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Sums of logarithmic weights involving *r*-full numbers

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Abstract
Let (n, q) denote the greatest common divisor of positive integers n and q, and let f_r denote the characteristic function of r-full numbers. We consider several asymptotic formulas for sums of the modified square-full ($r = 2$) and cube-full numbers ($r = 3$), which is $\sum_{n\le y}\sum_{q\le x}\sum_{d|(n,q)} df_r\left(\frac{q}{d}\right)\log\frac{x}{q}$ with any positive real numbers x and y. Moreover, we derive the asymptotic formula of the above with $r = 2$ under the Riemann Hypothesis.

Keywords Square full numbers · Cube full numbers · Riemann zeta-function · Divisor function · Riemann hypothesis · Asymptotic results on arithmetical functions

Mathematics Subject Classification 11A25 · 11N37 · 11P99

1 Introduction

Let $s = \sigma + it$ be the complex variable, and let $\zeta(s)$ be the Riemann zeta-function. Let $r(\ge 2)$ be an integer, we call n an r-full or r-free integer if $p|n \Rightarrow p^r|n$ or $p|n \Rightarrow p^r \nmid n$, respectively. In the special case when $r = 2$ or 3 integer. We call n a square-full or cube-full numbers, respectively. Let $G(r)$ denote the set of r-full numbers, and let (n, q) denote the greatest common divisor of positive integers n and q. Define

$$f_r(n) := \begin{cases} 1 & \text{if } n \in G(r), \\ 0 & \text{if } n \notin G(r), \end{cases}$$

and

$$s_q^{(r)}(n) := \sum_{d|(n,q)} df_r\left(\frac{q}{d}\right). \tag{1.1}$$

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It is worth mentioning that the above sum is an analogue of the Ramanujan sum $c_q(n) = \sum_{d|(n,q)} d\mu\,(q/d)$, with μ being the Möbius function. For the case $r = 2$ and $r = 3$, the Dirichlet series of the function $s_q^{(r)}(n)$ is given by

$$\sum_{q=1}^{\infty} \frac{s_q^{(2)}(n)}{q^s} = \sigma_{1-s}(n)\frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} \tag{1.2}$$

for Re $s > \frac{1}{2}$, and

$$\sum_{q=1}^{\infty} \frac{s_q^{(3)}(n)}{q^s} = \sigma_{1-s}(n)\frac{\zeta(3s)\zeta(4s)\zeta(5s)\kappa_9(s)}{\zeta(8s)} \tag{1.3}$$

for Re $s > \frac{1}{3}$. Here $\sigma_{1-s}(n) = \sum_{d|n} d^{1-s}$, and the function $\kappa_9(s)$ is absolutely convergent for Re $s > \frac{1}{9}$, that is

$$\kappa_9(s) = \frac{\zeta(13s)\zeta(14s)\zeta(21s)\zeta^2(22s)\zeta^2(23s)\zeta(24s)\cdots}{\zeta(9s)\zeta(10s)\zeta(17s)\zeta(18s)\zeta(19s)\zeta(25s)\zeta^3(26s)\cdots}$$

(see (1.96), (1.97) and (1.98) in [5]). For any large positive real numbers x and y, and any non-negative integer k, we are interested by studying the double sums

$$S_k^{(r)}(x, y) := \frac{1}{k!}\sum_{n\le y}\sum_{q\le x} s_q^{(r)}(n)\left(\log\frac{x}{q}\right)^k. \tag{1.4}$$

In this paper, we shall consider the asymptotic formulas for $S_k^{(r)}(x, y)$ when $r = 2, 3$. In the case $k = 0$ and $r = 2$, the author [6] used the method of Chan and Kumchev [1] (see also [9], [11]) [1] and the theory of exponent pairs (see [3], [5])) to deduce the asymptotic formula to $S_0^{(2)}(x, y)$. It is shown that

$$S_0^{(2)}(x, y) = \frac{\zeta(2)\zeta(3)}{\zeta(6)}xy - \frac{\zeta(4)\zeta(6)}{4\zeta(12)}x^2 + O\left(x^{\frac{1}{2}}y + xy^{\frac{1}{3}} + \frac{x^3}{y}\right) \tag{1.5}$$

holds, where x and y are large real numbers such that $x \ll y \ll x^{\frac{3}{2}}$.

Recently, the author [7] gave a more precise asymptotic for $S_0^{(2)}(x, y)$ by using Lemma 2.2 below and some properties of the Riemann zeta-function. He proved that

$$S_0^{(2)}(x, y) = \frac{\zeta(2)\zeta(3)}{\zeta(6)}xy + \frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)}x^{\frac{1}{2}}y - \frac{\zeta(4)\zeta(6)}{4\zeta(12)}x^2$$

[1] Cohen–Ramanujan sums were first developed in [9] and then their moments were studied in [11] following the technique pioneered by Chan and Kumchev [1].

$$+ O\left(x^{\frac{4}{9}} y \log^4 x + x y^{\frac{1}{3}} \log^2 y + x^2 \left(\frac{x}{y}\right)^{\frac{1}{2}} \log^{\frac{3}{2}} x\right) \tag{1.6}$$

holds, where x and y are large real numbers such that $x^{\frac{4}{3}} \log x \ll y \ll \frac{x^{\frac{14}{9}}}{\log^4 x}$. Moreover, for $k = 0$ and $r = 3$, the author [8] showed that

$$S_0^{(3)}(x, y) = \frac{\zeta(3)\zeta(4)\zeta(5)\kappa_9(1)}{\zeta(8)} xy - \frac{\zeta(6)\zeta(8)\zeta(10)\kappa_9(2)}{4\zeta(16)} x^2$$
$$+ O\left(x^{\frac{1}{3}} y + x y^{\frac{1}{3}} + \frac{x^3}{y}\right) \tag{1.7}$$

