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ON THE INVERSE PROBLEMS OF NONLINEAR ACOUSTICS AND ACOUSTIC TURBULENCE

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We consider the problem of retrieval of the radiated acoustic signal parameters from the measured wave field in some cross section of the nonlinear medium. The possibilities of solving regular and statistical inverse problems are discussed on the basis of the solution of the Burgers equation for zero and infinitesimal viscosities.

1. INTRODUCTION

Retrieval of the radiated-signal parameters from the measured acoustic field far from the source is one of the important problems in nonlinear acoustics. Generally, the problem consists in retrieval of the radiated-signal shape or determination of the initial-wave amplitude if the signal shape is known. Among the inverse nonlinear problems, we mention the problems of nonlinear diagnostics of the medium. The stages of the formation of the corresponding research line, that is, nonlinear acoustic diagnostics, were discussed in review papers [1, 2]. Nonlinear diagnostics has acquired a special role as applied to medicine [3].

In this paper, we restrict ourselves to the one-dimensional case where the propagation of a plane acoustic wave is described by the nonlinear diffusion equation, or Burgers equation [4–7]:

$$\frac{\partial p}{\partial z} - \frac{\varepsilon}{c_0^3 \rho_0} p \frac{\partial p}{\partial \tau} = \frac{b}{2 c_0^3 \rho_0} \frac{\partial^2 p}{\partial \tau^2}. \tag{1}$$

Here, $p(\tau, z)$ is the acoustic pressure, ε and b are the medium nonlinearity and dissipation parameters, respectively, z is the coordinate along the beam axis, $\tau = t - z/c_0$ is the time in the reference frame moving at the sound speed c_0, and ρ_0 is the unperturbed density of the medium. In dimensionless variables, this equation is written as

$$\frac{\partial V}{\partial Z} - V \frac{\partial V}{\partial \theta} = \Gamma \frac{\partial^2 V}{\partial \theta^2}. \tag{2}$$

Here,

$$Z = \frac{\varepsilon P_0 z}{c_0^3 \rho_0 t_0}, \qquad \theta = \frac{\tau}{t_0}, \qquad V = \frac{p}{P_0}, \qquad \Gamma = \frac{b}{2 \varepsilon t_0 P_0}, \tag{3}$$

P_0 is the characteristic initial amplitude of perturbation, and t_0 is the characteristic time of the input-signal variation.

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The inverse problem of nonlinear acoustics consists in retrieval of the time profile of the source field $p_0(\tau) = p(\tau, z = 0)$ from the measured field $p(\tau, z)$ at a distance z from the source. The nonlinear diffusion equation in its classical form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \mu\frac{\partial^2 v}{\partial x^2} \tag{4}$$

was proposed by Burgers [8] as a model equation of the turbulence theory and described the effect of the inertial nonlinearity and viscosity on characteristics of strong hydrodynamic turbulence. Dynamical and statistical properties of the solution of the one- (and, recently, three-dimensional, too) Burgers equation have been the subject of many papers (see, e. g., the bibliography in monographs and review papers [4, 9–16]. Despite the fact that the Burgers equation has an exact solution (Hopf–Cole solution), the analysis of deterministic and, especially, random fields is a complex mathematical problem. In this regard, the first major results [17] for the cases of the Brownian initial potential was obtained only 39 years after the Burgers equation itself appeared [8].

It is apparent that by trivial replacement of the variables $V \to v$, $Z \to t$, and $\theta \to -x$, Eq. (2) reduces to its classical form (4). From the point of view of the theory of differential equations, both Eq. (4) with the initial condition $v(x, t = 0) = v_0(x)$ and Eq. (2) with the initial condition $V(\theta, Z = 0) = V_0(\theta)$ represent the Cauchy problem for the parabolic equation. From the point of view of information transfer, the problem consists in considering the impact of the nonlinear effects on the information parameters of the acoustic signal. In particular, if a modulated signal with amplitude and phase (frequency) modulation at the boundary of a nonlinear medium is specified in the form $p_0(\tau) = a(\tau)\sin[\omega_0\tau + \varphi(\tau)]$, then the inverse problem consists in retrieval of the parameters $a(\tau)$ and $\varphi(\tau)$ of the initial signal from the field $p(\tau, z)$ measured in the cross section z. Correspondingly, for Eq. (4) this problem is formulated as retrieval of the parameters of the initial quasiperiodic field $v_0(x) = a(x)\sin[k_0x + \varphi(x)]$ from the field $v(x, t)$ at the instant t.

For weak perturbations, when the nonlinearity can be neglected, Eq. (4) converts into a linear diffusion equation. In this case, as have a classical inverse problem for the heat conduction equation [18].

In this paper, on the basis of the classical Burgers equation (4) we consider a fundamental possibility for retrieval of the $v_0(x)$ profile from the field which is known at the instant t. The use of the Burgers equation in form (4) in the inverse problems is very convenient since for the infinitesimal viscosity ($\mu \to 0$) its solution is equivalent to the evolution of a flow of freely moving particles with absolutely inelastic collisions [9, 10, 14, 15]. The interpretation of the solution of the Burgers equation as a flow of coalescing particles makes it apparent that the complete solution of the inverse problem is impossible at times after a discontinuity forms in the initially continuous field. In the second part of the work we will discuss the inverse problem for the acoustic turbulence, namely, the solution of the Burgers equation for random initial conditions.

