DOI 10.1007/s11141-019-09932-7
Radiophysics and Quantum Electronics, Vol. 61, No. 10, March, 2019
(Russian Original Vol. 61, No. 10, October, 2018)

INFLUENCE OF THE CHOICE OF BOUNDARY CONDITIONS ON THE DISTRIBUTION OF THE ELECTRIC FIELD IN MODELS OF THE GLOBAL ELECTRIC CIRCUIT

N. A. Denisova[1*] and A. V. Kalinin[2]

UDC 537.531.2.098+
537.531.2:524.31.084-337

We obtain a new analytical representation of the solution for the classical model of the Roble—Hays global electric circuit, where the connection between the values of the electric potential and the current at magnetically conjugate points of the upper boundary of the atmosphere is allowed for in the boundary conditions. Using this representation, we analyze the influence of various boundary conditions at the upper boundary of the atmosphere on the potential distribution and present an estimate of perturbations of the electric field by thunderstorm sources at magnetically conjugate points.

1. INTRODUCTION

Studying the distribution of electric currents is a main task of the theory of atmospheric electricity [1–3]. It is connected with the study of spatial dependences of the electric fields in the atmosphere, which are due to external currents that model charge separation currents in a thunderstorm cloud. Some basic models of the global electric circuit and the relevant literature are discussed in [1–9]. In particular, a sufficiently comprehensive list of references with analytical results is presented in [3].

Problem statements for most of the existing models of the global electric circuit do not allow for the effect of coupling of electric fields at magnetically conjugate points at the upper boundary of the atmosphere. Specifically, the models with the equipotential upper boundary neglect this effect inherently.

The problem in which the relationships that relate the electric potential and currents at magnetically conjugate points were proposed as boundary conditions at the upper boundary of the atmosphere was studied for the first time in the classical work by Roble and Hays [10]. The Roble—Hays model is still relevant, since it was an attempt to allow for the influence of the magnetosphere on the global electric circuit. However, in [10] the distributions of the potential and the field were studied only numerically, and the influence of the conditions at the upper boundary on the distribution of the electric field was not analyzed.

The purpose of this work is to study analytically the influence of the choice of the boundary conditions (first of all, at the upper boundary of the atmosphere) on the distributions of the electric potential and the field in the models of the global electric circuit and to estimate the perturbations introduced by thunderstorm generators at magnetically conjugate points.

* natasha.denisova@mail.ru

[1] N. I. Lobachevsky State University of Nizhny Novgorod; [2] Institute of Applied Physics of the Russian Academy of Sciences, Nizhny Novgorod, Russia. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 61, No. 10, pp. 831–842, October 2018. Original article submitted May 17, 2018; accepted October 29, 2018.

2. PROBLEM STATEMENT AND MAIN RESULTS

In this paper, we consider a problem about the distribution of the electric potential for the atmosphere in the form of a spherical layer with $r_1 < r < r_{\mathrm{m}}$, whose conductivity grows exponentially along its radius, with a distributed stationary vertical current. Its mathematical formulation has the form [10]

$$\mathrm{div}(\sigma \mathrm{grad}\phi) = \mathrm{div}\mathbf{J}^{\mathrm{ext}}, \tag{1}$$

$$\phi(r,\theta,\varphi)|_{r=r_{\mathrm{m}}} = \phi(r,\pi-\theta,\varphi)|_{r=r_{\mathrm{m}}}, \tag{2}$$

$$(2)\frac{\partial\phi(r,\theta,\varphi)}{\partial r}\Big|_{r=r_{\mathrm{m}}} = -\frac{\partial\phi(r,\pi-\theta,\varphi)}{\partial r}\Big|_{r=r_{\mathrm{m}}}, \tag{3}$$

$$\left(\phi - \Delta R\sigma_1\frac{\partial\phi}{\partial r}\right)\Big|_{r=r_1} = 0. \tag{4}$$

Here, $\sigma(r) = \sigma_0 \exp[(r - r_0)/H]$ is the electric conductivity of the atmosphere, σ_0 is the electric conductivity near the spherical surface of the Earth, r is the distance from the Earth's center, r_0 is the Earth's radius (the following values were used in the numerical calculations: $r_0 = 6370$ km, $H = 6$ km), $\mathbf{J}^{\mathrm{ext}}$ is the density of external electric currents created by thunderstorm generators, and r, θ, and φ are the spherical coordinates. Boundary conditions (2) and (3) are written at the bottom boundary of the magnetosphere over the Earth, where anisotropy of the electric conductivity is significant, and electric currents run along the lines of the magnetic field. Boundary condition (4) of the third kind relates the potential and the current at the boundary $r = r_1$ [10]. The column resistance ΔR allows for the orography of the firm land on average, and σ_1 is the value of electric conductivity at a certain $r = r_1$.

It should be noted that problem (1)–(4) is not a classical problem of mathematical physics by virtue of boundary conditions (2) and (3). The issues of mathematical correctness of this model and other models of the global electric circuit are discussed in [11–13].

In [10], the numerical algorithm used to solve problem (1)–(4) is based on the expansion of the solution into a sum with respect to spherical functions with a special method of summation of the series terms (Cesàro summation). As a result of numerical experiments (allowing for the 37 terms of the series), the influence of the parameters and positions of the sources on the distribution of the electric potential in the atmosphere is analyzed under the assumption that the lower boundary of the magnetosphere, at which the upper boundary conditions are specified, is located at the height $h_{\mathrm{m}} = 105$ km over the Earth's surface, where $h_{\mathrm{m}} = r_{\mathrm{m}} - r_0$. In [14], in the special case, when $\Delta R = 0$, the formulas are obtained for the coefficients of expansion of the electric potential with respect to the spherical functions, and the questions of the influence of the source parameters on the electric field are discussed along with the possibility of allowing for orography within the framework of the considered model.

In this work, analytical consideration is also based on the expansion of the solution and the sources with respect to spherical functions. At $\Delta R \geq 0$, the formulas for the series coefficients are obtained, which are similar to those found in [14]. Using the theorem about summation of associated Legendre polynomials, we find the new formula for the electric potential in the form of superposition of two fields (see the Appendix), one of which coincides at $h_{\mathrm{m}} \to \infty$ with the field produced by the source of the right-hand side of Eqs. (1)–(4) for the boundary condition

$$\phi|_{r\to\infty} = V_\infty, \tag{5}$$

which is used instead of conditions (2) and (3). This expansion allowed us to separate explicitly the excitation of the electric field at points that are conjugate to the source location and compare the solutions of two different problems for Eq. (1), specifically, the problems with boundary conditions (2)–(4) and with conditions (4) and (5).