holds, where x and y denote large real numbers such that $x \ll y \ll x^{\frac{5}{3}}$. From the above, we notice that it is difficult to improve the error because the term $O\left(x^{\frac{1}{3}} y\right)$ is absorbed into all error terms. For this reason, in this paper, we consider asymptotic formulas for $S_1^{(r)}(x, y)$ and give the interesting relation between $S_0^{(r)}(x, y)$ and $S_1^{(r)}(x, y)$ for $r = 2, 3$. It is the most interesting problem for us to derive asymptotic formulas of (1.4) when $k = 0, 1$, and by a similar argument, we may prove that any cases $k(\geq 2)$. Before going into the statements of our theorems, we denote the Fourier integrals $\nu(u)$ and $\xi(u)$ defined by

$$\nu(u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\zeta(-\frac{1}{2} - it)\zeta(5 + 2it)\zeta(\frac{15}{2} + 3it)}{\zeta(15 + 6it)} \frac{e^{itu}}{(\frac{5}{2} + it)^2(\frac{1}{2} + it)} dt, \tag{1.8}$$

and

$$\xi(u) :=$$
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\zeta(-\frac{1}{2} - it)\zeta(\frac{15}{2} + 3it)\zeta(10 + 4it)\zeta(\frac{25}{2} + 5it)\kappa_9(\frac{5}{2} + it)}{\zeta(20 + 8it)} \frac{e^{itu}}{(\frac{5}{2} + it)^2(\frac{1}{2} + it)} dt \tag{1.9}$$

with $u = \log \frac{x}{y}$, respectively. It follows from (3.6) and (5.5) below that

$$|\nu(u)| \leq \frac{4}{(2\pi)^2} \frac{\zeta(\frac{3}{2})\zeta(5)\zeta(\frac{15}{2})\zeta(15)}{\zeta(30)} \left(\frac{\pi}{4} + 1\right)$$

and

$$|\xi(u)| \leq \frac{4}{(2\pi)^2} \frac{\zeta(\frac{3}{2})\zeta(\frac{15}{2})\zeta(10)\zeta(\frac{25}{2})\zeta(20)\kappa_9(\frac{5}{2})}{\zeta(40)} \left(\frac{\pi}{4} + 1\right)$$

hold. Here the integrals are computable constants, and, strictly speaking, that is enough for the purpose of this paper. Then we have the following results:

Theorem 1 *Let the notation be as above. Let x and y be large real numbers such that $x\log^3 x \ll y \ll x^{\frac{14}{9}}$. Then we have*

$$S_1^{(2)}(x,y) = \frac{\zeta(2)\zeta(3)}{\zeta(6)}xy + 2\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)}x^{\frac{1}{2}}y + 3\frac{\zeta(\frac{1}{3})\zeta(\frac{2}{3})}{\zeta(2)}x^{\frac{1}{3}}y - \frac{1}{8}\frac{\zeta(4)\zeta(6)}{\zeta(12)}x^2 - x^2\left(\frac{x}{y}\right)^{\frac{1}{2}}\nu(u) + E_1^{(2)}(x,y), \tag{1.10}$$

where the function $\nu(u)$ is given by (1.8) and the error term $E_1^{(2)}(x,y)$ is estimated by

$$E_1^{(2)}(x,y) = O\left(yx^{\frac{1}{6}}\exp\left(-C\frac{(\log x)^{\frac{1}{3}}}{(\log\log x)^{\frac{1}{3}}}\right) + xy^{\frac{1}{3}}\log^2 y\right) \tag{1.11}$$

with C being a positive constant.

Remark 1.1 Using (1.5) and (1.10) we deduce the relation

$$\frac{1}{xy}\left(S_1^{(2)}(x,y) - S_0^{(2)}(x,y)\right) = \frac{1}{8}\frac{\zeta(4)\zeta(6)}{\zeta(12)}\frac{x}{y} + O\left(x^{-\frac{1}{2}} + \frac{x^2}{y^2}\right)$$

for $x\log^3 x \ll y \ll x^{\frac{3}{2}}$. It follows from (1.10) and (1.11) that

$$\frac{1}{xy}\sum_{n\le y}\sum_{q\le x} s_q^{(2)}(n)\log\frac{x}{q} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} + 2\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)}x^{-\frac{1}{2}} + 3\frac{\zeta(\frac{1}{3})\zeta(\frac{2}{3})}{\zeta(2)}x^{-\frac{2}{3}} - \frac{1}{8}\frac{\zeta(4)\zeta(6)}{\zeta(12)}\frac{x}{y} + O\left(x^{-\frac{5}{6}}\log^5 x + y^{-\frac{2}{3}}\log^2 y\right)$$

holds. This means that the logarithmic average order of $s_q^{(2)}(n)$ is $\frac{\zeta(2)\zeta(3)}{\zeta(6)}$ where q and n satisfying the condition $q\log^3 q \ll n \ll q^{\frac{14}{9}}$.

In fact, it is suspected that there is a deep relationship between a zero-free region of the Riemann zeta-function and the order of magnitude of the error term (1.11). Then we immediately obtain

Conjecture 1 *We may conjecture that*

$$E_1^{(2)}(x,y) = O\left(yx^{\frac{1}{6}}\exp\left(-C\frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right) + xy^{\frac{1}{3}}\log^2 y\right)$$

holds with an absolute constant $C > 0$.

Theorem 2 *Let the notation be as above. Let x and y be large real numbers such that* $x^{\frac{6}{5}}\log^3 x \ll y \ll x^{\frac{19}{12}}$. *Then we have*