2. INVERSE PROBLEM FOR A ZERO VISCOSITY

For a zero viscosity, the Burgers equation (4) converts into the Riemann equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0, \tag{5}$$

which is also known as the Hopf equation, or a simple-wave equation [4, 19]. To consider the inverse problems of nonlinear acoustics, it is very useful to interpret Eq. (5) as the law of evolution of the Eulerian velocity of a flow of noninteracting particles. In the Lagrangian representation, Eq. (5) is equivalent to the system of characteristics

$$\frac{\partial V}{\partial t} = 0, \qquad \frac{\partial X}{\partial t} = V, \tag{6}$$

the solution of which

$$V(y, t) = v_0(y), \qquad X(y, t) = y + tv_0(y) \tag{7}$$

describes the particles moving freely at a constant velocity. Here, y is the Lagrangian coordinate of a particle (i. e., its initial coordinate at $t = 0$). Equation (7) specifies the velocity field in parametric form. To find the Eulerian velocity, it is needed to solve the nonlinear equation

$$x = y + v_0(y)t. \tag{8}$$

Then the velocity field can be represented in the form

$$v(x,t) = \frac{x - y(x,t)}{t}. \tag{9}$$

Here, $y(x,t)$ is the Lagrangian coordinate of a particle that entered the point x at the instant t. From Eqs. (8) and (9), the known implicit solution

$$v(x,t) = v_0[x - tv(x,t)] \tag{10}$$

of Eq. (5) follows. The solution of the inverse problem can also be represented in parametric form. It follows from Eqs. (7)–(9) that for the known field $v(x,t)$ at the point x at the instant t the initial velocity field $v_0(y)$ is determined by the relations

$$v_0 = v(x,t), \qquad y = x - tv(x,t). \tag{11}$$

Equation (5) describes the acoustic waves only at limited times until its solution is single-valued. At the later times, it is necessary to allow for the high-frequency decay and pass from Eq. (5) to the Burgers equation (4).

3. INVERSE PROBLEM FOR AN INFINITESIMAL VISCOSITY

By the Hopf–Cole replacement, the Burgers equation (4) reduces to the linear diffusion equation, and therefore has an exact solution. In the limit of an infinitesimal viscosity $\mu \to 0$, the integrals in the exact solution can be calculated by the saddle-point method. As a result, the asymptotic solution of the Burgers equation can, as previously, be written in form (9):

$$v_{\mathrm{w}}(x,t) = \frac{x - y_{\mathrm{w}}(x,t)}{t}. \tag{12}$$

However, now $y_{\mathrm{w}}(x,t)$ is the coordinate of the absolute minimum of the function

$$G(y,x,t) = s_0(y) + \frac{(x-y)^2}{2t}, \tag{13}$$

where

$$s_0(y) = \int\limits_{-\infty}^{y} v_0(y')\,\mathrm{d}y',$$

on the Lagrangian coordinate y [9, 10, 14, 15]. Taking into account that the extremum points of the function $G(y,x,t)$ on the variable y are determined from the equation

$$\partial G/\partial y = v_0(y) + (y - x)/t = 0,$$

it is seen that the asymptotic solution of the Burgers equation for $\mu \to 0$ consists of branches of the solution of the Riemann equation (5), and is therefore a weak solution of the Riemann equation.

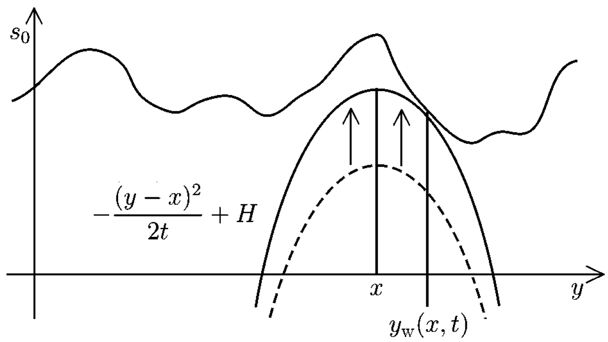

Fig. 1. A graphic illustration of the procedure of seeking the coordinate $y_{\rm w}(x,t)$ of the absolute minimum of the function $G(y,x,t)$ on the coordinate y; the minimum is reached at the tangent point of the parabola $P(y,x,t)$ and the initial potential $s_0(y)$.

To construct a qualitative pattern of the evolution of the velocity field, it is expedient to use a graphic procedure for seeking the absolute minimum of the function $G(y,x,t)$ on the variable y. Apparently, the position of the indicated extremum point coincides with the coordinate of the first point of tangency of the initial potential $s_0(y)$ and the parabola

$$P(y,x,t)=H-\frac{(x-y)^2}{2t} \tag{14}$$

when the quantity H increases from very large negative values (see Fig. 1). Substituting the coordinate of the tangent point $y_{\rm w}(x,t)$ into Eq. (12), we obtain the velocity field $v_{\rm w}(x,t)$ at the point x at the instant t. Note that the dependence $v_{\rm w}(x,t)$ for different values of x will result when the coordinate x of the vertex of the parabola sliding along the initial potential $s_0(y)$ is varied.

The type of tangency of the parabola and the initial potential depends on their curvature ratio, which is equal to $1/t$ and $\partial v_0(y)/\partial y$, respectively. If the condition $1+t\partial v_0(y)/\partial y>0$ is fulfilled for any y, then the parabola slides along the curve $s_0(y)$. In this case, $y_{\rm w}(x,t)$, and therefore $v_{\rm w}(x,t)$, are continuous functions of x, and the asymptotic solution of the Burgers equation coincides with the solution of the Riemann equation. Solution (11) of the inverse problem exists, and is single-valued.

As the time t increases, the parabola $P(y,x,t)$ becomes flatter, resulting in the occurrence of such values of $x=x_k$ at which the parabola is tangent to the initial profile of the potential $s_0(y)$ at two points at once, $y_k^+(x_k,t)$ and $y_k^-(x_k,t)$, $y_k^+>y_k^-$. In what follows, we call this parabola critical. When the coordinate of the parabola vertex passes through the point $x=x_k$, the coordinates of the tangent point of the parabola and the initial profile of the potential, and therefore the field $v_{\rm w}(x,t)$, experience a discontinuity (see Fig. 2).