Let us consider an individual source of the external current, which has the number s. In the case of several current sources (e.g., thunderstorm generators), one should perform summation with respect to the variable s in the formulas below. The density of the external radial electric current will be written as

$$\mathbf{J}^{\text{ext}} = \frac{I_{s0}}{r^2 \sin\theta}[\theta(r - r_{s0}) - \theta(r - r_{s1})]\delta_{N_s}(\theta, \theta_{s0}, \varphi, \varphi_{s0})\mathbf{e}_r, \tag{6}$$

where r_{s1} and r_{s0} are the radial distances corresponding to the positive and negative charges of the thunderstorm generator, while $r_{s0} < r_{s1}$, and I_{s0} is the current strength. The function $\theta(r)$ denotes the Heaviside function, and $(1/\sin\theta)\delta_{N_s}(\theta, \theta_{s0}, \varphi, \varphi_{s0})$ is a term of the delta-shaped sequence [16] with the number N_s that has the form

$$\frac{1}{\sin\theta}\delta_{N_s}(\theta, \theta_{s0}, \varphi, \varphi_{s0}) = \sum_{n=0}^{N_s}\sum_{k=0}^{n} \frac{Y_{n,k}^{(1)}(\theta,\varphi)Y_{n,k}^{(1)}(\theta_{s0},\varphi_{s0}) + Y_{n,k}^{(2)}(\theta,\varphi)Y_{n,k}^{(2)}(\theta_{s0},\varphi_{s0})}{||Y_{n,k}||^2}$$

$$= \frac{1}{4\pi}\sum_{n=0}^{N_s}(2n+1)P_n(\cos\gamma). \tag{7}$$

Here, γ is the angle between the radial direction towards the observation point and the dipole axis, and

$$\cos\gamma = \cos\theta\cos\theta_{s0} + \sin\theta\sin\theta_{s0}\cos(\varphi - \varphi_{s0}).$$

Sequence (7) converges at $N_s \to \infty$ to the generalized function $(\sin/\theta)\delta(\theta - \theta_{s0})\delta(\varphi - \varphi_{s0})$, where $\delta(x)$ is the Dirac delta function.

The spherical functions $Y_{n,k}^{(1)}(\theta,\varphi)$ and $Y_{n,k}^{(2)}(\theta,\varphi)$ are related to the associated Legendre functions by the formulas

$$Y_{n,k}^{(1)}(\theta,\varphi) = P_n^k(\cos\theta)\cos(k\varphi), \qquad Y_{n,k}^{(2)}(\theta,\varphi) = P_n^k(\cos\theta)\sin(k\varphi)$$

and have the form

$$||Y_{n,k}||^2 = \frac{2\pi\varepsilon_k(n+k)!}{(2n+1)(n-k)!},$$

where $\varepsilon_k = 1$ at $k > 0$, and $\varepsilon_0 = 2$. Figure 1 shows the plots of two sequence terms at two different values of N. The amplitudes of these functions take the value $(N_s+1)^2/(4\pi)$ and are found from formula (7), when $\cos\gamma = 1$ is substituted into it.

The boundary-value problems for Eq. (1) have not yet been studied analytically in full detail. As a rule, one considers the equation whose coefficients are little different from the coefficients of Eq. (1) [10, 14]. In the spherical system of coordinates,

$$\text{div}(\sigma\text{grad}\phi) = \sigma\left[\frac{\partial^2\phi}{\partial r^2} + \left(\frac{2}{r} + \frac{1}{H}\right)\frac{\partial\phi}{\partial r} + \frac{1}{r^2\sin\theta}\Delta_{\theta,\varphi}\phi\right] \tag{8}$$

and in the second summand, the term $2/r$ is omitted. In this paper, as in [10], we replace $1/r^2$ in the third term in Eq. (8) with $1/r_0^2$. In the obtained equation, we perform replacement of variables

$$\sigma = \sigma_0\exp[(r - r_0)/H], \qquad \mu = \cos\theta. \tag{9}$$

As a result of this replacement, we obtain the equation

$$\frac{1}{H^2}\frac{\partial}{\partial\sigma}\left(\sigma^2\frac{\partial\phi}{\partial\sigma}\right) + \frac{1}{r_0^2}\Delta_{\mu,\varphi}\phi = \frac{1}{\sigma}\text{div}\mathbf{J}^{\text{ext}}, \qquad \sigma_1 < \sigma < \sigma_{\text{m}}, \tag{10}$$

where $\sigma_{\text{m}} = \sigma(r_{\text{m}})$ and

$$\Delta_{\mu,\varphi} = \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right] + \frac{1}{1-\mu^2}\frac{\partial^2\phi}{\partial\mu^2}.$$

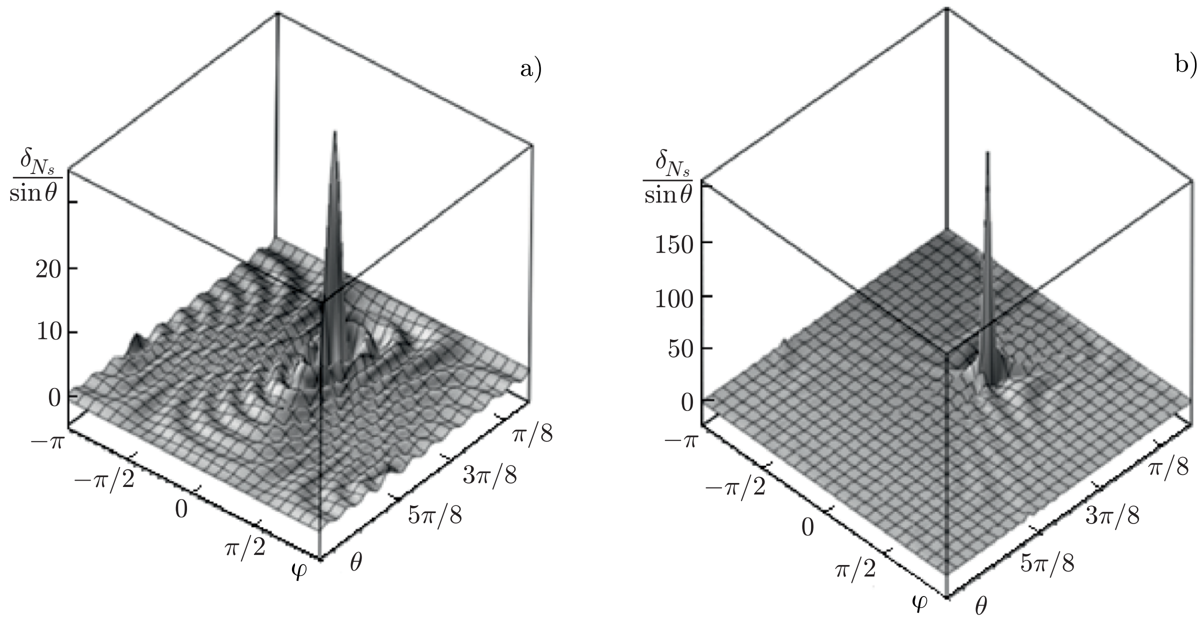

Fig. 1. Plot of the function $(1/\sin\theta)\delta_{N_s}(\theta, \pi/3, \varphi, 0)$ at $N_s = 20$ and 50 (*a* and *b*, respectively).

