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Indefinite q-integrals from a method using q-Riccati equations

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Abstract
In an earlier work, a method was introduced for obtaining indefinite q-integrals of q-special functions from the second-order linear q-difference equations that define them. In this paper, we reformulate the method in terms of q-Riccati equations, which are nonlinear and first order. We derive q-integrals using fragments of these Riccati equations, and here only two specific fragment types are examined in detail. The results presented here are for the q-Airy function, the Ramanujan function, the discrete q-Hermite I and II polynomials, the q-hypergeometric functions, the q-Laguerre polynomials, the Stieltjes-Wigert polynomial, the little q-Legendre and the big q-Legendre polynomials.

Keywords q-integrals, q-Bernoulli fragment · q-Linear fragment · Simple algebraic form · q-Airy function · Ramanujan function

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1 Introduction and preliminaries

In [14], we introduced a method to obtain indefinite q-integrals of the form

$$\begin{aligned}&\int f(x)\Big(\frac{1}{q}D_{q^{-1}}D_q h(x)+p(x)D_{q^{-1}}h(x)+r(x)h(x)\Big)y(x)\mathrm{d}_q x\\&=f(x/q)\Big(y(x)D_{q^{-1}}h(x)-h(x)D_{q^{-1}}y(x)\Big)\\&=f(x/q)\Big(y(x/q)D_{q^{-1}}h(x)-h(x/q)D_{q^{-1}}y(x)\Big),\end{aligned}\tag{1.1}$$

where the functions $p(x)$ and $r(x)$ are continuous functions in an interval I and the function $y(x)$ is a solution of the second-order q-difference equation

$$\frac{1}{q}D_{q^{-1}}D_q y(x)+p(x)D_{q^{-1}}y(x)+r(x)y(x)=0,\tag{1.2}$$

$f(x)$ is a solution of

$$\frac{1}{q}D_{q^{-1}}f(x)=p(x)f(x)\tag{1.3}$$

and $h(x)$ is an arbitrary function. We also introduced

$$\begin{aligned}&\int F(x)\Big(\frac{1}{q}D_{q^{-1}}D_q k(x)+p(x)D_q k(x)+r(x)k(x)\Big)y(x)\mathrm{d}_q x\\&=F(x)\Big(y(x)D_{q^{-1}}k(x)-k(x)D_{q^{-1}}y(x)\Big),\end{aligned}\tag{1.4}$$

where $y(x)$ is a solution of

$$\frac{1}{q}D_{q^{-1}}D_q y(x)+p(x)D_q y(x)+r(x)y(x)=0.\tag{1.5}$$

$F(x)$ is a solution of

$$D_q F(x)=p(x)F(x),\tag{1.6}$$

and $k(x)$ is an arbitrary function. The indefinite q-integral

$$\int f(x)\mathrm{d}_q x=F(x),\tag{1.7}$$

means that $D_q F(x)=f(x)$, where D_q is the Jackson's q-difference operator, which is defined in (1.13) below. The indefinite q-integrals in (1.1) and (1.4) generalize Conway's indefinite integral

$$\int f(x)\left(\frac{\mathrm{d}^2h}{\mathrm{d}x^2}+p(x)\frac{\mathrm{d}h}{\mathrm{d}x}+r(x)h(x)\right)y(x)\mathrm{d}x = f(x)\left(\frac{\mathrm{d}h}{\mathrm{d}x}y(x)-h(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right), \tag{1.8}$$

where $y(x)$ is a solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+p(x)\frac{\mathrm{d}y}{\mathrm{d}x}+r(x)y(x)=0, \tag{1.9}$$

$f(x)$ is a solution of $f'(x)=p(x)f(x)$ and $h(x)$ is an arbitrary function. See [2–7, 10]. Conway in [8, 9] reformulated (1.8) to take the form

$$\int f(x)h(x)\left(u'(x)+u^2(x)+p(x)u(x)+r(x)\right)y(x)\mathrm{d}x = f(x)h(x)\left(u(x)y(x)-y'(x)\right), \tag{1.10}$$

where

$$h(x)=\exp\left(\int u(x)\mathrm{d}x\right),$$

and $u(x)$ is an arbitrary function. Then, he derived many indefinite integrals by considering fragments of the Riccati equation

$$u'(x)+u^2(x)+p(x)u(x)+r(x)=0,$$

of the form

$$u'(x)+u^2(x)+p(x)u(x)=0, \tag{1.11}$$

or

$$u'(x)+p(x)u(x)+r(x)=0. \tag{1.12}$$

He identified (1.11) as the Bernoulli fragment, and (1.12) as the linear fragment. This paper is organized as follows. In the remainder of this section, we present the q-notations and concepts required in the next sections. In Sect. 2, we provide a q-analogue of Conway's indefinite integral formula in (1.10) to the q-setting, along with applications to q-hypergeometric functions, q-Legendre polynomials, discrete q-Hermite I and II polynomials, the q-Airy function, and the Ramanujan function. Section 3 contains applications to the discrete q-Hermite I and II polynomials, the q-Airy function, and the Ramanujan function. In Sect. 4, we introduce new q-integrals by setting $u(x)=\frac{a}{x}+b$, with appropriate choice of a and b in (6.2) and (6.4). Finally, we added an appendix for all q-special functions, we used in this paper.

Throughout this paper, q is a positive number less than 1, $\mathbb{N}$ is the set of positive integers, and $\mathbb{N}_0$ is the set of non-negative integers. We use I to denote an interval with zero or infinity as an accumulation point. We follow Gasper and Rahman [11] for the definitions of the q-shifted factorial, q-gamma, q-beta function, and q-hypergeometric series.

A q-natural number $[n]_q$ is defined by $[n]_q = \frac{1-q^n}{1-q}$, $n \in \mathbb{N}_0$. Jackson's q-derivative of a function f is denoted by $D_q f(x)$ and is defined as

$$D_q f(x) = \begin{cases} \frac{f(x)-f(qx)}{(1-q)x}, & \text{if } x \neq 0; \\ f'(0), & \text{if } x = 0, \end{cases} \tag{1.13}$$

provided that $f'(0)$ exists (see [13–15]). Jackson's q-integral of a function f is defined by

$$\int_0^a f(t) \mathrm{d}_q t := (1-q)a \sum_{n=0}^{\infty} q^n f(aq^n), \ a \in \mathbb{R}, \tag{1.14}$$

provided that the corresponding series in (1.14) converges, see [16].

The fundamental theorem of q-calculus [1, Eq. (1.29)]

$$\int_0^a D_q f(t) \mathrm{d}_q t = f(a) - \lim_{n\to\infty} f(aq^n). \tag{1.15}$$

If f is continuous at zero, then

$$\int_0^a D_q f(t) \mathrm{d}_q t = f(a) - f(0).$$

2 *q*-Integrals from Riccati fragments

In this section, we extend Conway's result (1.10) to functions satisfying homogenous second-order q-difference equation of the form (1.2) or (1.5). Consider the q-Riccati equations

$$\frac{1}{q} D_{q^{-1}} u(x) + \frac{1}{q} u(x)u(x/q) + A(x)u(x/q) + r(x) = 0, \tag{2.1}$$

and

$$D_q u(x) + u(x)u(qx) + \tilde{A}(x)u(qx) + r(x) = 0, \tag{2.2}$$

where $A(x)$ and $\tilde{A}(x)$ are defined as in (2.5) and (2.11), respectively. We can prove that Eqs. (2.1), (2.2) are equivalent to Eqs. (1.2), (1.5) by setting $\frac{D_q y(x)}{y(x)} = u(x)$ $\left(\frac{D_{q^{-1}} y(x)}{y(x)} = u(x)\right)$, respectively. This leads to Theorems 2.1 and 2.2 below.

Theorem 2.1 *Let $y(x)$ and $f(x)$ be solutions of Eqs. (1.2) and (1.3) in an open interval I, respectively. Let $u(x)$ be a continuous function on I and $h(x)$ be an arbitrary function satisfying*

$$D_q h(x) = u(x)h(x) \quad (x \in I). \tag{2.3}$$

Then,

$$\begin{aligned}&\int f(x)h(x/q)\Big(\frac{1}{q}D_{q^{-1}}u(x) + \frac{1}{q}u(x)u(x/q) + A(x)u(x/q) + r(x)\Big)y(x)\mathrm{d}_q x\\&\quad = f(x/q)h(x/q)\Big(y(x/q)u(x/q) - D_{q^{-1}}y(x)\Big),\end{aligned} \tag{2.4}$$

where the functions $p(x)$, $r(x)$ are defined as in (1.2) and

$$A(x) = p(x) - \frac{1}{q}x(1-q)r(x). \tag{2.5}$$

Proof Equation (1.1) can be written as

$$\begin{aligned}&\int f(x)h(x/q)\left[\frac{1}{q}\frac{D_{q^{-1}}D_q h(x)}{h(x/q)} + p(x)\frac{D_{q^{-1}}h(x)}{h(x/q)} + \frac{r(x)h(x)}{h(x/q)}\right]y(x)\mathrm{d}_q x\\&\quad = f(x/q)h(x/q)\left[y(x/q)\frac{D_{q^{-1}}h(x)}{h(x/q)} - D_{q^{-1}}y(x)\right].\end{aligned} \tag{2.6}$$

Then, from (2.3), we get

$$D_{q^{-1}}u(x) = D_{q^{-1}}\left(\frac{D_q h(x)}{h(x)}\right) = \frac{h(x)D_{q^{-1}}D_q h(x) - D_q h(x)D_{q^{-1}}h(x)}{h(x)h(x/q)}.$$

Hence,

$$\begin{aligned}\frac{D_{q^{-1}}D_q h(x)}{h(x/q)} &= D_{q^{-1}}u(x) + \frac{D_{q^{-1}}h(x)}{h(x/q)}u(x)\\&= D_{q^{-1}}u(x) + u(x)u\left(\frac{x}{q}\right).\end{aligned} \tag{2.7}$$