As was mentioned, the asymptotic solution of the Burgers equation can be interpreted as the law of evolution of a flow of noninteracting particles with absolutely inelastic collisions. The formation of a discontinuity corresponds to the coalescence of the particles filling the interval $[y_k^+(x_k,t),y_k^-(x_k,t)]$ with a unit initial density and the formation of a heavy macro particle with the mass $m_k=y_k^+(x_k,t)-y_k^-(x_k,t)$, whose coordinate and velocity coincide with the coordinate and velocity of the discontinuity. Thus, it is apparent that the formation of a discontinuity results in loss of information on the fine structure of the initial velocity field in the interval $[y_k^+(x_k,t),y_k^-(x_k,t)]$. Note that the case is the same for the media with other types of nonlinearity [20, 21].

We now discuss which characteristics of the initial field $v_0(x)$ can be retrieved from known characteristics of the field $v_{\rm w}(x,t)$ at the discontinuity stage. If $v_{\rm w}(x_k-0,t)$ and $v_{\rm w}(x_k+0,t)$ are the values of the velocity field $v_{\rm w}(x,t)$ on the left and on the right of the discontinuity with the coordinate x_k, then the boundaries of the interval $[y_k^+,y_k^-]$ of the initial field $v_0(x)$, from which the discontinuity formed, are determined by the equations

$$y_k^-=x_k-tv_{\rm w}(x_k-0,t),\qquad y_k^+=x_k-tv_{\rm w}(x_k+0,t). \tag{15}$$

The length of the interval is determined by the discontinuity amplitude:

$$y_k^+-y_k^-=tV_k=t[v_{\rm w}(x_k-0,t)-v_{\rm w}(x_k+0,t)]. \tag{16}$$

It follows from Fig. 2 that the arbitrary variation of the initial velocity field $v_0(x)$ in the interval $[y_k^+,y_k^-]$, which is limited by the condition that the initial potential $s_0(y)$ in this interval is above the critical parabola $P(y,x_k,t)$, does not affect the velocity field at the instant t. Thus, after the formation of discontinuities,

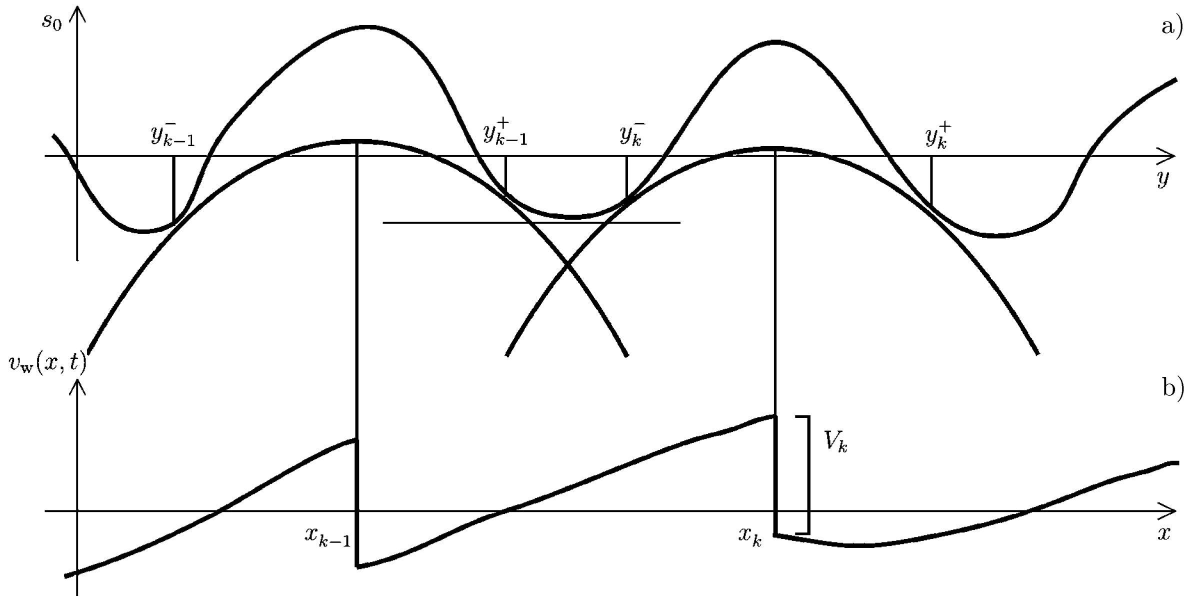

Fig. 2. Diagrams of the initial distribution of the potential and two critical parabolas (a) and a diagram of the asymptotic solution of the Burgers equation for $\mu \to 0$, whose discontinuity coordinates coincide with the coordinates of the vertices of the critical parabolas (b).

it is fundamentally impossible to solve the inverse problem over the entire space interval. It can easily be shown that the discontinuity parameters carry information on only the integral characteristics of the initial field within the interval $[y_k^+, y_k^-]$. From the condition of double tangency of the initial potential distribution $s_0(y)$ by the parabola $P(y, x_k, t)$ it follows that the discontinuity coordinate is equal to

$$x_k(t) = \frac{y_k^+ + y_k^-}{2} + t\frac{s_0(y_k^+) - s_0(y_k^-)}{y_k^+ - y_k^-}. \tag{17}$$

Comparing Eqs. (17) and (15), we find that from the discontinuity velocity

$$U_k(t) = \frac{v_{\rm w}(x_k + 0, t) - v_w(x_k - 0, t)}{2} = \frac{s_0(y_k^+) - s_0(y_k^-)}{y_k^+ - y_k^-} \tag{18}$$

it is possible to retrieve the integral of the initial velocity field $v_0(x)$ over the interval $[y_k^+, y_k^-]$:

$$\int\limits_{y_k^-}^{y_k^+} v_0(x)\,{\rm d}x = U_k(t)(y_k^+ - y_k^-). \tag{19}$$

Note that when the analogy between the solution of the Burgers equation and the evolution of a flow of coalescing particles is used, relation (19) is exactly the momentum conservation law. It should be borne in mind that, although the coordinates in Eq. (17) are time dependent, the equality $U_k(t) = {\rm d}x_k(t)/{\rm d}t$ is fulfilled.