In new variables, the right-hand side of the equation has the form

$$\mathrm{div}\mathbf{J}^{\mathrm{ext}} = \frac{I_{s0}}{H}\left[\frac{1}{r_{s0}^2}\delta(\sigma - \sigma_{s0}) - \frac{1}{r_{s1}^2}\delta(\sigma - \sigma_{s1})\right]\delta_{N_s}(\mu, \mu_{s0}, \varphi, \varphi_{s0}). \tag{11}$$

Let us write down transformed boundary conditions (3)–(5):

$$\phi(\sigma_{\mathrm{m}}, \mu, \varphi) = \phi(\sigma_{\mathrm{m}}, -\mu, \varphi), \tag{12}$$

$$\frac{\partial\phi(\sigma_{\mathrm{m}}, \mu, \varphi)}{\partial\sigma} = -\frac{\partial\phi(\sigma_{\mathrm{m}}, -\mu, \varphi)}{\partial\sigma}, \tag{13}$$

$$\left(\phi - \frac{\Delta R\sigma_1^2}{H}\frac{\partial\phi}{\partial\sigma}\right)\Big|_{\sigma=\sigma_1} = 0. \tag{14}$$

The solution of problem (10)–(14), which generalizes the solution of the corresponding problem in [14] for the case of $\Delta R \geq 0$, is presented in the Appendix. Using the theorem about summation of associated Legendre polynomials, a new formula is obtained for the electric potential, which has the form of a superposition of two fields (Eqs. (A10)–(A16)). Further simplification of the formula for it depends on the problem parameters H, r_0, h_1, h_{s0}, h_{s1}, h_{m}, ΔR, and N_s, where the height h is determined from the formula $h = r - r_0$, so that $h_1 = r_1 - r_0$, $h_{s1} = r_{s1} - r_0$, $h_{s2} = r_{s2} - r_0$, and $h_{\mathrm{m}} = r_{\mathrm{m}} - r_0$. In what follows, we assume that $h_1 = 0$, $\Delta R = 0$, $h_{s0} = 5$ km, and $h_{s1} = 10$ km (the values of H and r_0 have been given above). Below, formulas (A10)–(A16) are analyzed as a function of the parameter h_{m}. Formulas (A13)–(A16) contain sums of terms having the form $(\sigma_{sj}/\sigma_{\mathrm{m}})^{\xi_n}$ and $(\xi_n - 1)/(\xi_n + 1)$, where ξ_n is determined by formula (A7). As the height h_{m} increases, the first term decreases and, starting with a certain value, becomes significantly less than the second term, i.e., formulas (A13)–(A16) can be simplified.

The height

$$h_{\mathrm{m}}^* = h_{s1} + 2H\ln\left(\frac{r_0}{\sqrt{2}H}\right) + H\ln\left(1 + \frac{4H^2}{r_0^2}\right) \approx h_{s1} + 2H\ln\left(\frac{r_0}{\sqrt{2}H}\right) \tag{15}$$

plays an important role for the further study and comparison of Eq. (1) with boundary conditions (2)–(4), and (5), (6). For the chosen values of the parameters h_{s1}, H, and r_0, the height is $h_{\rm m}^* = 89.5$ km.

Consider the case of $h_{\rm m} = 120$ km ($h_{\rm m} > h_{\rm m}^*$). Then, the following inequalities are valid:

$$\left(\frac{\sigma_{s1}}{\sigma_{\rm m}}\right)^{\xi_n} < \frac{\sigma_{s1}}{\sigma_{\rm m}} \ll \frac{\sigma_{s1}}{\sigma(h_{\rm m}^*)} = \frac{2H^2}{r_0^2 + 4H^2} < \frac{\xi_1 - 1}{\xi_1 + 1} \leq \frac{\xi_n - 1}{\xi_n + 1}. \tag{16}$$

Therefore, the ratios $\sigma_1/\sigma_{\rm m}$ and $\sigma_{sj}/\sigma_{\rm m}$ in formulas (A13)–(A16) can be neglected. Within this approximation, $\bar{T}_n^{(j)}(\sigma) \approx 0$ at all $\sigma_1 < \sigma < \sigma_{s1}$, and $\bar{\bar{T}}_n^{(j)}(\sigma) \approx 0$, if $\sigma_{s0} < \sigma < \sigma_{s1}$. Therefore, the values of the potential under the sources and between them do not depend on the choice of the boundary conditions. Over the sources, in the region $\sigma_{s1} < \sigma < \sigma_{\rm m}$, formula (A10) takes the form

$$\phi_s = V_{\infty,s} + \sum_{n=1}^{N_s} (2n+1)\,[A_n(\sigma) P_n(\cos\gamma) + B_n(\sigma) P_n(\cos\gamma_1)], \tag{17}$$

where

$$A_n(\sigma) = \frac{1}{\xi_n}\left[\frac{1}{\sigma} C_n(\sigma) + B_n(\sigma)\right], \qquad B_n(\sigma) = \frac{1}{\sigma(\xi_n - 1)}\left(\frac{\sigma}{\sigma_{\rm m}}\right)^{\xi_n} C_n(\sigma), \tag{18}$$

$$C_n(\sigma) = \frac{Q_s}{4\pi}\left\{\left(\frac{\sigma_{s1}}{\sigma}\right)^{(\xi_n-1)/2}\left[1 - \left(\frac{\sigma_0}{\sigma_{s1}}\right)^{\xi_n}\right] - \left(\frac{\sigma_{s0}}{\sigma}\right)^{(\xi_n-1)/2}\left[1 - \left(\frac{\sigma_0}{\sigma_{s0}}\right)^{\xi_n}\right]\right\}. \tag{19}$$

At the ionospheric altitudes, at $\sigma = \sigma_{\rm m}$, coefficients (18) are identical and, therefore, electric potential (17) has identical values at magnetically conjugate points $(\sigma_{\rm m}, \mu_{s0}, \varphi_{s0})$ and $(\sigma_{\rm m}, -\mu_{s0}, \varphi_{s0})$. Moreover, since $C_n(\sigma) > 0$, the value of the potential above the sources changes significantly as determined by the choice of the parameter N_s. Since the inequality $C_n(\sigma) \leq \xi_n \sigma_0 V_{\infty,s}$, is valid for the function $C_n(\sigma)$ at all values $\sigma_{s1} < \sigma < \sigma_{\rm m}$, we obtain the following estimates for coefficients (18):

$$B_n(\sigma) \leq \frac{\xi_1}{(\xi_1 - 1)}\frac{\sigma_0}{\sigma_{\rm m}} V_{\infty,s}, \qquad A_n(\sigma) \leq \left[\frac{1}{\sigma} + \frac{1}{(\xi_1 - 1)\sigma_{\rm m}}\right]\sigma_0 V_{\infty,s}. \tag{20}$$