Also,

$$r(x)\frac{h(x)}{h(x/q)} = \frac{r(x)}{h(x/q)}\Big(h(x/q) + \big(1-\frac{1}{q}\big)xD_{q^{-1}}h(x)\Big)$$
$$= r(x)\Big(1 + \big(1-\frac{1}{q}\big)xu\big(\frac{x}{q}\big)\Big). \tag{2.8}$$

Substituting with (2.7) and (2.8) into (2.6), we get (2.4) and completes the proof. □

Theorem 2.2 *Let $y(x)$ and $F(x)$ be solutions of Eqs.* (1.5) *and* (1.6) *in an open interval I, respectively. Let $u(x)$ be a continuous function on I and $k(x)$ be an arbitrary function satisfying*

$$D_{q^{-1}}k(x) = u(x)k(x) \quad x \in I. \tag{2.9}$$

Then,

$$\int F(x)k(qx)\Big(D_q u(x) + u(x)u(qx) + \tilde{A}(x)u(qx) + r(x)\Big)y(x)\mathrm{d}_q x$$
$$= F(x)k(x)\Big(y(x)u(x) - D_{q^{-1}}y(x)\Big), \tag{2.10}$$

where the functions $p(x)$, $r(x)$ are defined as in (1.5) *and*

$$\tilde{A}(x) = p(x) + x(1-q)r(x). \tag{2.11}$$

Proof The proof follows similarly as the proof of Theorem 2.1 and is omitted. □

The q-integrals presented in the sequel are obtained by choosing the function $u(x)$ to be a solution of a fragment of the q-Riccati equations (2.1) or (2.2). Bernoulli and linear fragments of (2.1) are defined as

$$\frac{1}{q}D_{q^{-1}}u(x) + \frac{1}{q}u(x)u(x/q) + A(x)u(x/q) = 0, \tag{2.12}$$
$$\frac{1}{q}D_{q^{-1}}u(x) + p(x)u(x/q) + r(x) = 0, \tag{2.13}$$

respectively. Similarly, the Bernoulli and linear fragments of (2.2) are defined as

$$D_q u(x) + u(x)u(qx) + \tilde{A}(x)u(qx) = 0, \tag{2.14}$$

and

$$D_q u(x) + p(x)u(qx) + r(x) = 0, \tag{2.15}$$

respectively. The trivial solution $u(x)=0$ of (2.12) implies that $h(x)=c$ is a solution of (2.3), where c is a non-zero constant. Then, (2.4) becomes

$$\int f(x)r(x)y(x)\mathrm{d}_q x=-f(x/q)D_{q^{-1}}y(x). \tag{2.16}$$

Similarly, the trivial solution $u(x)=0$ of (2.14) implies that $k(x)=c$ is a solution of (2.9), where c is a non-zero constant. Then, (2.10) becomes

$$\int F(x)r(x)y(x)\mathrm{d}_q x=-F(x)D_{q^{-1}}y(x). \tag{2.17}$$

Theorem 2.3 *If $g(x)$ is a solution of the first-order q-difference equation*

$$\frac{1}{q}D_{q^{-1}}g(x)=A(x)g(x),\quad g(0)=1, \tag{2.18}$$

where $A(x)$ is the function which is defined in (2.5). Then,

$$u(x)=\frac{1}{g(x)\int_0^x \frac{1}{g(t)}\mathrm{d}_q t},\quad x\in I, \tag{2.19}$$

is a solution of (2.12) and (2.4) takes the form

$$\int f(x)h(x/q)r(x)y(x)\mathrm{d}_q x=f(x/q)h(x/q)\Big(y(x/q)u(x/q)-D_{q^{-1}}y(x)\Big). \tag{2.20}$$

Proof In Theorem 2.1, we choose $u(x)$ to be a solution of (2.12). This produces (2.20). But one can verify that if we set $u(x)=\frac{1}{v(x)}$, then (2.12) takes the form

$$D_{q^{-1}}v(x)-qA(x)v(x)=1, \tag{2.21}$$

which can be rewritten as $D_{q^{-1}}\left(\frac{v(x)}{g(x)}\right)=\frac{1}{g(x/q)}$ or equivalently, $D_q\left(\frac{v(x)}{g(x)}\right)=\frac{1}{g(x)}$. Hence, from (1.15), we get $v(x)=g(x)\int_0^x \frac{1}{g(t)}\mathrm{d}_q t$. Hence, $u(x)=\frac{1}{v(x)}$ is defined as in (2.19). □

Theorem 2.4 *Assume that $g(x)$ is defined as in Theorem 2.3 in an interval I containing zero. Then,*

$$g(x)h(x)=\frac{1}{u(x)}.$$

Proof From (2.3),

$$\frac{D_q h(x)}{h(x)}=u(x)=\frac{\frac{1}{g(x)}}{\int_0^x \frac{1}{g(t)}\mathrm{d}_q t}.$$

Hence,

$$D_q\left(\frac{h(x)}{\int_0^x \frac{1}{g(t)}\mathrm{d}_q t}\right) = 0.$$

Therefore,

$$h(x) = c\int_0^x \frac{1}{g(t)}\mathrm{d}_q t,$$

where c is a constant, we can choose $c = 1$. Hence,

$$g(x)h(x) = g(x)\int_0^x \frac{1}{g(t)}\mathrm{d}_q t = \frac{1}{u(x)}.$$

□

Theorem 2.5 *Let I be an interval containing zero. Let $p(x)$ and $r(x)$ be continuous functions at zero. If $f(x)$ is a solution of Eq. (1.3), then*

$$u(x) = \frac{-1}{f(x)}\int_0^{qx} f(t)r(t)\mathrm{d}_q t, \tag{2.22}$$

is a solution of Eq. (2.13) in I and (2.4) takes the form

$$\begin{aligned}&\int f(x)h(x/q)\left(\frac{1}{q}u(x)u(x/q) + \frac{1}{q}xr(x)(q-1)u(x/q)\right)y(x)\mathrm{d}_q x\\ &\quad = f(x/q)h(x/q)\left(y(x/q)u(x/q) - D_{q^{-1}}y(x)\right).\end{aligned} \tag{2.23}$$

Proof Multiplying both sides of (2.13) by $f(x)$, we obtain

$$D_{q^{-1}}\Big(f(x)u(x)\Big) = -qf(x)r(x),$$

or equivalently

$$D_q\Big(f(x)u(x)\Big) = -qf(qx)r(qx).$$

Hence, from (1.15), we get (2.22). If $u(x)$ is a solution of the q-linear fragment (2.13), then from (2.4), we obtain (2.23) and completes the proof. □

3 *q*-Integrals from the Bernoulli fragment

This section contains indefinite q-integrals that are derived from the q-Bernoulli fragment (2.12).

Theorem 3.1

$$\int x\cos(x;q)\,{}_2\phi_1\left(0,q;q^3;q^2,\frac{-x^2}{q}(1-q)^2\right)\mathrm{d}_q x$$
$$=-\frac{q(q;q^2)_\infty\cos\left(\frac{x}{q};q\right)}{(1-q)\left(\frac{-x^2}{q}(1-q)^2;q^2\right)_\infty}+\sqrt{q}x\sin\left(q^{\frac{-1}{2}}x;q\right){}_2\phi_1\left(0,q;q^3;q^2,\frac{-x^2}{q}(1-q)^2\right), \tag{3.1}$$

$$\int x\sin(x;q)\,{}_2\phi_1\left(0,q;q^3;q^2,\frac{-x^2}{q}(1-q)^2\right)\mathrm{d}_q x$$
$$=-x\cos(q^{\frac{-1}{2}}x;q)\,{}_2\phi_1\left(0,q;q^3;q^2,\frac{-x^2}{q}(1-q)^2\right)-\frac{q(q;q^2)_\infty\sin(x/q;q)}{(1-q)\left(\frac{-x^2}{q}(1-q)^2;q^2\right)_\infty}. \tag{3.2}$$

Proof By comparing Eq. (A4) with Eq. (1.2), we get $p(x)=0$ and $r(x)=-1$. Then, $f(x)=1$ is a solution of (1.3) and $g(x)=(-q(1-q)^2x^2;q^2)_\infty$ is a solution of (2.18) with $A(x)=\frac{x}{q}(1-q)$. By Theorem 2.3,

$$u(x)=\frac{1}{x(1-q)\,{}_2\phi_1(-q(1-q)^2x^2,q^2;0;q^2,q)},$$

using (A19), we get

$$u(x)=\frac{(q;q^2)_\infty}{x(1-q)(-q(1-q)^2x^2;q^2)_\infty\,{}_2\phi_1(0,q;q^3;q^2,-q(1-q)^2x^2)}.$$

By Theorem 2.4,

$$h(x)=\frac{x(1-q)\,{}_2\phi_1(0,q;q^3;q^2,-q(1-q)^2x^2)}{(q;q^2)_\infty}.$$

Substituting with $u(x)$, $f(x)$, and $h(x)$ into (2.20) and using the q-difference equations (A7) and (A8), we get (3.1) and (3.2), respectively. □