On intervals between the discontinuities, the retrieval of the initial field is determined by Eqs. (11) for the Riemann wave. Only the expounding parts of the profile "survive" at sufficiently long times. In the vicinity of the minimum initial potential $x = y_k$, the velocity field $v_0(x)$ can be approximated as

$$v_0(x) \approx \frac{\partial v_0(y)}{\partial y}\Big|_{y=y_k} (x - y_k) = \alpha_k(x - y_k). \tag{20}$$

Correspondingly, at long times the velocity field on the intervals between the discontinuities has the form

$$v_0(x) = \frac{\alpha_k(x - y_k)}{1 + t\alpha_k} = \alpha_k(t)(x - y_k). \tag{21}$$

From Eq. (21) it is seen that for $t\alpha_k \gg 1$ the gradient $\alpha_k(t) \approx 1/t$ is virtually independent of the initial value of the gradient α_k. Let us estimate the possibility of determining the initial gradient of the velocity field. If the field gradient $\alpha_k(t)$ is measured at the instant t, then the initial gradient α_k is determined by the relation

$$\alpha_k = \frac{\alpha_k(t)}{1 - t\alpha_k(t)}. \tag{22}$$

If the field gradient at the instant t is measured with a relative error δ, i. e., $\alpha_k^*(t) = \alpha_k(t)(1+\delta)$, then from Eq. (22) for the relative error ε of the estimate of the initial gradient $\alpha_k^* = \alpha_k(1 + \varepsilon)$ we have

$$\varepsilon = \delta \left[1 + \frac{t\alpha_k(1 + \delta)}{1 - t\alpha_k\delta}\right]. \tag{23}$$

From relation (23) it follows that for $\delta > 0$ the error ε increases with the time, and for $t \to 1/(\alpha_k\delta)$ the error tends to the infinity, and therefore the initial value of the gradient cannot be determined. For single-scale input signals with the spatial period L, from the initial gradient α_k one can determine the amplitude a_0 of the initial perturbation of the field $v_0(x)$ as $a_0 \approx \alpha_k L$.

Let us estimate the possibility of retrieval of the parameters of a quasiharmonic signal $v_0(x) = a_0(x)\sin[k_0x+\varphi(x)]$ with amplitude and frequency modulation. From the distance between the neighboring nulls of the signal $L_k = y_{k+1}-y_k$ (here, $k_0y_k+\varphi(y_k) = 2\pi k$) one can estimate the input-signal frequency: $k(y_k) \approx 2\pi/L_k$. The field gradient at the point of a zero field is equal to $\alpha_k \approx a_0(y_k)k(y_k)$. Consequently, by measuring the field gradient at the instant t it is possible to retrieve the initial amplitude $a_0(y_k) = \alpha_k/k(y_k)$ and energy of the input signal. The relative error of determining the amplitude is, as previously, determined by relation (23), where $\alpha_k = 1/t_{\rm nl}$, and $t_{\rm nl}$ is the time of formation of the harmonic-signal discontinuity.

One of the important inverse problems in nonlinear acoustics is determination of the initial-signal shape minimizing the energy loss of a signal as it propagates to a certain fixed point. For a periodic signal, the wave energy $E(t) = \int_0^L v_2(x,t)\,\mathrm{d}x$, according to the Burgers equation, varies as

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\gamma(t), \qquad \gamma(t) = \mu\int_0^L \left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x, \tag{24}$$

where L is the wave period. In the case of an infinitesimal viscosity, the rate of decrease in the energy is determined by the discontinuity amplitude alone [4, 5, 14, 15]:

$$\gamma(t) = \frac{2}{3}U^3(t). \tag{25}$$

Thus, the maximum absorption of the acoustic-wave energy is reached at the point where the discontinuity amplitude is the maximum. For a harmonic signal, the time of the discontinuity formation is equal to $t_{\rm nl}$, and the maximum of $U(t)$ is equal to the double amplitude of the initial wave.

It has already been mentioned that for an infinitesimal viscosity, the solution of the Burgers equation can be interpreted as a flow of noninteracting particles with absolutely inelastic collisions. In this case, the discontinuity formation corresponds to the coalescence of particles and the formation of a heavy macro particle, whose position and velocity coincide with the position and velocity of the discontinuity and whose mass is proportional to the discontinuity amplitude. Using the analogy between the solution of the Burgers equation and the flow of particles, it can easily be seen that the maximum amplitude of the discontinuity is reached at the time when all the particles coalesce simultaneously. It follows from Eq. (7) that the distance between the neighboring particles varies as

$$J(y,t) = \frac{\partial X(y,t)}{\partial y} = 1 + t\partial v_0(y)/\partial y. \tag{26}$$

Here, $J(y,t)$ is the Jacobian of transformation from the Lagrangian to Eulerian coordinates. It is seen from Eq. (26) that all the particles collide at the same instant of time if the initial velocity profile has an inverse sawtooth form at the period $-L/2 < x < L/2$:

$$v_0(x) = -\beta x. \tag{27}$$

For such a profile, $v(x,t) = -\beta x/(1 - t\beta)$, the discontinuity is formed at $t_{\rm nl} = 1/\beta$, and its amplitude is the maximum at this instant and is equal to βL. The inverse sawtooth wave is specific in that a finite-amplitude discontinuity is formed and the wave profile is universal both before and after the discontinuity formation. On the basis of these properties, a model of acoustic turbulence as the superposition of inverse sawtooth waves having different scales with a Weierstrass–Mandelbrot spatial spectrum was constructed in [22].