$$\begin{aligned} S_1^{(3)}(x,y) &= \frac{\zeta(3)\zeta(4)\zeta(5)\kappa_9(1)}{\zeta(8)}xy + 3\frac{\zeta(\frac{1}{3})\zeta(\frac{4}{3})\zeta(\frac{5}{3})\kappa_9\left(\frac{1}{3}\right)}{\zeta(\frac{8}{3})}x^{\frac{1}{3}}y + 4\frac{\zeta(\frac{1}{4})\zeta(\frac{3}{4})\zeta(\frac{5}{4})\kappa_9\left(\frac{1}{4}\right)}{\zeta(\frac{8}{4})}x^{\frac{1}{4}}y \\ &+ 5\frac{\zeta(\frac{1}{5})\zeta(\frac{3}{5})\zeta(\frac{4}{5})\kappa_9\left(\frac{1}{5}\right)}{\zeta(\frac{8}{5})}x^{\frac{1}{5}}y - \frac{1}{8}\frac{\zeta(6)\zeta(8)\zeta(10)\kappa_9(2)}{\zeta(16)}x^2 - x^2\left(\frac{x}{y}\right)^{\frac{1}{2}}\xi(u) \\ &+ O\left(yx^{\frac{1}{8}}\exp\left(-C\frac{(\log x)^{\frac{1}{3}}}{(\log\log x)^{\frac{1}{3}}}\right) + xy^{\frac{1}{3}}\log^2 y\right), \end{aligned} \tag{1.12}$$

where the function $\xi(u)$ *is given by (1.9) and* C *is a positive constant.*

Remark 1.2 Similarly as in Remark 1.1, we have

$$S_1^{(3)}(x,y) - S_0^{(3)}(x,y) = \frac{\zeta(6)\zeta(8)\zeta(10)\kappa_9(2)}{8\zeta(16)}x^2 + O\left(yx^{\frac{1}{3}} + xy^{\frac{1}{3}}\log^2 y + \frac{x^3}{y}\right)$$

for $x^{\frac{6}{5}}\log^3 x \ll y \ll x^{\frac{5}{3}}$, and the logarithmic average order of $s_q^{(3)}(n)$ is derived by $\frac{\zeta(3)\zeta(4)\zeta(5)\kappa_9(1)}{\zeta(8)}$ under q and n satisfying the condition $q^{\frac{6}{5}}\log^3 q \ll n \ll q^{\frac{19}{12}}$.

Next, we assume the truth of the unproved Riemann Hypothesis, that all the complex zeros of the Riemann zeta-function $\zeta(s)$ lie on the line $\sigma = \frac{1}{2}$. We consider the precise asymptotic formula concerning $S_1^{(2)}(x,y)$. Then we derive the following

Theorem 3 *Assume that the Riemann Hypothesis is true. Let x and y be large real numbers such that* $x\log^3 x \ll y \ll x^{\frac{29}{18}}\exp\left(-A\frac{\log x}{\log\log x}\right)$. *Then the error term* $E_1^{(2)}(x,y)$ *of (1.10) is estimated by*

$$E_1^{(2)}(x,y) = O\left(yx^{\frac{1}{12}}\exp\left(A\frac{\log x}{\log\log x}\right) + xy^{\frac{1}{3}}\log^2 y\right) \tag{1.13}$$

with A being a positive constant.

In addition, we assume that all the zeros ρ of the Riemann zeta-function $\zeta(s)$ on the critical line are simple, where $\rho = \frac{1}{2} + i\gamma$ denotes a nontrivial zero of the Riemann zeta-function, and γ denotes the imaginary part of zero on the critical line. Then we may derive a sum involving the zeros ρ of $\zeta(s)$ concerning $E_1^{(2)}(x,y)$. To improve the order of magnitude of its sum, we make use of the Gonek-Hejhal Hypothesis (Gonek [2], and Hejhal [4] independently conjectured), namely

$$J_{-\lambda}(T) := \sum_{0<\gamma\le T}\frac{1}{|\zeta'(\rho)|^{2\lambda}} \asymp T(\log T)^{(\lambda-1)^2}$$

for real number $\lambda < \frac{3}{2}$, where $\zeta'(s)$ is the first derivative of $\zeta(s)$, then we may deduce a new estimate of $E_1^{(2)}(x,y)$, which will be done elsewhere.

Notations. Throughout this paper, we use the following notations: The Riemann zeta-function $\zeta(s)$, defined by $\sum_{n=1}^{\infty}\frac{1}{n^s}$ for $\sigma > 1$, admits of analytic continuation over the whole complex plane having as its only singularity a simple pole with residue 1 at $s = 1$. In what follows, C donotes any arbitrarily positive number, not necessarily the same ones at each occurrence.

2 Some Lemmas

Lemma 2.1 *Suppose that the Dirichlet series* $\alpha(s) := \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ *converges for* $\operatorname{Re} s > \sigma_c$. *If* $\sigma_0 > \max(0, \sigma_c)$ *and* $x > 1$, *then*

$$\sum_{n\le x} a_n \log\frac{x}{n} = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\alpha(s)\frac{x^s}{s^2}ds.$$

Proof This is Riesz typical means of Perron's formula. For more details, see (5.20)–(5.22) in [10]. □

Lemma 2.2 *Let* $\operatorname{Re} z \le 0$, *and let* $\sigma_{z,b}(n)$ *denote the generalization of the divisor function defined by* $\sigma_{z,b}(n) = \sum_{d^b|n} d^{bz}$. *Then we have*

$$\sideset{}{'}\sum_{n\le x}\sigma_{z,b}(n) = D_{z,b}(x) + \Delta_{z,b}(x),$$

where $\sum'$ *indicates that the last term is to be halved if* x *is an integer, and*

$$\Delta_{z,b}(x) = O\left(x^{\frac{1}{3}}\log^2 x\right)$$

uniformly for $b \ge 1$ *and* $D_{z,b}(x)$ *is given by the following*

(i) If $b = 1, 2$ *and* $-\frac{2}{3b^2} < \operatorname{Re} z \le 0$, *then*

$$D_{z,b}(x) = \zeta(b(1-z))x + \frac{1}{1+bz}\zeta\left(z+\frac{1}{b}\right)x^{z+\frac{1}{b}}.$$

(ii) If $b \ge 3$ *and* $-1 < \operatorname{Re} z \le 0$, *then*

$$D_{z,b}(x) = \zeta(b(1-z))x.$$

Proof The proof of this result can be found in [Theorem 1.4, [11]]. □

Lemma 2.3 *There is an absolute constant* $C > 0$ *such that* $\zeta(s) \ne 0$ *for*

$$\sigma \ge 1 - C(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}} \quad (t \ge t_0).$$