Theorem 3.2 *Let* ${}_2\phi_1(q^a, q^b; q^c; q, x)$ *be the* q*-hypergeometric functions,* a*,* b*, and* c *are real numbers,* $c < 1$*,* $\delta > a + b - c$*, and* $c \neq q^{-n}$*,* $n \in \mathbb{N}_0$*. Then,*

$$\begin{aligned}&\int (x;q)_{a+b-c}\ {}_2\phi_1(q^{a+b-c-\delta+1}, q^{1-c}; q^{2-c}; q, q^{\delta-1}x)\, {}_2\phi_1(q^a, q^b; q^c; q, x)\, \mathrm{d}_q x \\ &= \frac{x}{[c]_q}\left(\frac{x}{q};q\right)_{a+b+1-c} {}_2\phi_1\left(q^{a+b-c-\delta+1}, q^{1-c}; q^{2-c}; q, q^{\delta-1}x\right) {}_2\phi_1 \\ &\quad \times \left(q^{a+1}, q^{b+1}; q^{c+1}; q, \frac{x}{q}\right) + \frac{\mu[c-1]_q}{[a]_q[b]_q}\left(\frac{x}{q};q\right)_\delta {}_2\phi_1\left(q^a, q^b; q^c; q, \frac{x}{q}\right),\end{aligned}$$

where $q^\delta = q^a + q^b - q^{a+b}$ *and* $\mu = q^{c+1-c(a+b-c)}$*.*

Proof By comparing (A17) with Eq. (1.2), we get

$$p(x) = \frac{[c]_q - [a+b+1]_q \frac{x}{q}}{q^c x(1 - q^{a+b-c}x)}, \quad \text{and} \quad r(x) = -\frac{[a]_q[b]_q}{q^c x(1 - q^{a+b-c}x)}.$$

Then,

$$f(x) = x^c \frac{(x;q)_\infty}{(xq^{a+b-c+1};q)_\infty} = x^c (x;q)_{a+b+1-c},$$

is a solution of (1.3)

$$g(x) = x^c \frac{(q^\delta x;q)_\infty}{(xq^{a+b-c+1};q)_\infty} = x^c (q^\delta x;q)_{a+b+1-\delta-c},$$

satisfies (2.18). By Lemmas 2.3 and A.1, we have

$$u(x) = \frac{q^{(1-c)(a+b-c)}x^{-c}}{(q^\delta x;q)_{a+b+1-\delta-c} B_q(1-c, c-a-b+\delta; q^{a+b-c}x)},$$

satisfies (2.21). Therefore, by Theorem 2.4, we obtain

$$h(x) = \frac{1}{g(x)u(x)} = q^{(c-1)(a+b-c)} B_q(1-c, c-a-b+\delta; q^{a+b-c}x).$$

By substituting with $f(x)$, $h(x)$, and $u(x)$ into (2.20), we get

$$\begin{aligned}&\int x^{c-1}(x;q)_{a+b-c}\ B_q\left(1-c, c-a-b+\delta; q^{a+b-c-1}x\right) {}_2\phi_1\left(q^a, q^b; q^c; q, x\right) \mathrm{d}_q x \\ &= \frac{x^c}{[c]_q}\left(\frac{x}{q};q\right)_{a+b+1-c} B_q\left(1-c, c-a-b+\delta; q^{a+b-c-1}x\right) {}_2\phi_1\left(q^{a+1}, q^{b+1}; q^{c+1}; q, \frac{x}{q}\right) \\ &\quad - \frac{q^{c-(c-1)(a+b-c)}}{[a]_q[b]_q}\left(\frac{x}{q};q\right)_\delta {}_2\phi_1\left(q^a, q^b; q^c; q, \frac{x}{q}\right),\end{aligned}$$

where $B_q(\alpha, \beta; x)$ is a function defined in (A18). Using (A18) and (A19), we get the desired result. □

Theorem 3.3 *If $y(x) = x^{1-c}\,{}_2\phi_1(q^{a+1-c}, q^{b+1-c}; q^{2-c}; q, x)$, $c < 1$, $\delta > a+b-c$, and $q^c \neq q^{n+2}$, $n \in \mathbb{N}_0$, is the q-hypergeometric functions. Then,*

$$\int x^{1-c}(x;q)_{a+b-c}\,{}_2\phi_1\left(q^{a+b-c-\delta+1}, q^{1-c}; q^{2-c}; q, q^{\delta-1}x\right){}_2\phi_1$$
$$\times\left(q^{a+1-c}, q^{b+1-c}; q^{2-c}; q, x\right)\mathrm{d}_q x$$
$$= \lambda x^{2-c}\left(\frac{x}{q};q\right)_{a+b+1-c}{}_2\phi_1\left(q^{a+b-c-\delta+1}, q^{1-c}; q^{2-c}; q, q^{\delta-1}x\right){}_2\phi_1$$
$$\times\left(q^{a+2-c}, q^{b+2-c}; q^{2-c}; q^2, \frac{x}{q}\right)$$
$$+\frac{\mu[c-1]_q}{[a]_q[b]_q}\left(\frac{x}{q};q\right)_\delta x^{1-c}\,{}_2\phi_1\left(q^{a+1-c}, q^{b+1-c}; q^{2-c}; q, \frac{x}{q}\right),$$

where $\lambda = \dfrac{[a+1-c]_q[b+1-c]_q}{q^{1-c}[a]_q[b]_q}$, $\mu = q^{-c(a+b-c-2)}$, *and* $q^\delta = q^a + q^b - q^{a+b}$.

Proof By substituting with $f(x)$, $h(x)$, and $u(x)$ as in Theorem 3.2 and $y(x) = x^{1-c}\,{}_2\phi_1(q^{a+1-c}, q^{b+1-c}; q^{2-c}; q, x)$ into (2.20) and using (A18) and (A19), we get the desired result. □

Theorem 3.4 *If $y(x) = x^{-a}\,{}_2\phi_1\left(q^a, q^{a+1-c}; q^{a+1-b}; q, \frac{q^{c-a-b+1}}{x}\right)$, $c < 1$, $\delta > a+b-c$, and $q^a \neq q^{b-n-1}$, $n \in \mathbb{N}_0$ is the q-hypergeometric functions. Then,*

$$\int (x;q)_{a+b-c}x^{-a}\,{}_2\phi_1\left(q^{a+b-c-\delta+1}, q^{1-c}; q^{2-c}; q, q^{\delta-1}x\right){}_2\phi_1$$
$$\times\left(q^a, q^{a+1-c}; q^{a+1-b}; q, \frac{q^{c-a-b+1}}{x}\right)\mathrm{d}_q x$$
$$= \lambda x^{3-a}\left(\frac{x}{q};q\right)_{a+b+1-c}{}_2\phi_1\left(q^{a+b-c-\delta+1}, q^{1-c}; q^{2-c}; q, q^{\delta-1}x\right){}_2\phi_1$$
$$\times\left(q^{a+2}, q^{a+2-c}; q^{a+2-b}; q, \frac{q^{c-a-b+1}}{x}\right)$$
$$+\frac{\mu[c-1]_q}{[a]_q[b]_q}\left(\frac{x}{q};q\right)_\delta x^{-a}\,{}_2\phi_1\left(q^a, q^{a+1-c}; q^{a+1-b}; q, \frac{q^{c-a-b+2}}{x}\right),$$

where $\lambda = \dfrac{-q^{2-a-b+c}[a+1]_q[a+1-c]_q}{[b]_q[a+1-b]_q}$, $\mu = q^{(a+1)+c(1-a-b+c)}$, *and* $q^\delta = q^a + q^b - q^{a+b}$.

Proof By substituting with $f(x)$, $h(x)$, and $u(x)$ as in Theorem 3.2 and $y(x) = x^{-a}\,{}_2\phi_1\left(q^a, q^{a+1-c}; q^{a+1-b}; q, \frac{q^{c-a-b+1}}{x}\right)$ into (2.20) and using (A18) and (A19), we get the desired result. □

4 *q*-Integrals from the linear fragment

In the following results, we obtain new indefinite q-integrals from the linear fragment (2.13).

Theorem 4.1 *If* $|\, a \,| < \sqrt{\frac{q}{1-q}}$, *then*

$$\int_0^a \frac{x^2}{(x^2q^{-1}(1-q); q^2)_\infty} \cos(x; q)\mathrm{d}_q x$$
$$= \frac{q}{\left(\frac{a^2}{q}(1-q); q^2\right)_\infty}\left(a\cos\left(\frac{a}{q}; q\right) + \sqrt{q}\sin\left(q^{\frac{-1}{2}}a; q\right)\right), \tag{4.1}$$
$$\int_0^a \frac{x^2}{\left(x^2q^{-1}(1-q); q^2\right)_\infty} \sin(x; q)\mathrm{d}_q x$$
$$= \frac{q}{\left(\frac{a^2}{q}(1-q); q^2\right)_\infty}\left(a\sin\left(\frac{a}{q}; q\right) - \cos(q^{\frac{-1}{2}}a; q)\right) + q, \tag{4.2}$$

where $\sin(x; q)$ *and* $\cos(x; q)$ *are defined in* (A5) *and* (A6)*, respectively.*

Proof From (A4), we have $p(x) = 0$ and $r(x) = -1$. Then, $f(x) = 1$ is a solution of (1.3). By Theorem 2.5, the function $u(x) = qx$ is a solution of (2.13). Hence,

$$h(x) = \frac{1}{(qx^2(1-q); q^2)_\infty}, \tag{4.3}$$

is a solution of (2.3). Substituting with $u(x)$ and $h(x)$ into (2.23) and using the q-difference equations (A7) and (A8), we get (4.1) and (4.2), respectively. □

Theorem 4.2 *Let $n \in \mathbb{N}$. If $p_n(x;-1;q)$ is the big q-Legendre polynomial which is defined in (A22), $r_n = \frac{2-q^{-n}-q^{n+1}}{1-q}$, then*

$$\int \frac{x^2\left(x^2;q^2\right)_\infty}{(r_n x^2;q^2)_\infty} p_n(x;-1;q)\mathrm{d}_q x$$
$$= \frac{q^{n+2}\left(\frac{x^2}{q^2};q^2\right)_\infty}{[n]_q[n+1]_q\left(\frac{r_n x^2}{q^2};q^2\right)_\infty}\left(-\frac{q^2-x^2}{1+q}p_{n-1}(x;-q;q) - xp_n\left(\frac{x}{q};-1;q\right)\right). \tag{4.4}$$