4. INVERSE PROBLEM FOR NOISE SIGNALS

In what follows, we discuss the inverse problem for the random fields satisfying the Burgers equation. Namely, we clarify which statistical characteristics of the initial field $v_0(x)$ can be retrieved from the known statistics of the field $v(x,t)$ at the instant t. Such random fields are often called the Burgers turbulence, or burgulence [23, 24], or acoustic turbulence in application to the evolution of intense acoustic noise. As was mentioned, the study of statistical properties of the solutions of the Burgers equation have been the subject of many papers (see, e. g., the bibliography in monographs and reviews [4, 9–16]). Here, we restrict ourselves to the cases of zero and infinitesimal viscosities.

At the initial stage, the wave propagation is described by the Riemann equation (5), which is equivalent to the Burgers equation (4) for a zero viscosity ($\mu = 0$). In the Lagrangian representation, Eq. (5) is equivalent to a system of characteristics (6), whose solution (7) describes the particles moving freely at a constant velocity. Thus, the statistical description of the motion of particles in the Lagrangian representation is trivial since the probability density of the velocity of a separate particle does not change with the time. Using the formulas of connection between the Lagrangian and Eulerian statistics, we are able to give a comprehensive description of the Eulerian statistics of the Riemann wave field $v(x,t)$, i. e., we find the evolution of the probability distributions, correlation functions, and energy spectra [9, 10, 14, 15]. To find the Eulerian statistics of the velocity field, the solution for the velocity and coordinate (7) should be supplemented with Eq. (26) for the Jacobian of transformation of the Lagrangian to Eulerian coordinates.

First of all, we note that the nonlinear distortion in a statistically uniform random field $v(x,t)$ does not affect the form of its single-point probability density:

$$w_{\rm e}(v;t) = w_0(v). \tag{28}$$

This statement, which seems unexpected at first glance, can easily be understood by using the well-known relation between the probability density of a statistically uniform random function and the behavior of its realizations [14, 15, 25]. As is known, the probability density of the field $v(x,t)$, which is statistically uniform and ergodic with respect to x, can be expressed through the limit of the relative length of stay of the realization $v(x,t)$ in the interval $[v, v + dv]$:

$$w_{\rm e}(v;t) = \lim_{\ell\to\infty} \sum_k {\rm d}x_k \frac{1}{\ell\,{\rm d}v},$$

where ${\rm d}x_k$ is the length of the x-axis intervals within which the values of the field $v(x,t)$ lie in the interval

$[v, v+\mathrm{d}v]$. Due to the linear time dependence of the Lagrangian field $X(y,t)$ (26), the length of each interval $\mathrm{d}x_k$ changes with the time as $\mathrm{d}x_k(t) = \mathrm{d}x_k(0) \pm t\,\mathrm{d}v$, where the minus sign corresponds to the intervals on the steepening parts and the plus sign, to those on the flattening parts of the field profile $v(x,t)$. It is apparent that until the solution of the Riemann equation (1) is single-valued, the sum of lengths of any of the two neighboring intervals is retained: $\mathrm{d}x_k + \mathrm{d}x_{k+1} = \mathrm{const}$, the relative length of stay of the field $v(x,t)$ in any given interval $[v, v+\mathrm{d}v]$ persists, and therefore the single-point probability density of a statistically uniform field remains unchanged, too. Thus, the solution of the inverse problem for the probability density is trivial: the probability density of the Riemann waves at the initial instant $w_0(v)$ coincides with the probability density $w_\mathrm{e}(v;t)$ at the time t. However, it should be mentioned that this statement is true only in the case where the condition $J(x,t) \geq 0$ for the Jacobian (26) is fulfilled in all realizations of the velocity field $v(x,t)$. Note that for the Gaussian statistics of the initial field, this condition is violated for an arbitrarily short time. Nevertheless, the use of the Riemann-wave approximation is possible until the number of discontinuities per unit length is small.

Much more information on the evolution of the spatiotemporal properties of random fields is contained in their spatial spectrum. At the initial stage, when the propagation of the acoustic wave is described by the Riemann equation, the spatial spectrum of a statistically uniform velocity field

$$E(k,t) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} B(s,t) \exp(-iks)\,\mathrm{d}s, \qquad B(s,t) = \langle v(x,t) v(x+s,t) \rangle, \tag{29}$$

where the angle brackets denote the statistical averaging, is equal to [9, 10, 14, 15]

$$E(k,t) = \frac{1}{2\pi (kt)^2} \int\limits_{-\infty}^{+\infty} \exp(-iks) [\theta_2(kt, -kt, s) - \theta_1(kt)\theta_1(-kt)]\,\mathrm{d}s. \tag{30}$$

Here,

$$\theta_2(k_1, k_2, s) = \langle \exp\{i[k_1 v_0(x) + k_2 v_0(x+s)]\} \rangle, \qquad \theta_1(k) = \langle \exp[ikv_0(x)] \rangle \tag{31}$$

are the double- and single-point characteristic functions of the initial field $v_0(x)$, respectively. Apparently, the solution of the inverse problem, namely, the retrieval of the characteristic function of the input signal from the observed-field spectrum, is impossible.

