)\right]d\phi$, for $j = 1, 2$, with the characteristic functions, f_1 and f_2, given by:

$$\begin{aligned} f_1 = exp\Bigg\{ & -\frac{\theta_v}{\sigma_v^2}\left[2\,\ln\left(1 - \frac{[\xi_v - \kappa_v + (1+i\phi)\rho\sigma_v]\,(1-e^{-\xi_v\tau})}{2\xi_v}\right)\right] \\ & -\frac{\theta_v}{\sigma_v^2}\,[\xi_v - \kappa_v + (1+i\phi)\rho\sigma_v)]\;\tau + i\phi r\tau + i\phi\ln[S(t)] \\ & + \frac{i\phi(i\phi+1)(1-e^{-\xi_v\tau})\,V(t)}{2\xi_v - [\xi_v - \kappa_v + (1+i\phi)\rho\sigma_v](1-e^{-\xi_v\tau})}\Bigg\}, \end{aligned}$$

$$\begin{aligned} f_2 = exp\Bigg\{ & -\frac{\theta_v}{\sigma_v^2}\left[2\,\ln\left(1 - \frac{[\xi_v^* - \kappa_v + i\phi\rho\sigma_v](1-e^{-\xi_v^*\tau})}{2\xi_v^*}\right)\right] \\ & -\frac{\theta_v}{\sigma_v^2}\left[\xi_v^* - \kappa_v + i\phi\rho\sigma_v\right]\tau + i\phi r\tau + i\phi\ln[S(t)] \\ & + \frac{i\phi(i\phi-1)(1-e^{-\xi_v^*\tau})\,V(t)}{2\xi_v^* - [\xi_v^* - \kappa_v + i\phi\rho\sigma_v](1-e^{-\xi_v^*\tau})}\Bigg\}. \end{aligned}$$

The put price is obtained by using the put-call parity.

Let $\Phi \equiv \{\kappa_v, \theta_v, \sigma_v, \rho\}$, and $\hat{O}_n(t,\tau_n,K_n)$ and $O_n(t,\tau_n,K_n;V(t),\Phi)$ be, respectively, the observed and the SV model price of the nth option at t. For a set of N observed option prices on day t, we choose values for Φ and $V(t)$ to minimize the sum of squared pricing errors:

$$SSE(t) \equiv \min_{V(t),\Phi}\sum_{n=1}^{N} \mid \hat{O}_n(t,\tau_n,K_n) - O_n(t,\tau_n,K_n;V(t),\Phi)\mid^2 .$$

Applying this procedure to each day, we obtain a time series for the spot volatility.

References

Bakshi, G., Cao, C., & Chen, Z. (1997). Empirical performance of alternative option pricing models. *Journal of Finance, 52*, 2003–2049.

Bakshi, G., Cao, C., & Chen, Z. (2000). Do call prices and the underlying stock always move in the same direction? *Review of Financial Studies, 13*, 549–584.