Proof By comparing (A23) with (1.2), we get

$$p(x) = \frac{-x(1+q)}{q^2(1-x^2)}, \quad r(x) = \frac{[n]_q[n+1]_q}{q^{1+n}(1-x^2)}.$$

Then, $f(x) = (1-x^2)$ is a solution of (1.3). From (2.22), we have

$$u(x) = -q^{-n}[n]_q[n+1]_q\frac{x}{1-x^2},$$

and $h(x) = \frac{\left(x^2;q^2\right)_\infty}{\left(r_n x^2;q^2\right)_\infty}$ is a solution of (2.3). By substituting with $u(x)$ and $h(x)$ into (2.23) and using the q-difference equation

$$D_{q^{-1}}p_n(x;-1;q) = \frac{q^{1-n}[n]_q[n+1]_q}{1+q}p_{n-1}(x;-q;q), \tag{4.5}$$

we get (4.4). □

Theorem 4.3 *Let $n \in \mathbb{N}$. If $p_n(x|q)$ is the little q-Legendre polynomials defined in (A24), $r_n = \frac{2-q^{-n}-q^{n+1}}{1-q}$, then*

$$\int \frac{x(qx;q)_\infty}{(qr_n x;q)_\infty}p_n(x|q)\mathrm{d}_q x$$
$$= \frac{q^n x(x;q)_\infty}{[n]_q[n+1]_q(r_n x;q)_\infty}\left(\frac{1}{q}(1-x)\,{}_2\phi_1\left(q^{-n+1},q^{n+2};q^2;q,x\right) - p_n\left(\frac{x}{q}|q\right)\right). \tag{4.6}$$

Proof By comparing Eq. (A25) with (1.2), we get

$$p(x)=\frac{qx+x-1}{qx(qx-1)},\quad r(x)=\frac{[n]_q[n+1]_q}{q^n x(1-qx)}.$$

Then, $f(x)=x(1-qx)$ is a solution of (1.3). From (2.22), we get $u(x)=\frac{-q^{1-n}[n]_q[n+1]_q}{(1-qx)}$, $h(x)$ satisfies the q-difference equation (2.3). Consequently, $h(x)=\frac{(qx;q)_\infty}{(r_n qx;q)_\infty}$. By substituting with $u(x)$ and $h(x)$ into (2.23) and using the q-difference equation

$$D_{q^{-1}}p_n(x|q)=-q^{-n}[n]_q[n+1]_q\,{}_2\phi_1(q^{1-n},q^{n+2};q^2;q,x), \tag{4.7}$$

we get (4.6). □

5 *q*-Integrals from arbitrary parts from Riccati equation

In this section, we discuss an approach that chooses $u(x)$ to be a solution of a fragment of the Riccati equation, where a fragment is an equation obtained from Riccati's equation by deleting one or more of the terms.

Theorem 5.1 *Let $n\in\mathbb{N}$ and c be a real number. If $h_n(x;q)$ is the discrete q-Hermite I polynomial of degree n which is defined in (A9), then*

$$\begin{aligned}&\int (q^2x^2;q^2)_\infty\left((cq+x)[n]_q-x\right)h_n(x;q)\mathrm{d}_qx\\&=q^{n-1}(1-q)(x^2;q^2)_\infty\left(qh_n\left(\frac{x}{q};q\right)-[n]_q(cq+x)h_{n-1}\left(\frac{x}{q};q\right)\right),\end{aligned} \tag{5.1}$$

$$\int x(q^2x^2;q^2)_\infty h_n(x;q)\mathrm{d}_qx=\frac{q^{n-1}(x^2;q^2)_\infty}{[n-1]_q}\left((1-q)h_n\left(\frac{x}{q};q\right)-\frac{1-q^n}{q}xh_{n-1}\left(\frac{x}{q};q\right)\right), \tag{5.2}$$

$$\begin{aligned}&\int \frac{(q^2x^2;q^2)_\infty}{(q^{-(n+1)}x^2;q^2)_\infty}h_n(x;q)\mathrm{d}_qx\\&=\frac{(x^2;q^2)_\infty}{[n+1]_q(q^{-(n+1)}x^2;q^2)_\infty}\left(xh_n\left(\frac{x}{q};q\right)-q^n(1-q^n)h_{n-1}\left(\frac{x}{q};q\right)\right),\end{aligned} \tag{5.3}$$

and

$$\int x^{n-2}\left(q^2x^2;q^2\right)_\infty h_n(x;q)\mathrm{d}_qx=\frac{x^n(x^2;q^2)_\infty}{[n-1]_q}\left(\frac{h_n\left(\frac{x}{q};q\right)}{x}-\frac{1}{q}h_{n-1}\left(\frac{x}{q};q\right)\right). \tag{5.4}$$

Proof The discrete q-Hermite I polynomial of degree n is defined in (A9) and satisfies the second-order q-difference equation (A10). By comparing (A10) with (1.2), we get

$$p(x) = -\frac{x}{1-q}, \qquad r(x) = \frac{q^{1-n}[n]_q}{1-q}. \tag{5.5}$$

Then,

$$f(x) = (q^2x^2; q^2)_\infty \tag{5.6}$$

is a solution of (1.3). Therefore Eq. (2.4) becomes

$$\int (q^2x^2; q^2)_\infty h(x/q)\Big(\frac{1}{q}D_{q^{-1}}u(x) + \frac{1}{q}u(x)u(x/q) - \frac{q^{-n}x}{1-q}u(x/q) + \frac{q^{1-n}[n]_q}{(1-q)}\Big)y(x)\mathrm{d}_qx$$
$$= (x^2; q^2)_\infty h(x/q)\Big(y(x/q)u(x/q) - D_{q^{-1}}y(x)\Big). \tag{5.7}$$

By taking the fragment

$$D_{q^{-1}}u(x) + u(x)u(x/q) = 0, \tag{5.8}$$

we get

$$u(x) = \frac{1}{x+c}. \tag{5.9}$$

Hence,

$$h(x) = \begin{cases} 1 + \frac{x}{c}, & \text{if} c \neq 0; \\ x, & \text{if } c = 0, \end{cases} \tag{5.10}$$

is a solution of (2.3). Substituting with the values of $h(x)$ into (5.7) and using

$$D_{q^{-1}}h_n(x; q) = [n]_q h_{n-1}\Big(\frac{x}{q}; q\Big), \tag{5.11}$$

see [17, Eq. (3.28.7)], we get (5.1) for $c \neq 0$ and (5.2) for $c = 0$. To prove (5.3), we consider the fragment

$$\frac{1}{q}u(x)u(x/q) - \frac{q^{-n}x}{1-q}u(x/q) = 0,$$

then $u(x) = \frac{q^{1-n}}{1-q}x$ and $h(x) = \dfrac{1}{(q^{1-n}x^2; q^2)_\infty}$ is a solution of (2.3). Substituting with $h(x)$ and $u(x)$ into (5.7) and using (5.11), we get (5.3). Finally, the proof of (5.4) follows by taking the fragment

$$-\frac{q^{-n}x}{1-q}u(x/q)+\frac{q^{1-n}[n]_q}{(1-q)}=0.$$

In this case, $u(x)=\frac{[n]_q}{x}$ and $h(x)=x^n$ is a solution of (2.3). Substituting with $h(x)$ and $u(x)$ into (5.7) and using (5.11), we get (5.4). □

Theorem 5.2 *Let $n\in\mathbb{N}$ and c be a real number. If $\widetilde{h}_n(x;q)$ is the discrete q-Hermite II polynomial of degree n which is defined in* (A11)*, then*

$$\begin{aligned}&\int\frac{1}{(-x^2;q^2)_\infty}\left(qx[n-1]_q+c[n]_q\right)\widetilde{h}_n(x;q)\mathrm{d}_qx\\&=\frac{1-q}{(-x^2;q^2)_\infty}\left(\widetilde{h}_n(x;q)-q^{1-n}[n]_q(c+x)\widetilde{h}_{n-1}(x;q)\right),\end{aligned}\tag{5.12}$$

$$\int\frac{x}{(-x^2;q^2)_\infty}\widetilde{h}_n(x;q)\mathrm{d}_qx=\frac{1-q}{[n-1]_q(-x^2;q^2)_\infty}\left(\frac{1}{q}\widetilde{h}_n(x;q)-q^{-n}[n]_qx\widetilde{h}_{n-1}(x;q)\right),\tag{5.13}$$

$$\begin{aligned}&\int\frac{(-q^{n+3}x^2;q^2)_\infty}{(-x^2;q^2)_\infty}\widetilde{h}_n(x;q)\mathrm{d}_qx\\&=\frac{(-q^{n+1}x^2;q^2)_\infty}{[n+1]_q(-x^2;q^2)_\infty}\left(q^nx\widetilde{h}_n(x;q)-q^{1-n}(1-q^n)\widetilde{h}_{n-1}(x;q)\right),\end{aligned}\tag{5.14}$$

and

$$\int\frac{x^{n-2}}{(-x^2;q^2)_\infty}\widetilde{h}_n(x;q)\mathrm{d}_qx=\frac{x^n}{[n-1]_q(-x^2;q^2)_\infty}\left(\frac{\widetilde{h}_n(x;q)}{x}-\widetilde{h}_{n-1}(x;q)\right).\tag{5.15}$$

Proof The discrete q-Hermite II polynomial of degree n is defined in (A11) and satisfies the second-order q-difference equation (A12). By comparing (A12) with (1.5), we get

$$p(x)=-\frac{x}{1-q},\qquad r(x)=\frac{[n]_q}{1-q}.$$

Then, $F(x)=\frac{1}{(-x^2;q^2)_\infty}$ is a solution of (1.6), and (2.10) becomes

$$\begin{aligned}&\int\frac{k(qx)}{(-x^2;q^2)_\infty}\left(D_qu(x)+u(x)u(qx)-\frac{q^nx}{(1-q)}u(qx)+\frac{[n]_q}{(1-q)}\right)y(x)\mathrm{d}_qx\\&=\frac{k(x)}{(-x^2;q^2)_\infty}\left(y(x)u(x)-D_{q^{-1}}y(x)\right).\end{aligned}\tag{5.16}$$

