The Ramanujan Journal (2024) 64:143–151
https://doi.org/10.1007/s11139-023-00817-y

Some new Ramanujan-type modular equations of degree 15

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Received: 21 July 2023 / Accepted: 21 November 2023 / Published online: 1 February 2024

Abstract
Ramanujan in his notebook recorded two modular equations involving multipliers with moduli of degrees (1,7) and (1,23). In this paper, we find some new Ramanujan-type modular equations involving multipliers with moduli of degrees (3,5) and (1,15), and give concise proofs by employing Ramanujan's multiplier functional equation.

Keywords Theta functions · Modular equations · Multiplier

Mathematics Subject Classification 11F20 · 33C05

1 Introduction

Ramanujan defined modular equations as follows: Suppose that

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}, \tag{1}$$

holds for some positive integer n or positive rational fraction $n = i/j$ (i and j are coprime). The relation between α and β induced by the above equation is called a modular equation of degree n, and we say that β over α is of degree n. Let

This work was supported by the National Natural Science Foundation of China (Grant No. 42174037).

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$$z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right),\quad z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right). \tag{2}$$

The multiplier m of the modular equation is defined by

$$m := \frac{z_1}{z_n}. \tag{3}$$

Before proceeding to the main theta function identity of this paper, we shall first recall certain known theta function identities which we need in the sequel. Throughout the paper, we assume $|q| < 1$. The following is the well-known Jacobi triple product identity:

$$\begin{aligned} f(a,b) : &= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab| < 1. \end{aligned} \tag{4}$$

The two particular facts of $f(a,b)$ (See [1], Entry 22, p. 36), are as follows

$$\begin{aligned} \varphi(q) &:= f(q,q) = \left(-q;q^2\right)_\infty^2 \left(q^2;q^2\right)_\infty, \\ \psi(q) &:= f\left(q,q^3\right) = \left(-q;q^2\right)_\infty \left(q^2;q^2\right)_\infty. \end{aligned} \tag{5}$$

Let the base q in the classical theory of elliptic functions be defined by

$$q = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}\right). \tag{6}$$

We have the following well-known identities (See [1], Entry 10 and 11, pp. 122–123)

$$z_1 = \varphi^2(q),\quad z_n = \varphi^2(q^n),\quad m = \frac{\varphi^2(q)}{\varphi^2(q^n)}, \tag{7}$$

and

$$\begin{aligned} &\frac{\sqrt[4]{q}\,\psi(q^2)}{\varphi(q)} = \frac{\sqrt[4]{\alpha}}{2},\quad \frac{\varphi(-q)}{\varphi(q)} = \sqrt[4]{1-\alpha},\quad \frac{q^{n/4}\psi(q^{2n})}{\varphi(q^n)} = \frac{\sqrt[4]{\beta}}{2}, \\ &\frac{\varphi(-q^n)}{\varphi(q^n)} = \sqrt[4]{1-\beta}, \end{aligned} \tag{8}$$

and

$$\frac{\sqrt[8]{q}\psi(q)}{\varphi(q)} = \frac{\sqrt[8]{\alpha}}{\sqrt{2}}, \quad \frac{\varphi(-q^2)}{\varphi(q)} = \sqrt[8]{1-\alpha}, \quad \frac{q^{n/8}\psi(q^n)}{\varphi(q^n)} = \frac{\sqrt[8]{\beta}}{\sqrt{2}},$$
$$\frac{\varphi(-q^{2n})}{\varphi(q^n)} = \sqrt[8]{1-\beta}. \tag{9}$$

Ramanujan found and recorded the following modular equations involving multipliers with moduli of degrees 7 and 23, in the case degree ($n \bmod 8) = 7$, respectively.