At any fixed N_s in the vicinity of the point $(\theta_{s0}, \varphi_{s0})$ specified by the inequality $\cos\gamma \geq \cos[1/(2N_s + 1)]$, all the Legendre polynomials $P_n(\cos\gamma)$, $n = 1, \ldots, N_s$, take positive values [15]. Comparing the coefficients $A_n(\sigma)$ and $B_n(\sigma)$, one can introduce another parameter,

$$h_{\rm p} \approx h_{\rm m} - 2H \ln[r_0/(2H)], \tag{21}$$

(21) such that at $h_{s1} < h \ll h_{\rm p}$

$$\frac{1}{\sigma} \gg \frac{1}{(\xi_1 - 1)\sigma_{\rm m}}.$$

Then, by virtue of estimates (20), electric-field potential (17) is determined by the expression

$$\phi_s \approx V_{\infty,s} + \sum_{n=1}^{N_s} (2n+1) \frac{C_n(\sigma)}{\sigma \xi_n} P_n(\cos\gamma). \tag{22}$$

Function (22) is the solution of Eq. (10) at $\sigma > \sigma_{s1}$ and satisfies the boundary condition $V|_{\sigma\to\infty} = V_{\infty,s}$.

All numerical results of work [10] were obtained for the case of $N_s = 37$. Let us show that the formulas for potential (22) can be simplified in the range $\sigma_{s1} < \sigma \ll \sigma_{\rm p}$ for $N_s < 63$ at $h_{\rm m} > h_{\rm m}^*$: (here, $\sigma_{\rm p}$ is the conductivity at the height $h_{\rm p}$). By virtue of the inequality $[4n\,(n+1)\,H^2]/r_0^2 < 1/100$, the variable $\xi_n \approx 1 + 2n\,(n+1)\,H^2)/r_0^2 \approx 1$ at all $1 \leq n \leq N_s$. Allowing for the terms of the same order of smallness, we

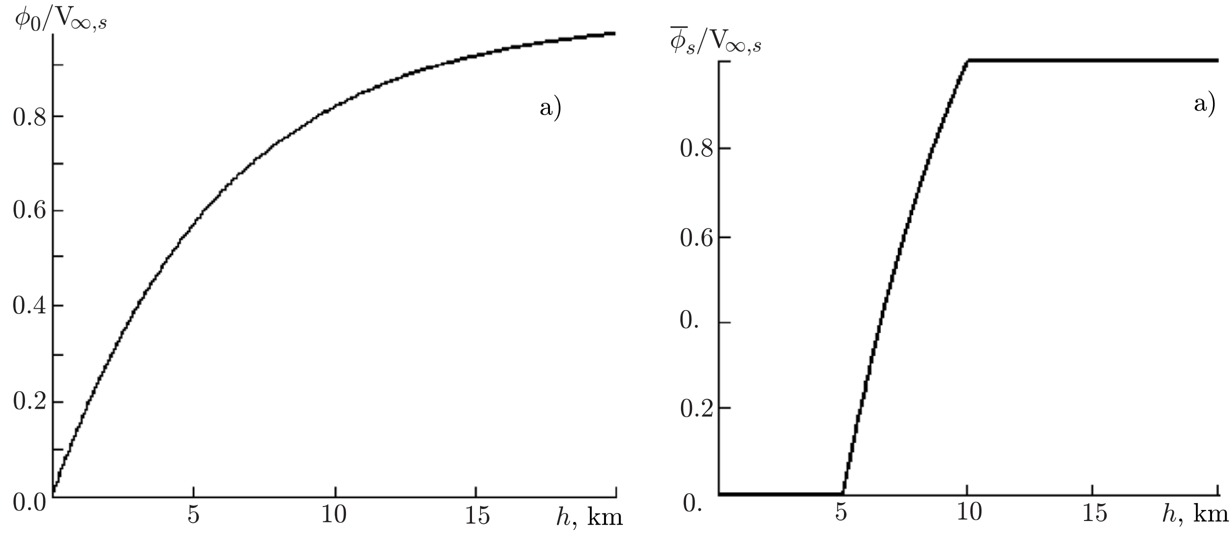

Fig. 2. Plot of the function $\phi_0(\sigma)/V_{\infty,s}$ (*a*) and the spherical mean of the function $\phi_s/V_{\infty,s}$ (*b*).

obtain a formula for the coefficient $C_n(\sigma)$, which is independent of the index n:

$$T_n(\sigma) \approx V_{\infty,s}\sigma_0/\sigma. \tag{23}$$

Using formulas (22), (23), and (7), in the range $\sigma_{s1} < \sigma \ll \sigma_\mathrm{p}$ we have

$$\phi_s \approx V_{\infty,s}\left[1 - \frac{\beta_0\sigma_1}{\sigma} + 4\pi\frac{\sigma_0}{\sigma}\delta_{N_s}(\mu,\mu_{s0},\varphi,\varphi_{s0})\right]. \tag{24}$$

Similar simplifications of formulas (A11) and (A12) at $\sigma_1 < \sigma < \sigma_{s0}$ and $\sigma_{s0} < \sigma < \sigma_{s1}$ yield the following result (at all $\sigma_0 < \sigma \ll \sigma_\mathrm{p}$):

$$\phi_s \approx \phi_0(\sigma) + \phi_1(\sigma)\delta_{N_s}(\mu,\mu_{s0},\varphi,\varphi_{s0}). \tag{25}$$

Here, we introduce the following notations:

$$\phi_0(\sigma) = V_{\infty,s}(1-\sigma_0/\sigma), \qquad \sigma_0 < \sigma \ll \sigma_\mathrm{p}, \tag{26}$$

$$\phi_1(\sigma) = \begin{cases} -4\pi V_{\infty,s}(1-\sigma_0/\sigma), & \sigma_0 < \sigma < \sigma_{s0}; \\ 4\pi V_{\infty,s}\dfrac{\sigma_{s0}}{\sigma}\left(\dfrac{\sigma_0}{\sigma} - \dfrac{\sigma_{s1}-\sigma}{\sigma_{s1}-\sigma_{s0}}\right), & \sigma_{s0} < \sigma < \sigma_{s1}; \\ 4\pi V_{\infty,s}\sigma_0/\sigma & \sigma_{s1} < \sigma \ll \sigma_\mathrm{p}. \end{cases} \tag{27}$$