Proof This lemma is given by Theorem 6.1 in [5]. □

Lemma 2.4 *For $|t| \geq 2$ and $\sigma \geq 1 - C(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}}$ we have*

$$\zeta(\sigma+it) \ll (\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}} \quad \text{and} \quad \frac{1}{\zeta(\sigma+it)} \ll (\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}.$$

Proof The first term of this lemma is a well-known result. The second term of this lemma is given by Lemma 12.3 in [5]. □

Lemma 2.5 *For $t \geq t_0 > 0$ uniformly in σ, we have*

$$\zeta(\sigma+it) \ll \begin{cases} t^{\frac{1}{6}(3-4\sigma)}\log t & \left(0 \leq \sigma \leq \frac{1}{2}\right), \\ t^{\frac{1}{3}(1-\sigma)}\log t & \left(\frac{1}{2} \leq \sigma \leq 1\right), \\ \log t & \left(1 \leq \sigma \leq \frac{3}{2}\right), \\ 1 & \left(\sigma > \frac{3}{2}\right). \end{cases}$$

Proof The proof of this lemma follows from Theorem II.3.8 in [12] (see also [5], [13]). □

Lemma 2.6 *For any positive number $T > 1$ we have*

$$\int_1^T |\zeta(\sigma+it)|^4 dt \ll \begin{cases} T^{3-4\sigma} & \left(0 < \sigma < \frac{1}{2}\right), \\ T\log^4 T & \left(\sigma = \frac{1}{2}\right). \\ T & \left(\sigma > \frac{1}{2}\right), \end{cases} \tag{2.1}$$

Proof The second and third terms of (2.1) are due to Theorem 5.1 and Theorem 8.5 in [5]. We use (2.2) below and the formula $\int_1^T |\zeta(\sigma+it)|^4 dt = O(T)$ for $\frac{1}{2} < \sigma \leq 1$ to deduce (2.1). □

Lemma 2.7 *Assume that the Riemann hypothesis is true. Then we have*

$$\zeta(\sigma+it) \ll t^{\varepsilon} \quad \text{and} \quad \frac{1}{\zeta(\sigma+it)} \ll t^{\varepsilon}$$

for every σ $(\frac{1}{2}+\delta \leq \sigma \leq 2)$ and $t \geq t_0$ being a sufficiently large real number.

Proof The first and second terms of this lemma are given by (14.2.5), (14.2.6), (14.14.1) and (14.16.2) in [13], respectively. □

The next lemma is a well-known result (see [5], [13]), that is

Lemma 2.8 *The functional equation of the Riemann zeta-function is given by*

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{2.2}$$

where $\chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)$. Thus in any bounded vertical strip, we have

$$|\chi(s)| \asymp \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma}\left(1 + O\left(\frac{1}{t}\right)\right).$$

3 Proof of Theorem 1

We assume that $1 \le y \le x^M$ for some constant M. Without loss of generality we can assume that $x,\ y \in \mathbb{Z} + \frac{1}{2}$. We apply Lemma 2.1 with (1.2) to get

$$\sum_{q \le x} s_q^{(2)}(n) \log \frac{x}{q} = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \sigma_{1-s}(n) \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} \frac{x^s}{s^2} ds + O\left(\sigma_0(n) \frac{x}{T}\right),$$

where $\alpha = 1 + \frac{1}{\log x}$. Let T be a real parameter at our disposal. We have

$$\sum_{n \le y} \sum_{q \le x} s_q^{(2)}(n) \log \frac{x}{q} = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \sum_{n \le y} \sigma_{1-s}(n) \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} \frac{x^s}{s^2} ds + O\left(\frac{x}{T} \sum_{n \le y} \sigma_0(n)\right).$$

Taking $b = 1$ and $z = 1 - s$ into Lemma 2.2 and using the estimate $\sum_{n \le y} \sigma_0(n) \ll y \log y$ we have

$$S_1^{(2)}(x, y) = K_1 + K_2 + O\left(x y^{\frac{1}{3}} \log^2 y\right) + O\left(\frac{xy}{T} \log y\right), \tag{3.1}$$

where

$$K_1 := \frac{y}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{\zeta(s)\zeta(2s)\zeta(3s)}{\zeta(6s)} \frac{x^s}{s^2} ds,$$

and

$$K_2 := \frac{y^2}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{\zeta(2 - s)\zeta(2s)\zeta(3s)}{\zeta(6s)} \frac{(x/y)^s}{s^2(2 - s)} ds.$$

Define

$$\varepsilon(T) := \frac{C}{100} (\log T)^{-\frac{2}{3}} (\log \log T)^{-\frac{1}{3}} \tag{3.2}$$

with C being the same as that in Lemma 2.3 and $T = x^2$. Let $\Gamma(\alpha, \beta, T)$ denote the contour consisting of the line segments $[\alpha - iT, \beta - iT]$, $[\beta - iT, \beta + iT]$ and $[\beta + iT, \alpha + iT]$.

In K_1, we move the integration, with respect to s, to $\Gamma(\alpha, \beta, T)$ with $\beta = \frac{1}{6} - \varepsilon(T)$. We denote the integrals over the horizontal line segments by $K_{1,1}$ and $K_{1,3}$, and the integral over the vertical line segment by $K_{1,2}$, respectively. We use Lemmas 2.3–2.5 to deduce

$$K_{1,1},\ K_{1,3}$$
$$\ll \frac{y}{T^2 \varepsilon(T)} \left(\int_\beta^{\frac{1}{3}} + \int_{\frac{1}{3}}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{\alpha}\right) |\zeta(\sigma + iT)||\zeta(2(\sigma + iT))||\zeta(3(\sigma + iT))| x^\sigma d\sigma$$