Consider the fragment

$$D_q u(x) + u(x)u(qx) = 0. \tag{5.17}$$

Hence,

$$u(x) = \frac{1}{x+c} \tag{5.18}$$

and

$$k(x) = \begin{cases} 1 + \frac{x}{c}, & \text{if} c \neq 0; \\ x, & \text{if } c = 0, \end{cases} \tag{5.19}$$

is a solution of (2.9). Substituting with $u(x)$ and the values of $k(x)$ into (5.16), and using [17, Eq. (3.29.7)] (with x is replaced by $\frac{x}{q}$)

$$D_{q^{-1}} \widetilde{h}_n(x; q) = q^{1-n}[n]_q \widetilde{h}_{n-1}(x; q), \tag{5.20}$$

we get (5.12) for $c \neq 0$ and (5.13) for $c = 0$. The fragment

$$u(x)u(qx) - \frac{q^n x}{1-q} u(qx) = 0.$$

Then, $u(x) = \frac{q^n}{1-q} x$ and $k(x) = (-q^{n+1}x^2; q^2)_\infty$ is a solution of (2.9). Substituting with $k(x)$ into (5.16), we get (5.14). Similarly, to prove (5.15), we consider the fragment

$$-\frac{q^n x}{1-q} u(qx) + \frac{[n]_q}{1-q} = 0,$$

then we obtain $u(x) = q^{1-n}[n]_q \dfrac{1}{x}$ and $k(x) = x^n$. Substituting with $u(x)$ and $k(x)$ into (5.16) yields (5.15). □

Theorem 5.3 *Let c be a real number. If $Ai_q(x)$ is the q-Airy function which is defined in (A13), then*

$$\sum_{n=0}^{\infty}(-1)^n q^n \left(c-1+q^2+q^n x\right) Ai_q(q^n x) = \frac{qc+x}{q(1+q)}\, {}_1\phi_1(0;-q^2;q,-x) - (1-q)Ai_q\left(\frac{x}{q}\right), \tag{5.21}$$

$$\sum_{n=0}^{\infty}(-1)^n q^n \left(1-q^2-q^n x\right) Ai_q(q^n x) = (1-q)Ai_q(\frac{x}{q}) - \frac{x}{q(1+q)}\, {}_1\phi_1(0;-q^2;q,-x), \tag{5.22}$$

$$\sum_{n=0}^{\infty} q^{n+1} \left(q-q^3-q^n x\right) (-q^{-3}x;q)_n Ai_q(q^n x) = -\frac{x}{1+q}\, {}_1\phi_1(0;-q^2;q,-x) + \left(q(1+q)+\frac{x}{q}\right) Ai_q\left(\frac{x}{q}\right). \tag{5.23}$$

Proof The q-Airy function is defined in (A13) and satisfies the second-order q-difference equation (A14). By comparing (A14) with (1.2), we get

$$p(x) = -\frac{1+q}{q(1-q)x}, \qquad r(x) = \frac{1}{q(1-q)^2 x}. \tag{5.24}$$

By taking the fragment (5.8), we get $u(x)$ and $h(x)$ as in (5.9) and (5.10), respectively. Therefore, (2.4) takes the form

$$\int f(t)h(t/q)\left(-\frac{q(1+q)+t}{q^2(1-q)t}u(t/q) + \frac{1}{q(1-q)^2 t}\right) y(t)\mathrm{d}_q t = f(x/q)h(x/q)\left(y(x/q)u(x/q) - D_{q^{-1}}y(x)\right). \tag{5.25}$$

Denote the right hand side of Eq. (5.25) by $H(x)$. I.e

$$H(x) = f(x/q)h(x/q)\left(y(x/q)u(x/q) - D_{q^{-1}}y(x)\right).$$

Then, from (1.15), we obtain

$$\int_0^x f(t)h(t/q)\left(-\frac{q(1+q)+t}{q^2(1-q)t}u(t/q) + \frac{1}{q(1-q)^2 t}\right) y(t)\mathrm{d}_q t = H(x) - \lim_{n\to\infty} H(q^n x). \tag{5.26}$$

From (1.3), we obtain $f(qx) = -qf(x)$,

$$\prod_{k=0}^{n-1} \frac{f(q^{k+1}x)}{f(q^k x)} = \prod_{k=0}^{n-1}(-q),$$

then we get

$$f(q^n x) = (-1)^n q^n f(x) \quad (n \in \mathbb{N}_0). \tag{5.27}$$

Since

$$D_{q^{-1}} Ai_q(x) = \frac{1}{1-q^2}\, {}_1\phi_1(0; -q^2; q, -x), \tag{5.28}$$

and using the value of $h(x)$ at $c \neq 0$, we get

$$H(x) = \frac{f(x)}{cq}\left(\frac{cq+x}{q\big(1-q^2\big)}\, {}_1\phi_1(0; -q^2; q, -x) - Ai_q(\frac{x}{q})\right). \tag{5.29}$$

Hence, $\lim_{n\to\infty} H(q^n x) = 0$. From (1.14),(5.9), (5.10) with $c \neq 0$ and (5.26), we obtain

$$\int_0^x f(t)h(t/q)\left(-\frac{q(1+q)+t}{q^2(1-q)t}u(t/q) + \frac{1}{q(1-q)^2 t}\right) y(t)\, \mathrm{d}_q t$$
$$= \frac{f(x)}{cq(1-q)} \sum_{n=0}^{\infty} (-q)^n \left(c - 1 + q^2 + q^n x\right) y(q^n x) = H(x). \tag{5.30}$$

Combining Eqs. (5.29) and (5.30) yields (5.21). Substituting with the value of $h(x) = x$ at $(c = 0)$ yields (5.22). Now, we prove (5.23), by taking the fragment

$$\frac{1}{q}u(x)u(x/q) - \frac{q(1+q)+x}{q^2(1-q)x}u(x/q) = 0,$$

which implies that $u(x) = \dfrac{q(1+q)+x}{q(1-q)x}$. Since $h(x)$ satisfies (2.3), then

$$h(qx) = -\left(q + \frac{x}{q}\right)h(x),$$

$$\prod_{k=0}^{n-1} \frac{h(q^{k+1}x)}{h(q^k x)} = \prod_{k=0}^{n-1} (-q)^{k+1}\left(1 + q^{k-2}x\right),$$

then we get

$$h(q^n x) = (-q)^{n+1}\left(\frac{-x}{q^2};q\right)_n h(x) \quad \left(n \in \mathbb{N}_0\right). \tag{5.31}$$

Substituting with $u(x)$ into (2.4) and using equations (5.27), (5.28), and (5.31), we get (5.23). □

Theorem 5.4 *Let $c \in \mathbb{R}$. If $A_q(x)$ is the Ramanujan function which is defined in (A15), then*

$$\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} x^n \left(1-q+qc+q^{n+2}x\right) A_q(q^n x) = (1-q)A_q\left(\frac{x}{q}\right) - (cq+x)A_q(qx), \tag{5.32}$$

$$\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} x^n \left(1-q+q^{n+2}x\right) A_q(q^n x) = (1-q)A_q\left(\frac{x}{q}\right) - xA_q(qx), \tag{5.33}$$

$$\sum_{n=0}^{\infty} \left(1-q^2+q^{n+2}x\right)(x;q)_n A_q(q^n x) = \frac{x(1+q)-q}{q} A_q\left(\frac{x}{q}\right) - \frac{x^2}{q} A_q(qx). \tag{5.34}$$

Proof The Ramanujan function is defined in (A15) and satisfies the second-order q-difference equation (A16). By comparing (A16) with (1.2), we get

$$p(x) = \frac{1-qx}{q(1-q)x^2}, \qquad r(x) = \frac{1}{(1-q)^2 x^2}. \tag{5.35}$$

By taking the fragment (5.8), we get $u(x)$ and $h(x)$ as in (5.9) and (5.10), respectively. Therefore, (2.4) takes the form

$$\begin{aligned} &\int f(t)h(t/q)\left(\frac{1-t(1+q)}{q(1-q)t^2}u(t/q) + \frac{1}{(1-q)^2t^2}\right) y(t)\mathrm{d}_q t \\ &= f(x/q)h(x/q)\left(y(x/q)u(x/q) - D_{q^{-1}}y(x)\right). \end{aligned} \tag{5.36}$$

Denote the right hand side of Eq. (5.36) by $G(x)$. That is

$$G(x) = f(x/q)h(x/q)\left(y(x/q)u(x/q) - D_{q^{-1}}y(x)\right).$$

Then, from (1.15), we get

$$\int_0^x f(t)h(t/q)\left(\frac{1-t(1+q)}{q(1-q)t^2}u(t/q) + \frac{1}{(1-q)^2t^2}\right) y(t)\mathrm{d}_q t = G(x) - \lim_{n\to\infty} G(q^n x). \tag{5.37}$$

From (1.3), we obtain $f(qx) = q^2 x f(x)$. Consequently,

$$\prod_{k=0}^{n-1} \frac{f(q^{k+1}x)}{f(q^k x)} = \prod_{k=0}^{n-1} q^{k+2}x \quad (n \in \mathbb{N}_0),$$

then

$$f(q^n x) = q^{2n+\frac{n(n-1)}{2}} x^n f(x) \quad \Big(n \in \mathbb{N}_0\Big). \tag{5.38}$$

Since

$$D_{q^{-1}} A_q(x) = \frac{q}{1-q} A_q(qx), \tag{5.39}$$

substituting with the value of $h(x)$ at $c \neq 0$, then

$$G(x) = \frac{f(x)}{cqx} \left(A_q\left(\frac{x}{q}\right) - \frac{(cq+x)}{1-q} A_q(qx) \right). \tag{5.40}$$