If the initial field $v_0(x)$ has the Gaussian statistics with a zero mean and a given correlation function $B_0(s)$, $B_0(0) = \sigma_0^2$, then the spectral density given by Eq. (30) takes the form [9, 10, 14, 15, 26, 27]

$$E(k,t) = \frac{1}{2\pi (kt)^2} \exp[-(\sigma_0 kt)^2] \int\limits_{-\infty}^{+\infty} \{\exp[B_0(s) k^2 t^2] - 1\} \exp(-iks)\,\mathrm{d}s. \tag{32}$$

Even in this case, the general solution of the inverse problem, i. e., the retrieval of the initial correlation function $B_0(x)$ from the field spectrum $E(k,t)$ at the instant t, is impossible. Nevertheless, the partial retrieval of information on the initial-spectrum characteristics is possible. Note that from Eq. (32) the relation

$$E(0,t) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} B_0(s)\,\mathrm{d}s = \mathrm{const} \tag{33}$$

follows, i. e., the spatial spectrum at the point $x = 0$ is retained, and therefore this integral characteristic of the initial field can be retrieved from the measured value of $E(0,t)$ at the instant t. This invariant plays a special role in the Burgers turbulence theory since at the discontinuity stage, the evolution of its spectral–correlation and probabilistic properties is qualitatively different for $E(0,t) \neq 0$ and $E(0,t) = 0$.

We also note that the rate of nonlinear self-action and generation of new harmonics of the field $v(x,t)$ depends on the spatial frequency k: the smaller k, the slower the processes. From this point of view, invariant (33) is a consequence of the infinite characteristic time of self-action and harmonic generation at a zero frequency. At $\sigma_0 kt \ll 1$, the exponents in Eq. (32) can be expanded in a Taylor series, allowing for only several first terms:

$$E(k,t) = E_0(k) + \frac{1}{2}(kt)^2\big[E_0(k)\otimes E_0(k) - 2\sigma_0^2 E_0(k)\big] + \dots\,, \tag{34}$$

where $E_0(k)$ is the spectrum of the initial field $v_0(x)$ and the $\otimes$ sign denotes the convolution operation. Allowance for only the first term on the right-hand side of Eq. (34) corresponds to neglecting the nonlinear effects, allowance for two terms corresponds to allowance for the nonlinear interaction of pairs of the initial-field harmonics, which leads to the appearance of spectral components with the difference and sum wave numbers (single interaction), and so on.

At the initial stage, expression (34) describes the evolution of the spectrum almost over the entire frequency range. However, even with allowance for the single interaction alone, the retrieval of the initial spectrum $E_0(k)$ from the spectrum $E(k,t)$ at the instant t requires the solution of the nonlinear integral equation. A method for the initial-spectrum retrieval based on the use of bispectra was proposed in [28, 29]. The equation for the correlation function $B(s,t) = \langle v(x,t)v(x+s,t)\rangle$ of a statistically uniform velocity field, which satisfies the Riemann equation (5), is a counterpart of the Karman–Howarth equation [30] in the turbulence theory and contains the third moments of the random field $v(x,t)$. Correspondingly, the equation for the spectral density of the field has the form

$$\frac{\partial E(k,t)}{\partial t} = T(k,t), \tag{35}$$

where the function $T(k,t)$ describes the energy transfer over the spectrum and is expressed through the velocity field bispectra. The bispectrum $S_2(k_1,k_2,t)$ is connected with the Fourier image $c(k,t)$ of the field $v(x,t)$ by the relation

$$S_2(k_1,k_2,t)\delta(k_1+k_2-k_3) = \langle c(k_1,t)c(k_2,t)c^*(k_3,t)\rangle, \tag{36}$$

where $\delta(k)$ is a Dirac delta function. For the Gaussian field, all the bispectra are equal to zero. From the definition of bispectrum (36) it is seen that it reflects in a natural way the nonlinear three-wave interaction of spatial harmonics. At the initial stage of evolution, it is possible to calculate the bispectrum variation and, therefore, find the function $T(k,t)$, which at this stage increases proportionally to the time t. This makes it possible to solve the inverse problem of retrieval of the initial spectral density of the velocity field. From the measured spectrum and bispectra of the velocity field $v(x,t)$ at the instant t using Eq. (35) we can retrieve the initial spectral density of the field [29]:

$$E_0(k) = E(k,t) - tT(k,t)/2. \tag{37}$$

We now consider the behavior of the spectral density of the field $v(x,t)$ for large wave numbers k. In this case, to calculate integral (32) one can make use of the saddle-point technique. Restricting ourselves to the expansion of the correlation function $B_0(s)$ into a Taylor series over s, $B_0(s) = \sigma_0^2(1 - k_1^2 s^2/2! + \dots)$, we find that the spectral density of the Riemann field for $k \to \infty$ decays according to the universal power law:

$$E(k,t) = \frac{\sigma_0^2}{k_1\sqrt{2\pi}\,(\kappa\tau)^3}\exp\left(-\frac{1}{2\tau^2}\right), \qquad \tau = \sigma_0 k_1 t, \qquad \kappa = k/k_1. \tag{38}$$

Such an asymptotic form of the spectrum is due to the presence of singularities of the type $v(x,t) \propto \sqrt{x}$ in the realizations of the field $v(x,t)$. Multi-valued solutions of the Riemann equation indeed have such singularities. Hence, formally, asymptotic form (38) is indicative of a multi-valued solution, and the exponential factor

$\exp[-1/(2\tau^2)]$ is proportional to the average number of multi-valued parts per unit length. Actually, this expression describes the spectrum of the single-valued field constructed in some way from the multi-valued Riemann solution. Expressions (30) and (32) for the spectra were obtained by passing to the Lagrangian coordinates in the expressions for the Fourier images of the velocity field, followed by averaging of the initial field velocity $v_0(x)$ that is statistically uniform with respect to x. Therefore, it is exactly expression (32) that describes the spectrum of the single-valued function, which is obtained by summing the branches of the multi-valued Riemann solution with respective signs [31]. Correspondingly, the Riemann wave energy calculated as the integral of (32) over spatial frequencies decays with the time. In [32], it was shown that expression (32) gives a qualitatively correct description of the random-wave decay because of the discontinuity formation.