Figure 2*a* shows the plot of the function $\phi_0(\sigma)/V_{\infty,s}$ depending on the height h, and Fig. 2*b* shows the spherical mean of function (25) normalized with respect to $V_{\infty,s}$, which is determined by the formula

$$\frac{\bar{\phi}_s}{V_{\infty,s}} = \frac{1}{4\pi}\int\limits_0^{2\pi}\mathrm{d}\varphi\int\limits_0^{\pi}(\phi_s/V_{\infty,s})\sin\theta\,\mathrm{d}\theta.$$

On the radial half-ray $\mu = \mu_{s0}$, $\varphi = \varphi_{s0}$, the function $\delta_{N_s}(\mu,\mu_{s0},\varphi,\varphi_{s0})$ takes the maximum value $(N_s+1)^2/(4\pi)$. Therefore, the maximum and minimum values of function (25) are determined by not only the ionospheric potential $V_{\infty,s}$ and the values of conductivity, but also by the parameter N_s. Figure 3 shows the plot of function (25) normalized with respect to $V_{\infty,s}$ on the radial half-ray of charge locations depending on the height h ($N_s = 20$). It is not difficult to plot functions (25) at a fixed value of the height h. In Fig. 4, they are shown in the regions below the sources, between the sources, and above the sources for the same values of the parameters.

Formula (25) yields a rather good approximation at the heights $0 < h \ll h_\mathrm{p}$. However, it cannot be used at $h_\mathrm{p} < h < h_\mathrm{m}$. Extrapolation of formula (25) to this interval for solving of problem (10)–(14) with the boundary condition at the height $h_\mathrm{m} = 100$ km yields the value of the potential $\phi_s \approx V_{\infty,s}$. At the same time, numerical calculations by formulas (A10) show that the potential ϕ_s depends on the angles θ and φ in the region $h_\mathrm{p} < h < h_\mathrm{m}$.

It is seen in Fig. 5 that the values of the function $\phi_s/V_{\infty,s}$ differ from unity at almost all θ. Two points are isolated in the plot, which correspond to the point of the source locations and the conjugate point. At the height $h_\mathrm{m} = 100$ km, the values of the potential ϕ_s coincide at these points.

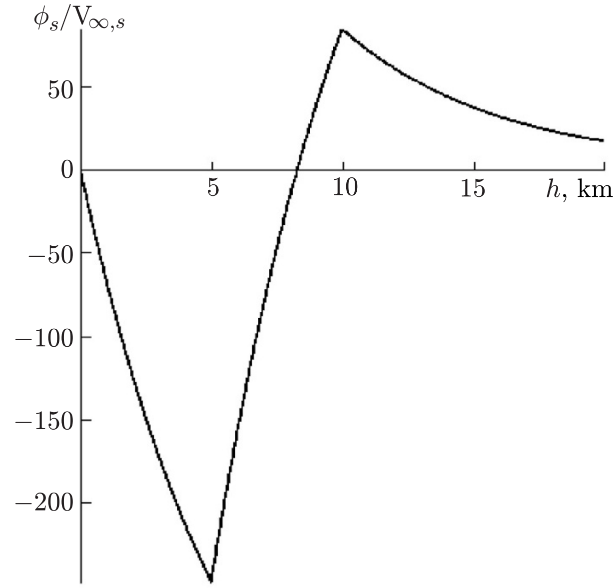

Fig. 3. Plot of the electric potential $\phi_s/V_{\infty,s}$ ($N_s = 20$, $\theta = \theta_{s0} = \pi/3$, and $\varphi = \varphi_{s0}$)

Now, consider the case where $h_\mathrm{m} = 70$ km ($h_\mathrm{m} < h^*_\mathrm{m}$). At this value of h_m, inequalities (16) are not fulfilled, and formulas (17) and (25) are inapplicable. Numerical calculations by formulas (A10)–(A16) show that the values of the electric potential at the center of a thunderstorm formation under the sources and between the sources are little different from those shown in Fig. 3 at $h_\mathrm{m} = 120$ km. The differences in the potential as compared with the case of $h_\mathrm{m} = 120$ km affect the area over the sources at all θ and φ and the vicinity of the conjugate point at all heights. The dependence of the potential at the conjugate point on the height is presented in Fig. 6.

Figure 7 shows plots of the potential $\phi_s/V_{\infty,s}$ calculated as a function of the angle θ by formulas (23) in the regions below the sources, between the sources, and above the sources. It is seen from the plot that the variations in the potential at the conjugate point appear at even sufficiently low heights. The solutions of problems (1)–(4) and (1), (5), and (4) differ significantly.

3. CONCLUSIONS

To conclude, we lay down the main results of this work.

The theorem about summation of associated Legendre polynomials is used to obtain a new formula for the electric potential in the form of a superposition of two fields (Eqs. (A10)–(A15)), which allows

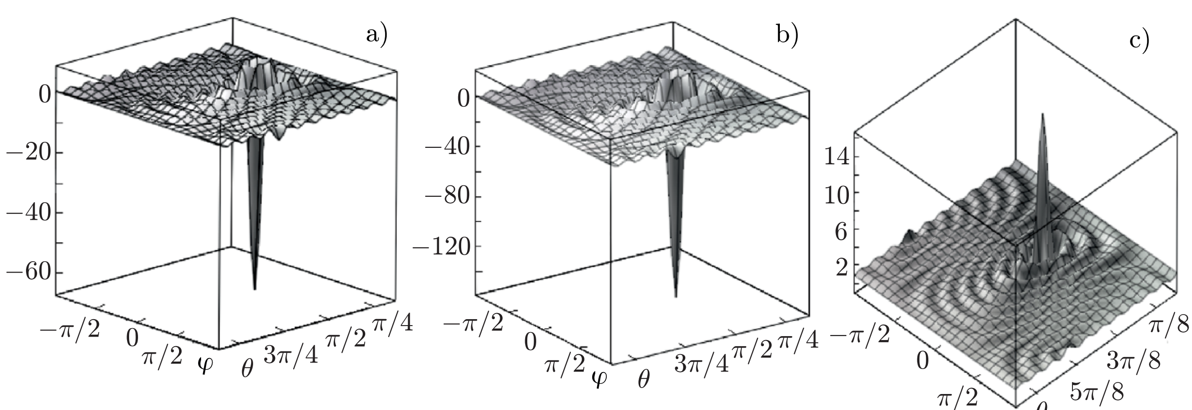

Fig. 4. Plots of the electric potential $\phi_s/V_{\infty,s}$ as a function of the angles θ and φ at fixed heights $h = 1$, 8, and 20 km (*a*, *b*, and *c*, respectively), $\theta_{s0} = \pi/3$ and $\varphi_{s0} = 0$.