Hence, $\lim_{n\to\infty} G(q^n x) = 0$. From (1.14), (5.9), (5.10) with $c \neq 0$ and (5.37), we obtain

$$\begin{aligned} &\int_0^x f(t)h(t/q) \left(-\frac{q(1+q)+t}{q^2(1-q)t} u(t/q) + \frac{1}{q(1-q)^2 t} \right) y(t)\, \mathrm{d}_q t \\ &= \frac{f(x)}{cq(1-q)} \sum_{n=0}^{\infty} q^{\frac{n(n-3)}{2}} x^{n-1} \Big(1 - q + qc + q^{n+2}x\Big) y(q^n x) = G(x). \end{aligned} \tag{5.41}$$

Combining equations (5.40) and (5.41) yields (5.32). Substituting with the value of $h(x) = x$ at $c = 0$ yields (5.33). Now, we prove (5.34), by taking the fragment

$$\frac{1}{q} u(x)u(x/q) + \frac{1 - x(1+q)}{q(1-q)x^2} u(x/q) = 0,$$

which implies that $u(x) = \frac{x(1+q)-1}{(1-q)x^2}$. Since $h(x)$ satisfies (2.3), then

$$h(qx) = \frac{1-qx}{x} h(x),$$

$$\prod_{k=0}^{n-1} \frac{h(q^{k+1})x}{h(q^k x)} = \prod_{k=0}^{n-1} \frac{1-q^{k+1}}{q^k x},$$

then we get

$$h(q^n x) = \frac{(qx;q)_n}{q^{\frac{n(n-1)}{2}} x^n} h(x) \quad (n \in \mathbb{N}_0). \tag{5.42}$$

Substituting with $u(x)$ into (2.4) and using (5.38), (5.39), and (5.42), we get (5.34). □

Theorem 5.5 *Let $n \in \mathbb{N}$. The following statements are true:*

(a) *If $h_n(x;q)$ is the discrete q-Hermite I polynomial of degree n which is defined in* (A9)*, then*

$$\int (q^2x^2;q^2)_\infty h_n(x;q) \mathrm{d}_q x = -q^{n-1}(1-q)(x^2;q^2)_\infty h_{n-1}\left(\frac{x}{q};q\right). \tag{5.43}$$

(b) *If $p_n(x;a,b;q)$ is the big q-Laguerre polynomial of degree n which is defined in* (A26)*, then*

$$\int \frac{(\frac{x}{a}, \frac{x}{b};q)_\infty}{(x;q)_\infty} p_n(x;a,b;q) \mathrm{d}_q x = \frac{abq^2(1-q)}{(1-aq)(1-bq)} \frac{\left(\frac{x}{aq}, \frac{x}{bq};q\right)_\infty}{(x;q)_\infty} p_{n-1}(x;aq,bq;q). \tag{5.44}$$

(c) *If $\alpha > -1$ and $L_n^\alpha(x;q)$ is the q-Laguerre polynomial of degree n which is defined in* (A28)*, then*

$$\int \frac{x^\alpha}{(-x;q)_\infty} L_n^\alpha(x;q) \mathrm{d}_q x = \frac{x^{\alpha+1}}{[n]_q(-x;q)_\infty} L_{n-1}^{\alpha+1}(x;q). \tag{5.45}$$

Proof The proof of (a) follows by substituting with $r(x)$ and $f(x)$ from (5.5) and (5.6), respectively, into (2.16). The proof of (b) follows by comparing (A27) with (1.2) to get

$$p(x) = \frac{x - q(a+b-qab)}{abq^2(1-q)(1-x)}, \qquad r(x) = -\frac{q^{-n-1}[n]_q}{ab(1-q)(1-x)}.$$

Hence, $f(x) = \dfrac{(\frac{x}{a}, \frac{x}{b};q)_\infty}{(qx;q)_\infty}$ is a solution of (1.3). Substituting with $r(x)$ and $f(x)$ into Eq. (2.16) and using

$$D_{q^{-1}} p_n(x;a,b;q) = \frac{q^{1-n}[n]_q}{(1-aq)(1-bq)} p_{n-1}(x;aq,bq;q), \tag{5.46}$$

see [17, Eq. (3.11.7)], we get (5.44). To prove (c), compare (A29) with (1.2) to obtain

$$p(x) = \frac{1-q^{\alpha+1}(1+x)}{q^{\alpha+1}x(1+x)(1-q)}, \qquad r(x) = \frac{[n]_q}{x(1-q)(1+x)}.$$

Hence, $f(x) = \dfrac{x^{\alpha+1}}{(-qx;q)_\infty}$ is a solution of (1.3). Finally, we prove (5.45) by substituting with $r(x)$ and $f(x)$ into (2.16) and using

$$D_{q^{-1}} L_n^{\alpha}(x;q) = \frac{-q^{\alpha+1}}{(1-q)} L_{n-1}^{\alpha+1}(x;q), \tag{5.47}$$

see [17, Eq. (3.21.8)]. □

Remark 1 (a) The indefinite q-integral (5.44) is nothing else but [14, Eq. (42)] or [17, Eq. (3.11.9)] (with n is replaced by $n-1$)

$$D_q\left(w(x;aq,bq;q)p_{n-1}(x;aq,bq;q)\right) = \frac{(1-aq)(1-bq)}{abq^2(1-q)} w(x;a,b;q)p_n(x;a,b;q),$$

where $w(x;a,b;q) = \frac{(\frac{x}{b},\frac{x}{a};q)_\infty}{(x;q)_\infty}$.

(b) The indefinite q-integral (5.45) is equivalent to [14, Eq. (46)] (if $m = n$) and to [17, Eq. (3.21.10)] (if $m = 0$) (with α is replaced by $\alpha+1$ and n is replaced by $n-1$)

$$D_q\left(w(x;\alpha+1;q)L_{n-1}^{\alpha+1}(x;q)\right) = [n]_q w(x;\alpha;q)L_n^{\alpha}(x;q),$$

where $w(x;\alpha;q) = \dfrac{x^{\alpha}}{(-x;q)_\infty}$.

Theorem 5.6 *The following statements are true:*

(a) *If $\widetilde{h}_n(x;q)$ is the discrete q-Hermite II polynomial of degree n which is defined in* (A11)*, then*

$$\int \frac{\widetilde{h}_n(x;q)}{(-x^2;q^2)_\infty} d_q x = -\frac{q^{1-n}(1-q)}{(-x^2;q^2)_\infty} \widetilde{h}_{n-1}(x;q). \tag{5.48}$$

(b) *If ν is a real number, $\nu > -1$, then*

$$\int \frac{qx^2 - q^{1-\nu}[\nu]_q^2}{x(-x^2(1-q)^2;q^2)_\infty} J_\nu^{(2)}(x|q^2) d_q x = \frac{-x}{(-x^2(1-q)^2;q^2)_\infty} D_{q^{-1}} J_\nu^{(2)}(x|q^2).$$

Proof The proof of (a) follows by substituting with $r(x)$ and $F(x)$ as in the proof of Theorems (5.2) into Eq. (2.17). To prove (b), compare (A21) with (1.5) to obtain

$$p(x) = \frac{1 - q\lambda^2 x^2(1-q)}{x}, \qquad r(x) = \frac{q\lambda^2 x^2 - q^{1-\nu}[\nu]_q^2}{x^2}.$$

Hence, $F(x) = \dfrac{x}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}$ is a solution of (1.6). Substituting with $r(x)$ and $F(x)$ into Eq. (2.17). □

Remark 2 The indefinite q-integral (5.48) is equivalent to [14, Eq. (68)] and to

$$D_q\left(w(x;q)\widetilde{h}_{n-1}(x;q)\right)=\frac{q^{n-1}}{1-q}w(x;q)\widetilde{h}_n(x;q),$$

where $w(x;q)=\frac{1}{(-x^2;q^2)}$, see [17, Eq. (3.29.9)].

Theorem 5.7 *The following statements are true:*

(a) *If $Ai_q(x)$ is the q-Airy function which is defined in (A13), then*

$$\sum_{k=0}^{\infty}(-q)^k Ai_q(q^k x)=\frac{1}{1+q}\,{}_1\phi_1(0;-q^2;q,-x). \tag{5.49}$$

(b) *If $A_q(x)$ is the Ramanujan function which is defined in (A15), then*

$$\sum_{k=0}^{\infty}q^{\frac{k(k+1)}{2}}x^k A_q(q^k x)=-\,{}_0\phi_1(-;0;q,-q^2x). \tag{5.50}$$

(c) *If $S_n(x;q)$ is the Stieltjes–Wigert polynomial of degree n ($n\in\mathbb{N}$) which is defined in (A30), then*

$$\sum_{k=0}^{\infty}q^{\frac{k(k+1)}{2}}x^k S_n(q^k x;q)=\frac{1}{1-q^n}S_{n-1}(qx;q). \tag{5.51}$$

Proof The proof of (a) follows by substituting with $r(x)$ from (5.24) into (2.16) and using (5.27) and (5.28). The proof of (b) follows by substituting with $r(x)$ from (5.35) into (2.16) and using (5.38) and (5.39). To prove (c), compare Eq. (A31) with (1.2) to get

$$p(x)=\frac{1-qx}{qx^2(1-q)},\qquad r(x)=\frac{[n]_q}{x^2(1-q)}.$$

From (1.3), we obtain $f(qx)=q^2xf(x)$. Consequently,

$$\prod_{j=0}^{k-1}\frac{f(q^{j+1})x}{f(q^jx)}=\prod_{j=0}^{k-1}q^{j+2}x\quad(n\in\mathbb{N}_0),$$

then

$$f(q^kx)=q^{2k+\frac{k(k-1)}{2}}x^kf(x)\quad(n\in\mathbb{N}_0). \tag{5.52}$$

Substituting with $r(x)$ into (2.16) and using (1.14), (5.52), and

$$D_{q^{-1}}S_n(x;q)=\frac{-q}{1-q}S_{n-1}(qx;q), \tag{5.53}$$

see [17, Eq. (3.27.7)], we get (5.51). ◻

6 q-Integrals from substitution of simple algebraic forms

In this section, we substitute into Eq. (2.4) with simple algebraic forms for $u(x)$ which involve arbitrary constants, such as

$$u(x)=\frac{a}{x}+b, \tag{6.1}$$

to derive indefinite q-integrals. Set

$$S_q(x):=\frac{1}{q}D_{q^{-1}}u(x)+\frac{1}{q}u(x)u(x/q)+A(x)u(x/q)+r(x). \tag{6.2}$$