Bakshi, G., Kapadia, N., & Madan, D. (2003). Stock return characteristics, skew laws, and differential pricing of individual equity options. *Review of Financial Studies, 16*, 101–143.
Bates, D. (1991). The crash of 87: Was it expected? The evidence from options markets. *Journal of Finance, 46*, 1009–1044.
Bates, D. (1996). Jumps and stochastic volatility: Exchange rate processes implicit in deutschemark options. *Review of Financial Studies, 9*, 69–108.
Bates, D. (2000). Post-87 crash fears in S&P 500 futures options. *Journal of Econometrics, 94*, 181–238.
Bekaert, G., & Urias, M. (1996). Diversification, integration and emerging market closed-end funds. *Journal of Finance, 51*, 835–869.
Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy, 81*, 637–659.
Bondarenko, O. (2003). *Why are put options so expensive?* Working paper, University of Illinois at Chicago.
Buraschi, A., & Jackwerth, J. (2001). The price of a smile: Hedging and spanning in option markets. *Review of Financial Studies, 14*, 495–527.
Chernov, M., & Ghysels, E. (2000). A study towards a unified approach to the joint estimation of objective and risk-neutral measures for the purpose of option valuation. *Journal of Financial Economics, 56*, 407–458.
Cochrane, J., & Saa-Requejo, J. (2000). Beyond arbitrage: Good-deal asset price bounds in incomplete markets. *Journal of Political Economy, 108*, 79–119.
DeSantis, G. (1995). *Volatility bounds for stochastic discount factors: Tests and implications from international stock returns*. Working paper, University of Southern California.
Dumas, B., Fleming, J., & Whaley, R. (1998). Implied volatility function: Empirical tests. *Journal of Finance, 53*, 2059–2106.
Ferson, W. (1995). Theory and empirical testing of asset pricing models in finance. In R. Jarrow, V. Maksimovic, & W. Ziemba (Eds.), *Handbooks in Operations Research and Management Science* (pp. 145–200). Elsevier.
Ferson, W., & Foerster, S. (1994). Finite sample properties of generalized methods of moments tests of conditional asset pricing models. *Journal of Financial Economics, 36*, 29–56.
French, K., Schwert, G. W., & Stambaugh, R. (1987). Expected stock return and volatility. *Journal of Financial Economics, 19*, 3–30.
Grinblatt, M., & Titman, S. (1987). The relation between mean–variance efficiency and arbitrage pricing. *Journal of Business, 60*, 97–112.
Hansen, L. (1982). Large sample properties of generalized Method of moment estimators. *Econometrica, 50*, 1029–1054.
Hansen, L., & Jagannathan, R. (1991). Implications of security market data for models of dynamic economies. *Journal of Political Economy, 99*, 225–262.
Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies, 6*, 327–343.
Huberman, G., Kandel, S., & Stambaugh, R. (1987). Mimicking portfolios and exact arbitrage pricing. *Journal of Finance, 42*, 1–9.
Huberman, G., & Kandel, S. (1987). Mean–variance spanning. *Journal of Finance, 42*, 373–388.
Hull, J., & White, A. (1987). The pricing of options with stochastic volatilities. *Journal of Finance, 42*, 281–300.
Ingersoll, J. (1987). Theory of financial decision making. Rowman and Littlefield.
Jones, C. (2006). A nonlinear factor analysis of S&P 500 index option returns. *Journal of Finance, 41*, 2325–2363.
Kan, R., & Zhou, G. (2001). *Tests of mean–variance spanning*. Working paper, University of Toronto and Washington University.
Knez, P., Litterman, R., & Scheinkman, J. (1994). Explorations into factors explaining money market returns. *Journal of Finance, 49*, 1861–1882.
Lehmann, B., & Modest, D. (1988). The empirical foundations of the arbitrage pricing theory. *Journal of Financial Economics, 21*, 213–254.
Longstaff, F. (1995). Option pricing and the martingale restriction. *Review of Financial Studies, 8*, 1091–1124.
Madan, D., Carr, P., & Chang, E. (1998). The variance gamma process and option pricing. *European Finance Review, 2*, 79–105.

Melino, A., & Turnbull, S. (1990). Pricing foreign currency options with stochastic volatility. *Journal of Econometrics, 45*, 239–265.

Merton, R. (1973). Theory of rational option pricing. *Bell Journal of Economics and Management Science, 4*, 141–183.

Newey, W., & West, K. (1987). A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix. *Econometrica, 55*, 703–708.

Rubinstein, M. (1994). Implied binomial trees. *Journal of Finance, 49*, 771–818.

Schwert, G. W. (1989). Why does stock market volatility change over Time? *Journal of Finance, 44*, 1115–1153.

Schwert, G. W. (1990). Stock market volatility and the crash of '87. *Review of Financial Studies, 3*, 77–102.

Scott, L. (1987). Option pricing when the variance changes randomly: Theory, estimators, and applications. *Journal of Financial and Quantitative Analysis, 22*, 419–438.

Scott, L. (1997). Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: Application of fourier inversion methods. *Mathematical Finance, 7*, 413–426.

Stein, E., & Stein, J. (1991). Stock price distributions with stochastic volatility. *Review of Financial Studies, 4*, 727–752.

Whaley, R. (1982). Valuation of american call options on dividend paying stocks. *Journal of Financial Economics, 10*, 29–58.

White, H. (1980). A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity. *Econometrica, 48*, 817–838.

Wiggins, J. (1987). Option values under stochastic volatilities. *Journal of Financial Economics, 19*, 351–372.