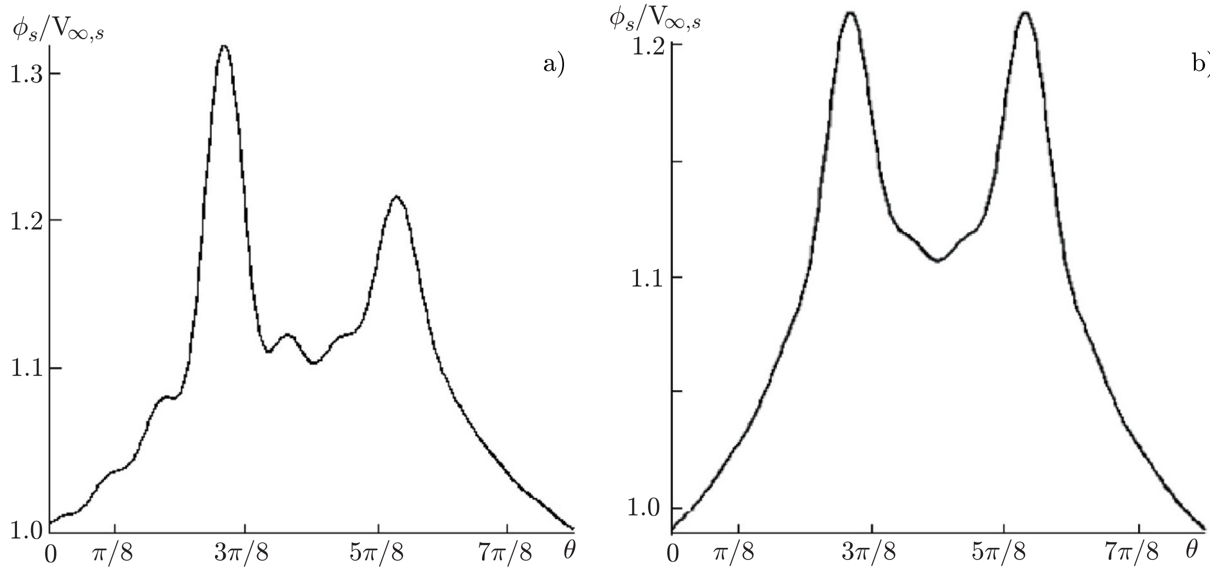

Fig. 5. Plot of the function $\phi_s/V_{\infty,s}$ as a function of the angle θ at fixed $\varphi = \varphi_{s0}$, $\theta_{s0} = \pi/3$, the height $h = 50$ and 100 km (*a* and *b*, respectively), and $N_s = 20$.

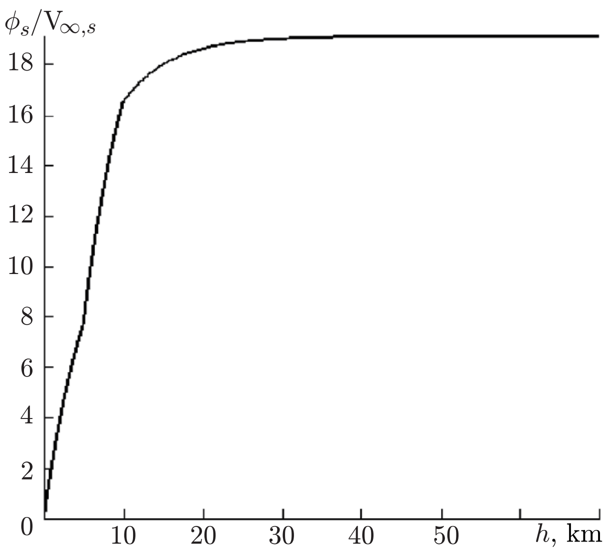

Fig. 6. Plot of the potential $\phi_s/V_{\infty,s}$ at the conjugate point $\theta = \pi - \theta_{s0}$ as a function of the height h ($N_s = 20$, $\theta = \theta_{s0} = \pi/3$, and $\varphi = \varphi_{s0}$).

one to analyze the influence of various boundary conditions at the upper boundary of the atmosphere on the distribution of the electric potential and estimate perturbations of the electric field by thunderstorm sources at magnetically conjugate points.

The fundamental parameters h_{m}^* and h_{p} are determined, which allow one to estimate the influence of the boundary conditions at the upper boundary h_{m} on the distribution of the electric field in the atmosphere. It is shown that at $h_{\mathrm{m}} > h_{\mathrm{m}}^*$, the boundary-value problem for Eq. (1) with simpler boundary conditions (5) and (4) can be used to find the electric field at the heights $0 < h < h_{\mathrm{p}}$. At $h_{\mathrm{m}} < h_{\mathrm{m}}^*$, boundary-value problems (1)–(4) and (1), (4) and (5) yield different results (for the case of $h_{\mathrm{m}}^* = 90$ considered in the paper).

It is shown that the ratio between the values of h_{m} and h_{m}^* also characterizes the perturbation of the electric field by thunderstorm sources at magnetically conjugate points. In particular, the perturbation of the electric field at $h_{\mathrm{m}} < h_{\mathrm{m}}^*$ will be significant at the magnetically conjugate point and at all heights below it.

The obtained results can be used to model the global electric circuit with allowance for the influence of the magnetosphere on the distribution of the electric field in the atmosphere.

The authors are grateful to E. A. Mareev, S. S. Davydenko, and N. N. Slunyaev for useful discussions of the results. This work was supported by the Russian Science Foundation (project No. 18–12–00441).

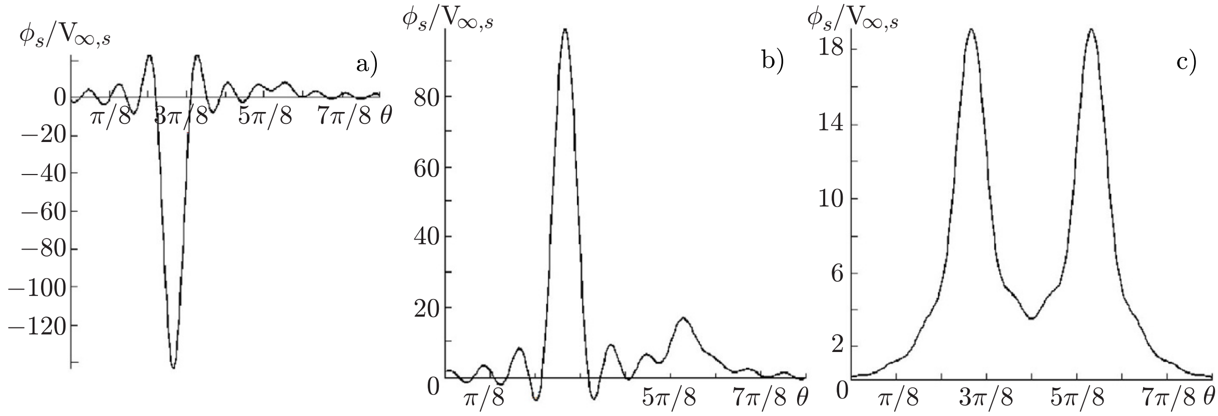

Fig. 7. Dependences of the potential $\phi_s/V_{\infty,s}$ on the angle θ at $h = 2.5$, 10, and 70 km (*a*, *b*, and *c*, respectively) ($h_\mathrm{m} = 70$ km, $\varphi = \varphi_{s0}$, $\theta = \theta_{s0} = \pi/3$, and $N_s = 20$).