Theorem 1 (See *[1]*, Entry 19, pp. 314–315) *If β has degree 7 over α, and m is the multiplier for degree 7, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1, \tag{10}$$

and

$$m - \frac{7}{m} = 2\left((\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}\right)\left(2 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right), \tag{11}$$

and Ramanujan's theta functions form is

$$\begin{aligned} \frac{\varphi^2(q)}{\varphi^2(q^7)} - 7\frac{\varphi^2(q^7)}{\varphi^2(q)} &= 2(2X - Y)\left(2 + (2X)^2 + Y^2\right), \\ X = q\frac{\psi(q)\psi(q^7)}{\varphi(q)\varphi(q^7)} &= q\frac{\psi(q^2)\psi(q^{14})}{\psi(q)\psi(q^7)}, \\ Y = \frac{\varphi(-q^2)\varphi(-q^{14})}{\varphi(q)\varphi(q^7)} &= \frac{\psi(-q)\psi(-q^7)}{\psi(q)\psi(q^7)}. \end{aligned} \tag{12}$$

The Eq. (11) has been proved by B. C. Berndt using the method of parametrization. Recently, K. R. Vasuki and R. G. Veeresha [2] have given the proof of the equivalent representation (12) by the theta function identity for degree 7.

Theorem 2 (See *[1]*, Entry 15, p. 411) *If β is of the 23 degree over α, and m is the multiplier for same degree, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1, \tag{13}$$

and

$$\begin{aligned} m - \frac{23}{m} = 2&\left((\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}\right)\left(11 - 13\cdot 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}\right. \\ &+ 18\cdot 2^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} - 14\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \\ &\left.+2^{5/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}\right). \end{aligned} \tag{14}$$

Berndt finished the proof of (14) by the theory of modular forms (See [1], Entry 15, pp. 415–416). Similarly, in this paper, we also find some new Ramanujan-type modular equations involving multipliers with moduli of degrees (3,5) and (1,15).

2 Some new Ramanujan-type modular equations of degrees 15 and $\frac{5}{3}$

The beautiful Ramanujan-type modular equations of degrees 15 and $\frac{5}{3}$ (See [1], Entry 21, p. 435): Let α and β have degrees $(n_1, n_2) = (3, 5)$; or $(1, 15)$, respectively. Then

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} \pm \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} = \sqrt{\frac{1}{2}(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)})}. \tag{15}$$

where the minus sign is chosen in the first case and the plus sign is selected in the last case.

For modular equations in the form of Russell, we redefine (See [3], Entry 21, p. 435),

$$\begin{cases} P = 1 + (-1)^{\frac{n1+n2}{8}} \left((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8}\right), \\ Q = 4\left((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + (-1)^{\frac{n1+n2}{8}} \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\right), \\ R = 4(\alpha\beta(1-\alpha)(1-\beta))^{1/8}. \end{cases} \tag{16}$$

If α and β have degrees $(n_1, n_2) = (1, 15)$ (See [3], Entry 21, p. 435), then

$$P(P^2 - Q) + R = 0, \tag{17}$$

and, if α and β have degrees $(n_1, n_2) = (3, 5)$,

$$P(P^2 + Q) + R = 0. \tag{18}$$

We can verify that these two modular equations (17) and (18) are equivalent formulations of (15).

Theorem 3 (new modular equations for degrees $(n_1, n_2) = (1, 15)$) *Let α and β have degrees $(n_1, n_2) = (1, 15)$, and $m = \frac{z_1}{z_{15}}$ is the multiplier for degree $n = 15$. Then*

1. *The natural form*

$$m-\frac{15}{m}=2\left((\alpha\beta)^{1/8}-\{(1-\alpha)(1-\beta)\}^{1/8}\right)\times\Big[1+3\left((\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}\right)+3\left((\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}\right)+2(\alpha\beta(1-\alpha)(1-\beta))^{1/8}\times\left(3+(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}\right)\Big], \tag{19}$$

2. *The emphatic form*

$$m-\frac{15}{m}=2\left((\alpha\beta)^{1/4}-\{(1-\alpha)(1-\beta)\}^{1/4}\right)\times\left[4\sqrt{\frac{1}{2}(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)})}+4-\left((\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}\right)\right], \tag{20}$$