$$\ll \frac{y}{T^2}\frac{\log^3 T}{\varepsilon(T)}\left(T^{\frac{3}{2}}\int_{\beta}^{\frac{1}{3}}\left(\frac{x}{T^4}\right)^{\sigma}d\sigma + T^{\frac{5}{6}}\int_{\frac{1}{3}}^{\frac{1}{2}}\left(\frac{x}{T}\right)^{\sigma}d\sigma + T^{\frac{1}{3}}\int_{\frac{1}{2}}^{\alpha}\left(\frac{x}{T^{\frac{1}{3}}}\right)^{\sigma}d\sigma\right)$$
$$\ll \frac{y}{T}\frac{\log^3 T}{\varepsilon(T)}\left(\frac{x^{\frac{1}{6}}}{T^{\frac{1}{6}}} + \frac{x^{\frac{1}{3}}}{T^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}}}{T^{\frac{2}{3}}} + \frac{x}{T}\right)$$

and

$$K_{1,2} \ll yx^{\frac{1}{6}-\varepsilon(T)}\left(\int_{|t|\le T_0} + \int_{T_0<|t|\le T}\right)\times$$
$$\times \frac{|\zeta(\frac{1}{6}-\varepsilon(T)+it)||\zeta(\frac{1}{3}-2\varepsilon(T)+2it)||\zeta(\frac{1}{2}-3\varepsilon(T)+3it)|}{|\zeta(1-6\varepsilon(T)+6it)|(1+|t|)^2}dt$$
$$\ll \frac{yx^{\frac{1}{6}-\varepsilon(T)}}{\varepsilon(T)} + yx^{\frac{1}{6}-\varepsilon(T)}\int_{T_0<|t|\le T}\frac{t^{\frac{5}{6}+4\varepsilon(T)}}{|\zeta(1-6\varepsilon(T)+6it)|t^2}dt$$
$$\ll \frac{yx^{\frac{1}{6}-\varepsilon(T)}}{\varepsilon(T)}.$$

It remains to evaluate the residues of the poles of the integrand, and there exist three simple poles at $s = 1$, $\frac{1}{2}$ and $\frac{1}{3}$ with residues $\frac{\zeta(2)\zeta(3)}{\zeta(6)}x$, $\frac{2\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)}x^{\frac{1}{2}}$, and $\frac{3\zeta(\frac{1}{3})\zeta(\frac{2}{3})}{\zeta(2)}x^{\frac{1}{3}}$, respectively. Therefore, we have

$$K_1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)}xy + \frac{2\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)}x^{\frac{1}{2}}y + \frac{3\zeta(\frac{1}{3})\zeta(\frac{2}{3})}{\zeta(2)}x^{\frac{1}{3}}y$$
$$+ O\left(yx^{\frac{1}{6}}\exp\left(-C\frac{(\log x)^{\frac{1}{3}}}{(\log\log x)^{\frac{1}{3}}}\right)\right) \tag{3.3}$$

by setting $T = x^2$, where C is a positive constant.

In K_2, we move the integration, with respect to s, to $\Gamma(\alpha, \frac{5}{2}, T)$. We denote the integrals over the horizontal line segments by $K_{2,1}$ and $K_{2,3}$, and the integral over the vertical line segment by $K_{2,2}$, respectively. Using Lemmas 2.5 and 2.8 we have

$$K_{2,1}, K_{2,3} \ll \frac{y^2}{T^3}\int_{\alpha}^{\frac{5}{2}}|\zeta(2-\sigma-iT)|\left(\frac{x}{y}\right)^{\sigma}d\sigma$$
$$\ll \frac{y^2}{T^3}\left(\int_{\alpha}^{2}|\zeta(2-\sigma-iT)|\left(\frac{x}{y}\right)^{\sigma}d\sigma + \int_{2}^{\frac{5}{2}}|\zeta(\sigma-1+iT)\chi(2-\sigma-iT)|\left(\frac{x}{y}\right)^{\sigma}d\sigma\right)$$
$$\ll \frac{y^2}{T^3}\log T\left(T^{-\frac{1}{2}}\int_{\alpha}^{2}\left(\frac{T^{\frac{1}{2}}x}{y}\right)^{\sigma}d\sigma + T^{-\frac{3}{2}}\int_{2}^{\frac{5}{2}}\left(\frac{Tx}{y}\right)^{\sigma}d\sigma\right)$$
$$\ll \frac{y^2\log T}{T^3}\left(\frac{x}{y} + T^{\frac{1}{2}}\left(\frac{x}{y}\right)^2 + T\left(\frac{x}{y}\right)^{\frac{5}{2}}\right), \tag{3.4}$$

and

$$K_{2,2} = \frac{y^2}{2\pi i}\int_{\frac{5}{2}-i\infty}^{\frac{5}{2}+i\infty} \frac{\zeta(2-s)\zeta(2s)\zeta(3s)}{\zeta(6s)} \frac{\left(\frac{x}{y}\right)^s}{s^2(2-s)}ds$$
$$+ O\left(x^2\left(\frac{x}{y}\right)^{\frac{1}{2}}\int_T^\infty \frac{|\zeta\left(\frac{3}{2}+it\right)||\zeta(5+2it)||\zeta\left(\frac{15}{2}+3it\right)|}{|\zeta\left(15+6it\right)|}\frac{|\chi\left(-\frac{1}{2}-it\right)|}{(1+t)^3}dt\right)$$
$$= -x^2\left(\frac{x}{y}\right)^{\frac{1}{2}}\nu(u) + O\left(x^2\left(\frac{x}{y}\right)^{\frac{1}{2}}\frac{1}{T}\right), \tag{3.5}$$

where $\nu(u)$ is given by

$$\nu(u) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta(-\frac{1}{2}-it)\zeta(5+2it)\zeta(\frac{15}{2}+3it)}{\zeta(15+6it)}\frac{e^{itu}}{(\frac{5}{2}+it)^2(\frac{1}{2}+it)}dt,$$

with $u = \log\frac{x}{y}$. We use Lemma 2.5 and the inequality $|\frac{1}{\zeta(s)}| \le \frac{\zeta(\sigma)}{\zeta(2\sigma)}$ for $\sigma > 1$ to obtain the absolute value of $\nu(u)$, that is