Then, (2.4) will be

$$\int f(x)h(x/q)S_q(x)y(x)d_qx=f(x/q)h(x/q)\left(y(x/q)u(x/q)-D_{q^{-1}}y(x)\right), \tag{6.3}$$

where the constants a and b in Eq. (6.1) are chosen so that $S_q(x)$ has a simple form. Also, we define

$$T_q(x):=D_qu(x)+u(x)u(qx)+\tilde{A}(x)u(qx)+r(x). \tag{6.4}$$

Then, (2.10) will be

$$\int F(x)k(qx)T_q(x)y(x)\mathrm{d}_qx=F(x)k(x)\left(y(x)u(x)-D_{q^{-1}}y(x)\right). \tag{6.5}$$

Theorem 6.1 *Let $n\in\mathbb{N}$, $n\geq 2$. Let $h_n(x;q)$ be the discrete q-Hermite I polynomial of degree n which is defined in (A9). Then,*

$$\int x(q^2x^2;q^2)_\infty h_n(x;q)\mathrm{d}_qx=\frac{(1-q)x(x^2;q^2)_\infty}{[n]_q-1}\left(\frac{q^n}{x}h_n\left(\frac{x}{q};q\right)\right.$$
$$\left.-q^{n-1}[n]_qh_{n-1}\left(\frac{x}{q};q\right)\right), \tag{6.6}$$

and

$$\int x^{n-2}(q^2x^2;q^2)_\infty h_n(x;q)\mathrm{d}_qx=\frac{x^n(x^2;q^2)_\infty}{[n-1]_q}\left(\frac{h_n(\frac{x}{q};q)}{x}-\frac{1}{q}h_{n-1}\left(\frac{x}{q};q\right)\right). \tag{6.7}$$

Proof From (A10),

$$p(x) = -\frac{x}{1-q}, \qquad r(x) = \frac{q^{1-n}[n]_q}{1-q}.$$

Hence, $f(x) = (q^2x^2; q^2)_\infty$ is a solution of Eq. (1.3). Set $u(x)$ as in (6.1). Then,

$$S_q(x) = \frac{a(a-1)}{x^2} + \frac{ab(1+q)}{qx} + \frac{q^{1-n}([n]_q - a) - q^{-n}bx}{1-q} + \frac{b^2}{q}. \tag{6.8}$$

If $a = 1$ and $b = 0$ in (6.8), then

$$S_q(x) = \frac{q^{2-n}[n-1]_q}{1-q},$$

and $h(x) = x$ is a solution of (2.3). By substituting with $u(x)$, $S_q(x)$, and $h(x)$ into (6.3) and using (5.11), we get (6.6). If $a = [n]_q$ and $b = 0$ in (6.8), then

$$S_q(x) = q[n]_q[n-1]_q \frac{1}{x^2}.$$

Hence, $h(x) = x^n$ is a solution of (2.3). Substituting with $u(x)$, $S_q(x)$, and $h(x)$ into (6.3) and using (5.11), we get (6.7). □

Remark 3 The indefinite q-integral (6.7) is equivalent to (5.4) in Theorem 5.1.

Theorem 6.2 *Let $n \in \mathbb{N}$, $n \geq 2$. Let $\widetilde{h}_n(x; q)$ be the discrete q-Hermite II polynomial of degree n which is defined in* (A11). *Then,*

$$\int \frac{x}{(-x^2; q^2)_\infty} \widetilde{h}_n(x; q) d_q x = \frac{(1-q)x}{[n-1]_q(-x^2; q^2)_\infty} \left(\frac{\widetilde{h}_n(x; q)}{qx} - q^{-n}[n]_q \widetilde{h}_{n-1}(x; q) \right), \tag{6.9}$$

and

$$\int \frac{x^{n-2}}{(-x^2; q^2)_\infty} \widetilde{h}_n(x; q) d_q x = \frac{x^n}{[n-1]_q(-x^2; q^2)_\infty} \left(\frac{\widetilde{h}_n(x; q)}{x} - \widetilde{h}_{n-1}(x; q) \right). \tag{6.10}$$

Proof From (A12),

$$p(x) = -\frac{x}{1-q}, \qquad r(x) = \frac{[n]_q}{1-q}.$$

Then, $F(x) = \frac{1}{(-x^2; q^2)_\infty}$ is a solution of (1.6). Set $u(x)$ as in (6.1), we get

$$T_q(x) = \frac{a(a-1)}{qx^2} + \frac{ab(1+q)}{qx} + \frac{[n]_q - q^n(q^{-1}a + bx)}{1-q} + b^2. \tag{6.11}$$

If $a = 1$ and $b = 0$ in (6.11), then

$$T_q(x) = \frac{[n-1]_q}{1-q}.$$

Therefore, $k(x) = x$ is a solution of (2.9). By substituting with $u(x)$, $T_q(x)$, and $k(x)$ into (6.5) and using Eq. (5.20), we get (6.9). If $a = q^{1-n}[n]_q$ and $b = 0$ in (6.11), then

$$T_q(x) = q^{1-2n}[n]_q[n-1]_q\frac{1}{x^2}.$$

Therefore, $k(x) = x^n$ is a solution of (2.9). By substituting with $u(x)$, $T_q(x)$, and $k(x)$ into (6.5) and using (5.20), we get (6.10). □

Remark 4 The indefinite q-integral (6.10) is equivalent to [14, Eq. (69)] (if $m = n$) and to (5.15) in Theorem 5.2.

Theorem 6.3 *Let $n \in \mathbb{N}$. If $S_n(x; q)$ is the Stieltjes-Wigert polynomial of degree n defined in* (A30)*, then*

$$\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}+nk} x^k S_n(q^k x; q) = S_n\left(\frac{x}{q}; q\right) + \frac{x}{1-q^n} S_{n-1}(qx; q), \tag{6.12}$$

and

$$\sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} x^k \left(1 + q^{2+k}[n-1]_q x\right) S_n(q^k x; q) = S_n\left(\frac{x}{q}; q\right) + \frac{x}{1-q} S_{n-1}(qx; q). \tag{6.13}$$

Proof By comparing (A31) with (1.2), we get

$$p(x) = \frac{1-qx}{qx^2(1-q)}, \qquad r(x) = \frac{[n]_q}{x^2(1-q)}.$$

Set $u(x)$ as in (6.1). Then,

$$\begin{aligned} S_q(x) = &\frac{b + q[n]_q}{q(1-q)x^2} + \frac{a(a-[n]_q)}{x^2} + \frac{ab(1+q) - qb[n-1]_q}{qx} \\ &+ \frac{a(1-x)}{(1-q)x^3} - \frac{b}{q(1-q)x} + \frac{b^2}{q}. \end{aligned} \tag{6.14}$$

We set $a = [n]_q$ and $b = 0$ in (6.14) then

$$S_q(x) = \frac{[n]_q}{(1-q)x^3}.$$

Therefore, $h(x) = x^n$ is a solution of (2.3). By substituting with $h(x)$, $S_q(x)$, and $u(x)$ into (6.3), using (5.52) and (5.53), we get (6.12).

If $a = 1$ and $b = 0$ in (6.14), then

$$S_q(x) = \frac{1 + q^2[n-1]_q x}{(1-q)x^3}.$$

Therefore, $h(x) = x$ ia a solution of (2.3). By substituting with $h(x)$, $S_q(x)$, and $u(x)$ into (6.3) and using (5.52) and (5.53), we get (6.13). □

7 Conclusions

A method of deriving q-integrals using fragments of q-Riccati equations has been presented. The method of fragmentation used is analogous to but not equivalent to that presented in [14]. Only two q-Riccati fragments have been presented here in detail, and these give the quadrature formulas presented in Eqs. (2.20) and (2.22)–(2.23).