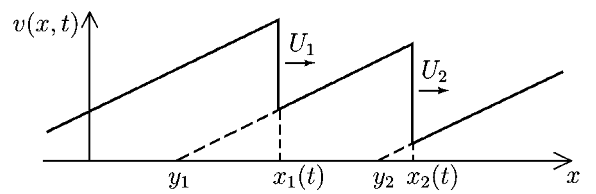

Fig. 3. Structure of the velocity field at the stage of developed discontinuities.

Let us discuss the behavior of the acoustic turbulence at times much greater than the characteristic time of the discontinuity formation. At long times, the parabola in Eq. (13) becomes a smooth function compared with the initial distribution of the potential $s_0(x)$, and the coordinates of its global maxima virtually coincide with the coordinates of some local minima of the dependence $s_0(x)$, i.e., the nulls of the velocity field $v_0(x)$ with a positive derivative. Consequently, at long times, the continuous random field transforms into a random sawtooth wave, that is, a sequence of cells with universal behavior of the field within the cell, $v(x,t) = (x - y_k)/t$, but with a random position of the discontinuities separating them (Fig. 3).

Thus, as a result of the discontinuity formation, the whole information on the fine structure of the field is lost, and by the discontinuity velocity it is possible to determine only the integral characteristic of the initial field, that is, the integral of the initial velocity field over the interval $[y_k^+, y_k^-]$ between the nulls of the initial field (see Eq. (19)). The multiple fusion of cells leads to an increase in the total scale of turbulence $L(t)$ (average mass of a macro particle) and loss of information on the fine structure of the spectrum. The rate of coalescence of discontinuities is determined by statistical characteristics of the discontinuity velocities, which, in turn, depend on the initial potential (see Eq. (18)). A common property of turbulence at large times is the onset of a statistical self-similarity [9, 10, 14, 15, 33]. In particular, for the energy spectrum there is a relationship

$$E(k,t) = \frac{L^3(t)}{t^2}\tilde{E}[kL(t)]. \tag{39}$$

For the turbulence energy, from Eq. (39) we obtain

$$E(t) = \langle v^2(x,t)\rangle = \int_{-\infty}^{+\infty} E(k,t)\,\mathrm{d}k = c^*\frac{L^2(t)}{t^2}, \tag{40}$$

where c^* is the constant determined by the integral of the dimensionless function $\tilde{E}(z)$, which describes the shape of the velocity field spectrum. Thus, at the stage of developed discontinuities, the properties of the turbulence are determined by a single parameter, namely, the outer scale of turbulence $L(t)$. In what follows, we discuss which statistical characteristics of the initial field can be retrieved from the turbulence characteristics at some instant t.

From the solution of the Burgers equation it is seen that at long times, the turbulence properties are determined by the statistical properties of the increments $\Delta s_0 = s_0(x+L) - s_0(x)$ for large separations L. For the Gaussian statistics of the initial field, the properties of the acoustic turbulence are determined by the asymptotic behavior of the structure function $D_{0\mathrm{s}}(x) = \langle [s_0(x+y) - s_0(y)]^2\rangle$ of the initial potential. Thus, the scenario of the turbulence development is determined by the behavior of the large-scale part of the initial energy spectrum of the velocity field. Assume that the spectral density at large scales (i.e., at small wave numbers k) is power-law,

$$E_0(k) = \alpha^2 \, |k|^n \, b_0(k), \tag{41}$$

where n is the power-law exponent, the function $b_0(k) \approx 1$ in the wave-number range $k < k_0$ and decays rather rapidly for $k > k_0$. The behavior of the structure function at long distances is determined by the low-frequency part of the velocity field spectrum. For $n > 1$, the initial potential $s_0(x)$ in Eq. (13) is statistically uniform and has a finite variance:

$$\sigma_{\rm s}^2 \equiv \langle s_0^2(x) \rangle = \int\limits_{-\infty}^{+\infty} \frac{E_0(k)}{k^2} \, {\rm d}k. \tag{42}$$

If $n < 1$, then the variance of the potential is unbounded, and the potential itself is characterized by a statistically uniform structure function, which exists only for $n > -1$:

$$D_{0{\rm s}}(x) \approx \alpha^2 \, |x|^{1-n}, \qquad |x| \to \infty. \tag{43}$$

Let us estimate qualitatively the law of increase in the turbulence scale $L(t)$ from the asymptotic solution of the Burgers equations (12) and (13). The growth rate of the potential s_0 can be estimated as the root of the structure function (43), and the characteristic scale $L(t) \approx |x - y|$. For a fixed coordinate x, the maximum in Eq. (13) corresponds to the point $y(x,t)$, for which the increments of the potential $s_0(y)$ are comparable with the variation in the parabolic term $(x - y)^2/(2t)$. This leads ot the following equation for the turbulence scale $L(t)$:

$$\sqrt{D_{0{\rm s}}(L)} \sim L^2/t. \tag{44}$$

Thus, depending on the exponent n of the initial spectrum (41) for long times, we have two qualitatively different laws of increase in the spatial scale $L(t)$ and, therefore, decay of the energy $E(t)$. For $n > 1$, we have the relations

$$L(t) \approx (\sigma_{\rm s} t)^{1/2}, \qquad E(t) \approx \sigma_{\rm s}/t. \tag{45}$$