$$|\nu(u)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\zeta(\frac{3}{2}+it)\zeta(5+2it)\zeta(\frac{15}{2}+3it)}{\zeta(15+6it)}\right|\left|\frac{\chi(-\frac{1}{2}-it)}{(\frac{5}{2}+it)^2(\frac{1}{2}+it)}\right|dt$$
$$\le \frac{2}{(2\pi)^2}\frac{\zeta(\frac{3}{2})\zeta(5)\zeta(\frac{15}{2})\zeta(15)}{\zeta(30)}\int_0^\infty \frac{t}{(t^2+\left(\frac{5}{2}\right)^2)\sqrt{t^2+\left(\frac{1}{2}\right)^2}}dt$$
$$\le \frac{2}{(2\pi)^2}\frac{\zeta(\frac{3}{2})\zeta(5)\zeta(\frac{15}{2})\zeta(15)}{\zeta(30)}\int_0^\infty \frac{t^{\frac{1}{2}}}{t^2+\left(\frac{1}{2}\right)^2}dt$$
$$\le \frac{4}{(2\pi)^2}\frac{\zeta(\frac{3}{2})\zeta(5)\zeta(\frac{15}{2})\zeta(15)}{\zeta(30)}\left(\frac{\pi}{4}+1\right), \tag{3.6}$$

hence $|\nu(u)|$ is an upper bound. Next, there exists a simple pole at $s = 2$ of the integral K_2 with residue $\frac{\zeta(4)\zeta(6)}{8\zeta(12)}\left(\frac{x}{y}\right)^2$ by using the value $\zeta(0) = -\frac{1}{2}$. Hence, we have

$$K_2 = -\frac{\zeta(4)\zeta(6)}{8\zeta(12)}x^2 - x^2\left(\frac{x}{y}\right)^{\frac{1}{2}}\nu(u) + O\left(\left(\frac{x}{y}\right)^{\frac{1}{2}}\right) \tag{3.7}$$

with $T = x^2$.

Combining (3.3) and (3.7) with (3.1) and taking $T = x^2$, we obtain the formula (1.10).

4 Proof of Theorem 3

In this section we assume that the Riemann Hypothesis is true, and $1 \le y \le x^M$ for some constant M. Without loss of generality we can assume that $x,\ y \in \mathbb{Z} + \frac{1}{2}$. The proof of this theorem follows by the same method as in Theorem 1, in addition to the Riemann Hypothesis. We start with (3.1), and set $\beta = \frac{1}{12} + \delta$ with $\delta = \frac{10}{\log\log T}$, and $\varepsilon = \frac{1}{\log\log T}$ in Lemma 2.7 and $T = x^2$.

In K_1, we move the integration, with respect to s, to $\Gamma(\alpha, \beta, T)$ with $\beta = \frac{1}{12} + \delta$. We denote the integrals over the horizontal line segments by $K_{1,1}$ and $K_{1,3}$, and the integral over the vertical line segment by $K_{1,2}$, respectively. We use Lemmas 2.5, 2.7, and 2.8 to deduce

$$\begin{aligned} K_{1,2} &\ll yx^{\frac{1}{12}+\delta}\left(\int_{|t|\le T_0} + \int_{T_0<|t|\le T}\right) \\ &\quad\times \frac{|\chi(\frac{1}{12}+\delta+it)||\chi(\frac{1}{6}+2\delta+2it)||\chi(\frac{1}{4}+3\delta+3it)|}{|\zeta(\frac{1}{2}+6\delta+6it)|}\times \\ &\quad\times \frac{|\zeta(\frac{11}{12}-\delta-it)||\zeta(\frac{5}{6}-2\delta-2it)||\zeta(\frac{3}{4}-3\delta-3it)|}{(1+|t|)^2}dt \\ &\ll yx^{\frac{1}{12}+\delta} + yx^{\frac{1}{12}+\delta}\int_{T_0<|t|\le T} t^{-1-6\delta+4\varepsilon}dt \\ &\ll yx^{\frac{1}{12}}\exp\left(A\frac{\log x}{\log\log x}\right) \end{aligned}$$

with A being a positive constant, and $T = x^2$. We use Lemmas 2.5, 2.7, 2.8, and the estimates of $K_{1,1}$ and $K_{1,3}$ in the proof of Theorem 1 to deduce

$$\begin{aligned} &K_{1,1}, K_{1,3} \\ &\ll \frac{y}{T^2}\left(\int_{\frac{1}{12}+\delta}^{\frac{1}{6}} + \int_{\frac{1}{6}}^{\alpha}\right)\frac{|\zeta(\sigma+iT)||\zeta(2(\sigma+iT))||\zeta(3(\sigma+iT))|}{|\zeta(6(\sigma+iT))|}x^{\sigma}d\sigma \\ &\ll \frac{y}{T^2}\left(T^{\frac{3}{2}+4\varepsilon}\int_{\frac{1}{12}+\delta}^{\frac{1}{6}}\left(\frac{x}{T^6}\right)^{\sigma}d\sigma + \frac{x^{\frac{1}{6}}}{T^{\frac{1}{6}}}\log^4 T\right) \\ &\ll \frac{y}{T}\left(\frac{x^{\frac{1}{6}}}{T^{\frac{1}{6}}}\log^4 T + \frac{x^{\frac{1}{12}+\delta}}{T^{6\delta-4\varepsilon}}\right). \end{aligned}$$

Therefore, by using Cauchy's residue theorem we have

$$K_1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)}xy + \frac{2\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)}x^{\frac{1}{2}}y + \frac{3\zeta(\frac{1}{3})\zeta(\frac{2}{3})}{\zeta(2)}x^{\frac{1}{3}}y + O\left(yx^{\frac{1}{12}}\exp\left(A\frac{\log x}{\log\log x}\right)\right) \tag{4.1}$$

by setting $T = x^2$, where A is a positive constant.

As for K_2, we make use of the same result in the proof of Theorem 1, that is

$$K_2 = -\frac{\zeta(4)\zeta(6)}{8\zeta(12)}x^2 - x^2\left(\frac{x}{y}\right)^{\frac{1}{2}}\nu(u) + O\left(\left(\frac{x}{y}\right)^{\frac{1}{2}}\right) \tag{4.2}$$

with $T = x^2$.