APPENDIX

The solution of problem (1)–(4), (6) is similar to that obtained in [2] and has the following form:

$$\phi_s = V_{\infty,s} + \sum_{n=1}^{N_s}\sum_{k=0}^{n} \frac{\tilde{R}^{(1)}_{n,k}(\sigma) - \tilde{R}^{(0)}_{n,k}(\sigma)}{||Y_{n,k}||^2} P^k_n(\mu) P^k_n(\mu_{s0}) \cos[k(\varphi - \varphi_{s0})], \qquad \sigma_{s1} < \sigma < \sigma_\mathrm{m}; \tag{A1}$$

$$\phi_s = \frac{Q_s}{4\pi}\left(\frac{1}{\sigma_{s0}} - \frac{1}{\sigma}\right) + \sum_{n=1}^{N_s}\sum_{k=0}^{n} \frac{R^{(1)}_{n,k}(\sigma) - \tilde{R}^{(0)}_{n,k}(\sigma)}{||Y_{n,k}||^2} P^k_n(\mu) P^k_n(\mu_{s0}) \cos[k(\varphi - \varphi_{s0})], \quad \sigma_{s0} < \sigma < \sigma_{s1}; \tag{A2}$$

$$\phi_s = \sum_{n=1}^{N_s}\sum_{k=0}^{n} \frac{R^{(1)}_{n,k}(\sigma) - R^{(0)}_{n,k}(\sigma)}{||Y_{n,k}||^2} P^k_n(\mu) P^k_n(\mu_{s0}) \cos[k(\varphi - \varphi_{s0})], \qquad \sigma_1 < \sigma < \sigma_{s0}. \tag{A3}$$

Here,

$$V_{\infty,s} = \frac{Q_s}{4\pi}\left(\frac{1}{\sigma_{s0}} - \frac{1}{\sigma_{s1}}\right), \qquad Q_s = \frac{I_{s0}H}{r_0^2}, \tag{A4}$$

and $V_{\infty,s}$ is the ionospheric potential. The functions depending on the variable σ have the form

$$\tilde{R}^{(j)}_{n,k}(\sigma) = \frac{Q_s}{\sqrt{\sigma\sigma_{sj}}} \frac{(\sigma_{sj}/\sigma)^{\xi_n/2}[\tau_{n,k} + (\sigma/\sigma_\mathrm{m})^{\xi_n}]\,[1 - \beta_n(\sigma_1/\sigma_{sj})^{\xi_n}]}{\xi_n[\tau_{n,k} + \beta_n(\sigma_1/\sigma_\mathrm{m})^{\xi_n}]}, \quad j = 0, 1; \tag{A5}$$

$$R^{(j)}_{n,k}(\sigma) = \frac{Q_s}{\sqrt{\sigma\sigma_{sj}}} \frac{(\sigma/\sigma_{sj})^{\xi_n/2}[\tau_{n,k} + (\sigma_{sj}/\sigma_\mathrm{m})^{\xi_n}]\,[1 - \beta_n(\sigma_1/\sigma)^{\xi_n}]}{\xi_n[\tau_{n,k} + \beta_n(\sigma_1/\sigma_\mathrm{m})^{\xi_n}]}, \quad j = 0, 1; \tag{A6}$$

where

$$\beta_n = \frac{2H + \Delta R\sigma_1(1 - \xi_n)}{2H + \Delta R\sigma_1(1 + \xi_n)}, \qquad \xi_n = \sqrt{1 + 4n(n+1)H^2/r_0^2}, \tag{A7}$$

$$\tau_{n,k} = \begin{cases} (\xi_n - 1)/(\xi_n + 1), & n + k\text{(iseven;} \\ -1, & n + k\text{(isodd.} \end{cases}$$

Solving of Eqs. (A1)–(A6) can be simplified. All functions $R^{(j)}_{n,k}(\sigma)$ are subdivided into two forms, specifically, those, for which the numbers $n + k$ are even or odd. Therefore, one can represent the inner sum with respect to k in formulas (A1)–(A6) in the form of two sums and use the formula

$$\sum_{k=0}^{n} \frac{1}{||Y_{n,k}||^2} P_n^k(\mu) P_n^k(\mu_{s0}) \cos[k(\varphi - \varphi_{s0})] = \frac{2n+1}{8\pi}[P_n(\cos\gamma) + P_n(\cos\gamma_1)], \tag{A8}$$

if in the summation with respect to k, only such values of the variable k are used at which the numbers $n+k$ are even, and the formula

$$\sum_{k=0}^{n} \frac{1}{||Y_{n,k}||^2} P_n^k(\mu) P_n^k(\mu_{s0}) \cos[k(\varphi - \varphi_{s0})] = \frac{2n+1}{8\pi}[P_n(\cos\gamma) - P_n(\cos\gamma_1)], \tag{A9}$$

if in the summation with respect to k, only such k are used, at which the numbers $n+k$ are odd. Formulas (A8) and (A9) follow from the theorem of summation for the associated Legendre functions. Here,

$$\cos\gamma = \mu\mu_{s0} + \sqrt{1-\mu^2}\sqrt{1-\mu_{s0}^2}\cos(\varphi - \varphi_{s0}), \quad \cos\gamma_1 = -\mu\mu_{s0} + \sqrt{1-\mu^2}\sqrt{1-\mu_{s0}^2}\cos(\varphi - \varphi_{s0}),$$

where γ_1 is the angle between the radial half-ray directed towards the observation point, and the radial half-ray that contains points conjugate with the points where the dipole charges are located. Formula (A8) contains the Legendre functions, which are even with respect to the variable μ, and formula (A9), those which are odd. Using formulas (A1)–(A9), we obtain the following expressions for the electric field potential:

$$\phi_s = V_{\infty,s} + \sum_{n=1}^{N_s}(2n+1)[\bar{\bar{R}}_n^{(1)}(\sigma) - \bar{\bar{R}}_n^{(0)}(\sigma)]\,P_n(\cos\gamma) + [\bar{\bar{T}}_n^{(1)}(\sigma) - \bar{\bar{T}}_n^{(0)}(\sigma)]\,P_n(\cos\gamma_1) \tag{A10}$$

in the range $\sigma_{s1} < \sigma < \sigma_{\rm m}$;