In this case, by measurements of the scale $L(t)$ or energy $E(t)$ at the instant t, we can estimate the variance $\sigma_{\rm s}^2$ of the initial potential. The spectrum of the velocity field has a universal asymptotic form both in the range of low ($E(k,t) \propto k^2 t^{1/2}$) and high spatial frequencies ($E(k,t) \propto k^{-2} t^{-3/2}$), and information on the shape of the initial spectrum is completely lost. The initial stage of the onset of the self-similar stage was observed in the experiments on propagation of intense acoustic noise in the pipes [34]. Note that for the one-dimensional Burgers turbulence with limited variance of the initial potential distribution (as well as the three-dimensional Burgers equation, which is used for a model description of the large-scale structure of the Universe), we are able to give a comprehensive statistical description [9, 10, 14, 15, 35]. In particular, we found the single- and double-point probability distributions of the velocity field, N-point probability distributions, and, correspondingly, multi-point moment functions of the velocity field. Such an analysis can be performed since at long times, the parabola in Eq. (13) becomes a smooth function compared with the initial distribution of the potential $s_0(x)$, and a large number of local minima of the dependence $s_0(x)$ compete for being the absolute minimum in this equation (13). This made it possible to use the statistical theory of extreme values [36] for analysis of the Burgers turbulence at long times. In a rigorous treatment, a slowly varying logarithmic factor appears in the expressions for the outer scale of turbulence and its energy (45).

For the unbounded initial variance ($n < 1$) from Eq. (44) we have the relations

$$L(t) \approx (\alpha t)^{2/(3+n)}, \qquad E(t) \approx (\alpha)^{4/(3+n)} t^{-2(n+1)/(3+n)}. \tag{46}$$

Thus, both an increase in the spatial scale of turbulence $L(t)$ (a decrease in the width k^* of the spatial spectrum $E(k,t)$, $k^*(t) \approx 1/L(t)$) and the energy decay occur according to the power law as t increases. Consequently, by these dependences, it is possible, in principle, to determine which type of initial conditions corresponds to these laws of increase in the scale $L(t)$ and decay of the energy $E(t)$, and find both the exponent n and "amplitude" α^2 of the initial spectrum for $n < 1$. Thus, in this case, we have information only on the low-frequency part of the spectrum, and the spectrum $E(k,t)$ in the range of low spatial frequencies repeats the initial spectrum $E_0(k)$. Moreover, for $n < 1$, the high-frequency part of the spectrum almost does not affect the evolution of the turbulence at long times [37], and not only the energy spectrum in the range of low spatial frequencies, but also the low-frequency components themselves are thus retained. In [37], it was shown that at long times, two signals with identical low-frequency part of the spectrum, but significantly different high-frequency components (such that their cross-correlation coefficient $r_{12}(0) \approx 0$) transforms into sawtooth waves with the cross-correlation coefficient $r_{12}(t) \approx 1$. This means that after the filtering of the sawtooth wave it is possible to retrieve the low-frequency component of the initial field.

A more rigorous treatment shows [38] that the case is nontrivial for $1 < n < 2$. At $n < 2$, the initial spectrum $E(k,t)$ is retained in the range of small wave numbers k, which is the spectral form of the "permanence of large eddies" (PLE) principle [23, 38]. In the Fourier space, the assumption of self-similarity (39) in combination with the PLE yield the same laws of the outer-scale increase and the turbulence-energy decay as those which follow from Eqs. (46). However, the applicability of this law is limited by the condition $n \leq 1$. Thus, for $1 < n < 2$, at long but finite times, the self-similarity is partially lost: an increase in the outer scale $L(t)$ and a decay of the turbulence energy $E(t)$ are determined, as previously, by expression (45), and the main part of the spectrum shows the same behavior as for $n \geq 2$. However, the low-frequency part of the spectrum is retained, $E(k,t) = E_0(k) = \alpha^2 \, |k|^n$, and transforms into the universal law $E(k,t) \propto k^2 t^{1/2}$. As the time proceeds, the boundary of the spectrum kink shifts towards the increasingly low spatial spectra faster than the coordinate of the maximum of the spectrum decreases, $k_{\mathrm{m}}(t) \propto 1/L(t) \propto 1/t$. Thus, a self-similar regime is asymptotically established in this case, too, and therefore from the laws of the outer-scale increase and turbulence-energy decay it is possible to retrieve only the variance of the initial potential distribution $s_0(x)$.

5. CONCLUSIONS

In this paper, we have considered the inverse problems of nonlinear acoustics and acoustic turbulence. On the basis of a one-dimensional Burgers equation for zero and infinitesimal viscosities, we discuss the fundamental possibility of retrieval of the input-signal parameters from measurements of the signal far from the source. At the initial stage, when the field is described by the Riemann equation, the complete retrieval of the initial wave is possible. At the discontinuity stage, from the field it is possible to retrieve only the integral characteristics of the initial field, i. e., information on the fine structure of the initial wave is lost. A common property of the noise signals is the onset of statistical self-similarity and, in particular, loss of information on the fine structure of the initial spectrum. It is shown that if the large-scale components of the spectrum are absent in the initial signal, then using the laws of the outer-scale increase and turbulence-energy decay, one can retrieve only the variance of the initial distribution of the potential, that is, the integral of the velocity field. If the large-scale components were fairly intense in the initial field, then, despite the strong nonlinear distortions of the wave profile and the discontinuity formation, it is possible to retrieve the large-scale components of the initial signal.

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