Combining (4.1) and (4.2) with (3.1) and taking $T = x^2$, we obtain the formula (1.13).

5 Proof of Theorem 2

Assume that $1 \le y \le x^M$ for some constant M. Without loss of generality we can assume that $x,\ y \in \mathbb{Z} + \frac{1}{2}$. The proof of this theorem follows by the same method as in Theorem 1. We apply Lemma 2.1 with (1.3) and substitute $b = 1$ and $z = 1 - s$ into Lemma 2.2 to deduce

$$S_1^{(3)}(x, y) = L_1 + L_2 + O\left(xy^{\frac{1}{3}}\log^2 y\right) + O\left(\frac{xy}{T}\log y\right), \tag{5.1}$$

where

$$L_1 := \frac{y}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} \frac{\zeta(s)\zeta(3s)\zeta(4s)\zeta(5s)\kappa_9(s)}{\zeta(8s)}\,\frac{x^s}{s^2}ds,$$

and

$$L_2 := -\frac{y^2}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} \frac{\zeta(2-s)\zeta(3s)\zeta(4s)\zeta(5s)\kappa_9(s)}{\zeta(8s)}\,\frac{(x/y)^s}{s^2(s-2)}ds.$$

Here $\alpha = 1 + \frac{1}{\log x}$ and T is a real parameter at our disposal.

We move the integration, with respect to s, to $\Gamma(\alpha, \beta, T)$ with $\beta = \frac{1}{8} - \varepsilon(T)$, where $\varepsilon(T)$ is given by (3.2). We denote the integrals over the horizontal line segments by $L_{1,1}$ and $L_{1,3}$, and the integral over the vertical line segment by $L_{1,2}$, respectively. Since the estimate $\kappa_9(\sigma + iT) \ll 1$ for $\sigma \ge \beta$, it follows from Lemmas 2.3–2.5 that

$$\begin{aligned}
&L_{1,1}, L_{1,3}\\
&\ll \frac{y}{T^2}\left(\int_\beta^{\frac{1}{5}} + \int_{\frac{1}{5}}^{\frac{1}{4}} + \int_{\frac{1}{4}}^{\frac{1}{3}} + \int_{\frac{1}{3}}^{\alpha}\right)\\
&\quad \frac{|\zeta(\sigma+iT)||\zeta(3(\sigma+iT))||\zeta(4(\sigma+iT))||\zeta(5(\sigma+iT))|}{|\zeta(8(\sigma+iT))|}x^\sigma d\sigma\\
&\ll \frac{y\log^5 T}{T^2}\left(T^{\frac{5}{3}}\int_\beta^{\frac{1}{5}}\left(\frac{x}{T^{\frac{17}{3}}}\right)^\sigma d\sigma\right.
\end{aligned}$$

$$+T^{\frac{7}{6}}\int_{\frac{1}{5}}^{\frac{1}{4}}\left(\frac{x}{T^3}\right)^{\sigma}d\sigma+T^{\frac{5}{6}}\int_{\frac{1}{4}}^{\frac{1}{3}}\left(\frac{x}{T^{\frac{5}{3}}}\right)^{\sigma}d\sigma+T^{\frac{1}{2}}\int_{\frac{1}{3}}^{\alpha}\left(\frac{x}{T^{\frac{2}{3}}}\right)^{\sigma}d\sigma\Bigg)$$
$$\ll\frac{y\log^5 T}{T}\left(\frac{x^{\frac{1}{8}}}{T^{\frac{1}{24}}}+\frac{x^{\frac{1}{5}}}{T^{\frac{13}{30}}}+\frac{x^{\frac{1}{4}}}{T^{\frac{7}{12}}}+\frac{x^{\frac{1}{3}}}{T^{\frac{13}{18}}}+\frac{x}{T}\right).$$

For $L_{1,2}$, we use integration by parts, Cauchy–Schwarz's inequality twice, Lemmas 2.3–2.6, and the estimate $\kappa_9(\frac{1}{8}+it)\ll 1$ to deduce

$$L_{1,2}\ll\frac{yx^{\frac{1}{8}-\varepsilon(T)}}{\varepsilon(T)}+yx^{\frac{1}{8}-\varepsilon(T)}\int_{T_0<|t|\le T}\frac{|\chi(\beta+it)||\chi(3\beta+3it)||\chi(4\beta+4it)|}{|\zeta(8\beta+8it)|}\times$$
$$\times\frac{|\zeta(1-\beta-it)||\zeta(1-3\beta-3it)||\zeta(1-4\beta-4it)||\zeta(5\beta+5it)|}{(1+|t|)^2}dt$$
$$\ll\frac{yx^{\frac{1}{8}-\varepsilon(T)}}{\varepsilon(T)}+\frac{yx^{\frac{1}{8}-\varepsilon(T)}}{\varepsilon(T)}\int_{T_0<|t|\le T}$$
$$\times\frac{|\zeta(\frac{7}{8}+\varepsilon(T)-it)||\zeta(\frac{5}{8}+3\varepsilon(T)-3it)||\zeta(\frac{1}{2}+4\varepsilon(T)-4it)||\zeta(\frac{5}{8}-5\varepsilon(T)+5it)|}{(1+|t|)^{\frac{3}{2}+8\varepsilon(T)}}dt$$
$$\ll\frac{yx^{\frac{1}{8}-\varepsilon(T)}}{\varepsilon(T)}+\frac{yx^{\frac{1}{8}-\varepsilon(T)}}{\varepsilon(T)}\left(\int_1^T\frac{\left|\zeta\left(\frac{7}{8}+iu\right)\right|^4}{(1+|u|)^{\frac{3}{2}+8\varepsilon(T)}}du\right)^{\frac{1}{4}}\left(\int_1^{3T}\frac{\left|\zeta\left(\frac{5}{8}+iu\right)\right|^4}{(1+|u|)^{\frac{3}{2}+8\varepsilon(T)}}du\right)^{\frac{1}{4}}$$
$$\times\left(\int_1^{4T}\frac{\left|\zeta\left(\frac{1}{2}+iu\right)\right|^4}{(1+|u|)^{\frac{3}{2}+8\varepsilon(T)}}du\right)^{\frac{1}{4}}\left(\int_1^{5T}\frac{\left|\zeta\left(\frac{9}{16}+iu\right)\right|^4}{(1+|u|)^{\frac{3}{2}+8\varepsilon(T)}}du\right)^{\frac{1}{4}}$$
$$\ll\frac{yx^{\frac{1}{8}-\varepsilon(T)}}{\varepsilon(T)}.$$