$$\phi_s = \frac{Q_s}{4\pi}\left(\frac{1}{\sigma_{s0}} - \frac{1}{\sigma}\right) + \sum_{n=1}^{N_s}(2n+1)[\bar{R}_n^{(1)}(\sigma) - \bar{\bar{R}}_n^{(0)}(\sigma)]\,P_n(\cos\gamma) + [\bar{T}_n^{(1)}(\sigma) - \bar{\bar{T}}_n^{(0)}(\sigma)]\,P_n(\cos\gamma_1) \tag{A11}$$

in the range $\sigma_{s0} < \sigma < \sigma_{s1}$; and

$$\phi_s = \sum_{n=1}^{N_s}(2n+1)[\bar{R}_n^{(1)}(\sigma) - \bar{R}_n^{(0)}(\sigma)]\,P_n(\cos\gamma) + [\bar{T}_n^{(1)}(\sigma) - \bar{T}_n^{(0)}(\sigma)]\,P_n(\cos\gamma_1) \tag{A12}$$

in the range $\sigma_1 < \sigma < \sigma_{s0}$. Here, the following notations are introduced:

$$\begin{aligned}\bar{\bar{R}}_n^{(j)}(\sigma) &= \frac{Q_s}{8\pi\sqrt{\sigma\sigma_{sj}}}\\ &\times \frac{(\sigma_{sj}/\sigma)^{\xi_n/2}[1 - \beta_n(\sigma_1/\sigma_{sj})^{\xi_n}]}{\xi_n}\left[\frac{(\sigma/\sigma_{\rm m})^{\xi_n} + (\xi_n-1)/(\xi_n+1)}{\beta_n(\sigma_1/\sigma_{\rm m})^{\xi_n} + (\xi_n-1)/(\xi_n+1)} + \frac{(\sigma/\sigma_{\rm m})^{\xi_n} - 1}{\beta_n(\sigma_1/\sigma_{\rm m})^{\xi_n} - 1}\right],\end{aligned} \tag{A13}$$

$$\begin{aligned}\bar{\bar{T}}_n^{(j)}(\sigma) &= \frac{Q_s}{8\pi\sqrt{\sigma\sigma_{sj}}}\\ &\times \frac{(\sigma_{sj}/\sigma)^{\xi_n/2}[1 - \beta_n(\sigma_1/\sigma_{sj})^{\xi_n}]}{\xi_n}\left[\frac{(\sigma/\sigma_{\rm m})^{\xi_n} + (\xi_n-1)/(\xi_n+1)}{\beta_n(\sigma_1/\sigma_{\rm m})^{\xi_n} + (\xi_n-1)/(\xi_n+1)} - \frac{(\sigma/\sigma_{\rm m})^{\xi_n} - 1}{\beta_n(\sigma_1/\sigma_{\rm m})^{\xi_n} - 1}\right],\end{aligned} \tag{A14}$$

$$\bar{R}_n^{(j)}(\sigma) = \frac{Q_s}{8\pi\sqrt{\sigma\sigma_{sj}}} \times \frac{(\sigma/\sigma_{sj})^{\xi_n/2}[1-\beta_n(\sigma_1/\sigma)^{\xi_n}]}{\xi_n}\left[\frac{(\sigma_{sj}/\sigma_\mathrm{m})^{\xi_n}+(\xi_n-1)/(\xi_n+1)}{\beta_n(\sigma_1/\sigma_\mathrm{m})^{\xi_n}+(\xi_n-1)/(\xi_n+1)}+\frac{(\sigma_{sj}/\sigma_\mathrm{m})^{\xi_n}-1}{\beta_n(\sigma_1/\sigma_\mathrm{m})^{\xi_n}-1}\right], \quad \text{(A15)}$$

$$\bar{T}_n^{(j)}(\sigma) = \frac{Q_s}{8\pi\sqrt{\sigma\sigma_{sj}}} \times \frac{(\sigma/\sigma_{sj})^{\xi_n/2}[1-\beta_n(\sigma_1/\sigma)^{\xi_n}]}{\xi_n}\left[\frac{(\sigma_{sj}/\sigma_\mathrm{m})^{\xi_n}+(\xi_n-1)/(\xi_n+1)}{\beta_n(\sigma_1/\sigma_\mathrm{m})^{\xi_n}+(\xi_n-1)/(\xi_n+1)}-\frac{(\sigma_{sj}/\sigma_\mathrm{m})^{\xi_n}-1}{\beta_n(\sigma_1/\sigma_\mathrm{m})^{\xi_n}-1}\right]. \quad \text{(A16)}$$

REFERENCES

1. E. A. Mareev, *Phys. Usp.*, **53**, No. 5, 504 (2010).
2. S. V. Anisimov and E. A. Mareev, *Izv. Phys. Solid Earth*, **44**, No. 10, 760 (2008).
3. V. N. Morozov, *Mathematical Modeling of Atmospheric Electric Processes Allowing for the Influence of Aerosol Particles and Radioactive Substances* [in Russian], Russian State Hydrometeorological Univ., St. Petersburg (2011).
4. S. S. Davydenko and A. E. Mareev, *J. Geophys. Res.*, **109**, No. D11, D11103 (2004).
5. A. J. G. Baumgaertner, J. P. Thayer, R. R. Neely, and G. Lucas, *J. Geophys. Res. Atmos.*, **118**, No. 16, 9221 (2013).
6. V. Bayona, N. Flyer, G. M. Lucas, and A. J. G. Baumgaertner, *Geosci. Model Dev.*, **8**, No. 10, 3007 (2015).
7. J. Jánský and V. P. Pasko, *J. Geophys. Res. Space Phys.*, **120**, No. 12, 10654 (2015).
8. G. M. Lucas, A. J. G. Baumgaertner, and J. P. Thayer, *J. Geophys. Res. Atmos.*, **120**, No. 23, 12054 (2015).
9. V. V. Denisenko, M. J. Rycroft, and R. G. Harrison, *Surveys Geophys.*, **40**, No. 1, 1 (2018).
10. P. B. Hays and R. G. Roble, *J. Geophys. Res.*, **84**, No. A7, 3291 (1979).
11. A. V. Kalinin, N. N. Slyunyaev, E. A. Mareev, and A. A. Zhidkov, *Izv. Atmos. Ocean. Phys.*, **50**, No. 3, 314 (2014).
12. N. N. Slyunyaev, E. A. Mareev, A. V. Kalinin, and A. A. Zhidkov, *J. Atmos. Sci.*, **71**, No. 11, 4382 (2014).
13. A. V. Kalinin and N. N. Slyunyaev, *J. Math. Anal. Appl.*, **450**, No. 1, 112 (2017).
14. T. Ogawa, *J. Geophys. Res.*, **90**, No. D4, 5951 (1985).
15. M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions*, Dover, New York (1972).
16. V. S. Vladimirov, *Equations of Mathematical Physics* [in Russian], Nauka, Moscow (1981).