8 Appendix A: q-Functions

Jackson introduced three q-analogues of Bessel functions [11, 16], they are defined by

$$J_\nu^{(1)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{(q, q^{\nu+1};q)_n} (z/2)^{2n+\nu}, \quad |z| < 2, \tag{A1}$$

$$J_\nu^{(2)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+\nu)}}{(q, q^{\nu+1};q)_n} (z/2)^{2n+\nu}, \quad z \in \mathbb{C}, \tag{A2}$$

$$J_\nu^{(3)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q, q^{\nu+1};q)_n} (z)^{2n+\nu}, \quad z \in \mathbb{C}. \tag{A3}$$

The solutions of the second-order q-difference equation, see [1],

$$\frac{1}{q} D_{q^{-1}} D_q y(x) - y(x) = 0 \quad (x \in \mathbb{R}), \tag{A4}$$

under the initial conditions

$$y(0) = 0, \quad D_q y(0) = 1, \quad \text{and} \quad y(0) = 1, \quad D_q y(0) = 0,$$

are the functions $\sin(x;q)$ and $\cos(x;q)$, respectively. The functions $\sin(z;q)$ and $\cos(z;q)$ are defined for $z \in \mathbb{C}$ by

$$\sin(z;q) := \frac{(q;q)_\infty}{(q^{1/2};q)_\infty} z^{1/2} J^{(3)}_{1/2}(z(1-q);q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2+n} z^{2n+1}}{\Gamma_q(2n+2)}, \tag{A5}$$

$$\cos(z;q) := \frac{(q;q)_\infty}{(q^{1/2};q)_\infty} (zq^{-1/2})^{1/2} J^{(3)}_{-1/2}(z(1-q)/\sqrt{q};q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2} z^{2n}}{\Gamma_q(2n+1)}. \tag{A6}$$

The q-trigonometric functions satisfy the q-difference equations

$$D_{q^{-1}} \sin(z;q) = \cos\left(q^{\frac{-1}{2}} z; q\right), \tag{A7}$$

$$D_{q^{-1}} \cos(z;q) = -q^{\frac{1}{2}} \sin\left(q^{\frac{-1}{2}} z; q\right). \tag{A8}$$

The discrete q-Hermite I polynomial of degree n

$$h_n(x;q) := q^{\binom{n}{2}} {}_2\phi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0, \end{matrix} \mid q; -qx\right), n \in \mathbb{N}_0 \tag{A9}$$

satisfies the second-order q-difference equation, see [17, Eq. (3.28.5)],

$$\frac{1}{q} D_{q^{-1}} D_q y(x) - \frac{x}{1-q} D_{q^{-1}} y(x) + \frac{q^{1-n}[n]_q}{1-q} y(x) = 0. \tag{A10}$$

The discrete q-Hermite II polynomials of degree n

$$\widetilde{h}_n(x;q) := x^n {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-n+1} \\ 0, \end{matrix} \mid q^2; \frac{-q^2}{x^2}\right), n \in \mathbb{N}_0, \tag{A11}$$

satisfies the second-order q-difference equation, see [17, Eq. (3.29.5)],

$$\frac{1}{q} D_{q^{-1}} D_q y(x) - \frac{x}{1-q} D_q y(x) + \frac{[n]_q}{1-q} y(x) = 0. \tag{A12}$$

The q-Airy function

$$Ai_q(x) := {}_1\phi_1(0; -q; q, -x), \tag{A13}$$

satisfies the second-order q-difference equation, see [19, Eq. (4)],

$$\frac{1}{q} D_{q^{-1}} D_q y(x) - \frac{1+q}{qx(1-q)} D_{q^{-1}} y(x) + \frac{1}{qx(1-q)^2} y(x) = 0. \tag{A14}$$

The Ramanujan function

$$A_q(x) := {}_0\phi_1(-; 0; q, -qx), \tag{A15}$$

satisfies the second-order q-difference equation, see [19, Eq. (5)],

$$\frac{1}{q} D_{q^{-1}} D_q y(x) + \frac{1-qx}{qx^2(1-q)} D_{q^{-1}} y(x) + \frac{1}{x^2(1-q)^2} y(x) = 0. \tag{A16}$$

The q-hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix} a_1 \; a_2 \; \ldots \; a_r \\ b_1 \; b_2 \; \ldots \; b_s, \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \ldots (a_r; q)_n}{(q; q)_n (b_1; q)_n (b_2; q)_n \ldots (b_s; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} z^n,$$

whenever the series converges, see [11].

The q-hypergeometric functions ${}_2\phi_1(q^a, q^b; q^c; q, x)$ satisfy the second-order q-difference equation [11]

$$\frac{1}{q} D_{q^{-1}} D_q y(x) + \frac{[c]_q - [a+b+1]_q \frac{x}{q}}{x(q^c - q^{a+b}x)} D_{q^{-1}} y(x) - \frac{[a]_q [b]_q}{x(q^c - q^{a+b}x)} y(x) = 0. \tag{A17}$$

The functions

$$y_1(x) = {}_2\phi_1\left(q^a, q^b; q^c; q, x\right), \quad c \neq q^{-n}, n \in \mathbb{N}_0,$$

$$y_2(x) = x^{1-c} {}_2\phi_1\left(q^{a+1-c}, q^{b+1-c}; q^{2-c}; q, x\right), \quad q^c \neq q^{n+2}, n \in \mathbb{N}_0,$$

$$y_3(x) = x^{-a} {}_2\phi_1\left(q^a, q^{a+1-c}; q^{a+1-b}; q, \frac{q^{c-a-b+1}}{x}\right), \quad q^a \neq q^{b-n-1}, n \in \mathbb{N}_0,$$

and

$$y_4(x) = x^{-b} {}_2\phi_1\left(q^b, q^{b+1-c}; q^{b+1-a}; q, \frac{q^{c-a-b+1}}{x}\right), \quad q^b \neq q^{a-n-1}, n \in \mathbb{N}_0,$$

are solutions of the basic hypergeometric q-difference Eq. (A17), see [11].

Lemma A.1 *Let α and β be complex numbers with positive real parts. Then,*

$$\begin{aligned} B_q(\alpha, \beta; x) &:= \int_0^x t^{\alpha-1} (qt; q)_{\beta-1} \, \mathrm{d}_q t \\ &= x^\alpha (1-q)(qx; q)_{\beta-1} \, {}_2\phi_1(q^\beta x, q; qx; q, q^\alpha), \end{aligned} \tag{A18}$$

where $(qt;q)_{\beta-1} = \dfrac{(qt;q)_\infty}{(q^\beta t;q)_\infty}$.

Proof From (1.14),

$$B_q(\alpha,\beta;x) = x^\alpha(1-q)\sum_{n=0}^{\infty} q^{n\alpha}\frac{(q^{n+1}x;q)_\infty}{(q^{n+\beta}x;q)_\infty}, \quad (\Re(\alpha>0)).$$

Since $(a;q)_n = \dfrac{(a;q)_\infty}{(aq^n;q)_\infty}$, then

$$\begin{aligned}B_q(\alpha,\beta;x) &= x^\alpha(1-q)\frac{(qx;q)_\infty}{(q^\beta x;q)_\infty}\sum_{n=0}^{\infty} q^{n\alpha}\frac{(q^\beta x;q)_n(q;q)_n}{(qx;q)_n(q;q)_n}\\ &= x^\alpha(1-q)(qx;q)_{\beta-1}\ {}_2\phi_1(q^\beta x,q;qx;q,q^\alpha).\end{aligned}$$

□

It is worth noting that from Lemma A.1,

$$B_q(\alpha,\beta;1) = B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}.$$

One of Heine's transformations of ${}_2\phi_1$ series

$${}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\,{}_2\phi_1(c/b,z;az;q,b), \tag{A19}$$

see [11, Eq. (III.1)]. The second Jackson q-Bessel function

$$J_\nu^{(2)}(x|q^2) := J_\nu^{(2)}(2x(1-q);q^2), \tag{A20}$$

satisfies the second-order q-difference equation [18]

$$\frac{1}{q}D_{q^{-1}}D_q y(x) + \frac{1-qx^2(1-q)}{x}D_q y(x) + \frac{qx^2 - q^{1-\nu}[\nu]_q^2}{x^2}y(x) = 0. \tag{A21}$$

The big q-Legendre polynomials

$$p_n(x;-1;q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}, x \\ q, -q, \end{matrix}\ \Big|\ q;q\right) \tag{A22}$$

satisfy the second-order q-difference equation, see [17, Eq. (3.5.17)],

$$\frac{1}{q}D_{q^{-1}}D_q y(x)+\frac{x(1+q)}{q^2(x^2-1)}D_{q^{-1}}y(x)-\frac{[n]_q[n+1]_q}{q^{1+n}(x^2-1)}y(x)=0. \tag{A23}$$

The little q-Legendre polynomials

$$p_n(x|q) := {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+1} \\ q, \end{matrix} \mid q; qx\right) \tag{A24}$$

satisfy the second-order q-difference equation, see [17, Eq. (3.12.16)],

$$\frac{1}{q}D_{q^{-1}}D_q y(x)+\frac{qx+x-1}{qx(qx-1)}D_{q^{-1}}y(x)+\frac{[n]_q[n+1]_q}{q^n x(1-qx)}y(x)=0. \tag{A25}$$

The big q-Laguerre polynomial

$$p_n(x;a,b;q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, 0, x \\ aq, bq, \end{matrix} \mid q; q\right) \tag{A26}$$

satisfies the second-order q-difference equation, see [17, Eq. (3.11.5)],

$$\frac{1}{q}D_{q^{-1}}D_q y(x)+\frac{x-q(a+b-qab)}{abq^2(1-q)(1-x)}D_{q^{-1}}y(x)-\frac{q^{-n-1}[n]_q}{ab(1-q)(1-x)}y(x)=0. \tag{A27}$$

The q-Laguerre polynomial of degree n

$$L_n^{\alpha}(x;q) := \frac{1}{(q;q)_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n}, -x \\ 0 \end{matrix} \mid q; q^{n+\alpha+1}\right), \qquad \alpha>-1, n\in\mathbb{N}, \tag{A28}$$

satisfies the second-order q-difference equation, see [17, Eq. (3.21.6)],

$$\frac{1}{q}D_{q^{-1}}D_q y(x)+\frac{1-q^{\alpha+1}(1+x)}{q^{\alpha+1}x(1+x)(1-q)}D_{q^{-1}}y(x)+\frac{[n]_q}{x(1-q)(1+x)}y(x)=0. \tag{A29}$$

The Stieltjes–Wigert polynomials

$$S_n(x;q) := \frac{1}{(q;q)_n}\,{}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0, \end{matrix} \mid q; -q^{n+1}x\right), \quad (n\in\mathbb{N}_0), \tag{A30}$$

satisfy the second-order q-difference equation, see [17, Eq. (3.27.5)],

$$\frac{1}{q}D_{q^{-1}}D_q y(x) + \frac{1-qx}{qx^2(1-q)}D_{q^{-1}}y(x) + \frac{[n]_q}{x^2(1-q)}y(x) = 0. \tag{A31}$$

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