It remains to evaluate the residues of the poles of the integrand, and there exist four simple poles at $s=1,\frac{1}{3},\frac{1}{4}$ and $\frac{1}{5}$ with residues $\frac{\zeta(3)\zeta(4)\zeta(5)\kappa_9(1)}{\zeta(8)}x$, $\frac{3\zeta(\frac{1}{3})\zeta(\frac{4}{3})\zeta(\frac{5}{3})\kappa_9\left(\frac{1}{3}\right)}{\zeta(\frac{8}{3})}x^{\frac{1}{3}}$, $\frac{4\zeta(\frac{1}{4})\zeta(\frac{3}{4})\zeta(\frac{5}{4})\kappa_9\left(\frac{1}{4}\right)}{\zeta(2)}x^{\frac{1}{4}}$, and $\frac{5\zeta(\frac{1}{5})\zeta(\frac{3}{5})\zeta(\frac{4}{5})\kappa_9\left(\frac{1}{5}\right)}{\zeta(\frac{8}{5})}x^{\frac{1}{5}}$, respectively. Therefore, we have

$$L_1=\frac{\zeta(3)\zeta(4)\zeta(5)\kappa_9(1)}{\zeta(8)}xy+\frac{3\zeta(\frac{1}{3})\zeta(\frac{4}{3})\zeta(\frac{5}{3})\kappa_9\left(\frac{1}{3}\right)}{\zeta(\frac{8}{3})}x^{\frac{1}{3}}y+\frac{4\zeta(\frac{1}{4})\zeta(\frac{3}{4})\zeta(\frac{5}{4})\kappa_9\left(\frac{1}{4}\right)}{\zeta(2)}x^{\frac{1}{4}}y$$
$$+\frac{5\zeta(\frac{1}{5})\zeta(\frac{3}{5})\zeta(\frac{4}{5})\kappa_9\left(\frac{1}{5}\right)}{\zeta(\frac{8}{5})}x^{\frac{1}{5}}y+O\left(yx^{\frac{1}{8}}\exp\left(-C\frac{(\log x)^{\frac{1}{3}}}{(\log\log x)^{\frac{1}{3}}}\right)\right)\tag{5.2}$$

by setting $T=x^2$, with C being a positive constant.

For L_2, we move the integration, with respect to s, to $\Gamma(\alpha,\frac{5}{2},T)$. We denote the integrals over the horizontal line segments by $L_{2,1}$ and $L_{2,3}$, and the integral over the vertical line segment by $L_{2,2}$, respectively. Following the same method as in (3.4) and (3.5) we have

$$L_{2,1}, L_{2,3} \ll \frac{y^2 \log T}{T^3} \left(\frac{x}{y} + T^{\frac{1}{2}} \left(\frac{x}{y}\right)^2 + T \left(\frac{x}{y}\right)^{\frac{5}{2}} \right),$$

and

$$\begin{aligned} L_{2,2} &= -\frac{y^2}{2\pi i} \int_{\frac{5}{2}-i\infty}^{\frac{5}{2}+i\infty} \frac{\zeta(2-s)\zeta(3s)\zeta(4s)\zeta(5s)\kappa_9(s)}{\zeta(8s)} \frac{(\frac{x}{y})^s}{s^2(s-2)} ds \\ &\quad + O\left(x^2 \left(\frac{x}{y}\right)^{\frac{1}{2}} \int_T^\infty \frac{|\zeta\left(-\frac{1}{2}-it\right)||\zeta(\frac{15}{2}+3it)||\zeta\,(10+4it)\,||\zeta\left(\frac{25}{2}+5it\right)||\kappa_9(\frac{5}{2}+it)|}{|\zeta(20+5it)|(1+t)^3} dt \right) \\ &= -x^2 \left(\frac{x}{y}\right)^{\frac{1}{2}} \xi(u) + O\left(x^2 \left(\frac{x}{y}\right)^{\frac{1}{2}} \frac{1}{T} \right), \end{aligned} \tag{5.3}$$

where $\xi(u)$ is given by

$$\xi(u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\zeta(-\frac{1}{2}-it)\zeta(\frac{15}{2}+3it)\zeta(10+4it)\zeta(\frac{25}{2}+5it)\kappa_9(\frac{5}{2}+it)}{\zeta(20+8it)} \frac{e^{itu}}{(\frac{5}{2}+it)^2(\frac{1}{2}+it)} dt \tag{5.4}$$

with $u = \log \frac{x}{y}$. Similarly as in (3.6), we have

$$|\xi(u)| \le \frac{4}{(2\pi)^2} \frac{\zeta(\frac{3}{2})\zeta(\frac{15}{2})\zeta(10)\zeta(\frac{25}{2})\zeta(20)\kappa_9(\frac{5}{2})}{\zeta(40)} \left(\frac{\pi}{4}+1\right). \tag{5.5}$$

There exists a simple pole at $s = 2$ of the integral L_2 with residue $\frac{\zeta(6)\zeta(8)\zeta(10)\kappa_9(2)}{8\zeta(16)} \left(\frac{x}{y}\right)^2$. Hence, we have

$$L_2 = -\frac{\zeta(6)\zeta(8)\zeta(10)\kappa_9(2)}{8\zeta(16)} x^2 - x^2 \left(\frac{x}{y}\right)^{\frac{1}{2}} \xi(u) + O\left(\left(\frac{x}{y}\right)^{\frac{1}{2}} \right) \tag{5.6}$$

by using $T = x^2$.

Combining (5.2) and (5.6) with (5.1) and taking $T = x^2$, the formula (1.12) is proved.

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