3. *Ramanujan's theta functions form:*

$$\frac{\varphi^2(q)}{\varphi^2(q^{15})}-15\frac{\varphi^2(q^{15})}{\varphi^2(q)}=2\left(4\frac{q^4\psi(q^2)\psi(q^{30})}{\varphi(q)\varphi(q^{15})}-\frac{\varphi(-q)\varphi(-q^{15})}{\varphi(q)\varphi(q^{15})}\right)\times\left[4\left(\frac{\varphi(q^2)\varphi(q^{30})}{\varphi(q)\varphi\left(q^{15}\right)}+4\frac{q^8\psi(q^4)\psi(q^{60})}{\varphi(q)\varphi\left(q^{15}\right)}\right)+4-\left(4\frac{q^4\psi(q^2)\psi(q^{30})}{\varphi(q)\varphi(q^{15})}+\frac{\varphi(-q)\varphi(-q^{15})}{\varphi(q)\varphi(q^{15})}\right)\right]. \tag{21}$$

Proof (i). According to Ramanujan's multiplier function equation (See [1], Entry 24(vi), p. 217), we can write

$$n\frac{\mathrm{d}\,\alpha}{\mathrm{d}\,\beta}=\frac{\alpha(1-\alpha)}{\beta(1-\beta)}m^2. \tag{22}$$

Let

$$\begin{aligned}x&:=x(t)=(\alpha\beta)^{1/8},\\ y&:=y(t)=\{(1-\alpha)(1-\beta)\}^{1/8},\end{aligned} \tag{23}$$

and employing the technique adopted in the proof of modular equation of degree 7 by Berndt ([1], p316–319), one can deduce that

$$\begin{aligned} \alpha := \alpha(t) &= \frac{1}{2}\left(1 + x^8 - y^8 + \sqrt{1 - 2x^8 - 2y^8 - 2x^8y^8 + x^{16} + y^{16}}\right), \\ \beta := \beta(t) &= \frac{1}{2}\left(1 + x^8 - y^8 - \sqrt{1 - 2x^8 - 2y^8 - 2x^8y^8 + x^{16} + y^{16}}\right), \\ &\quad (0 < \beta < \alpha < 1), \end{aligned} \tag{24}$$

and using the Eqs. (22)–(24), we deduce

$$\frac{n}{m^2} = \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \frac{\mathrm{d}\beta/\mathrm{d}t}{\mathrm{d}\alpha/\mathrm{d}t} = -\frac{\alpha y x'(t) + (1-\alpha) x y'(t)}{\beta y x'(t) + (1-\beta) x y'(t)}, \tag{25}$$

and

$$\left(m - \frac{n}{m}\right)^2 = -\frac{n\left(y(\alpha+\beta)x'(t) + x(-\alpha-\beta+2)y'(t)\right)^2}{(\alpha y x'(t) + (1-\alpha)xy'(t))\,(\beta y x'(t) + (1-\beta)xy'(t))}. \tag{26}$$

Again, from the Eq. (17), we have the equivalent modular equation of degree 15 in the form of x and y, it reads

$$1 - x - x^2 + x^3 - y - 2xy - x^2y - y^2 - xy^2 + y^3 = 0. \tag{27}$$

From the Eqs. (19) and (23), we also have

$$m - \frac{15}{m} = 2(x-y)(1 + 3x + 3x^2 + 3y + 6xy + 2x^2y + 3y^2 + 2xy^2). \tag{28}$$

Now set

$$x = \frac{1}{2}\left(\frac{1}{t} - \varrho\right), \quad y = \frac{1}{2}\left(\frac{1}{t} + \varrho\right), \tag{29}$$

using the Eq. (27), we obtain

$$t\varrho^2 = 1 + t - t^2. \tag{30}$$

Solving the above equation for ϱ and noticing that $x < y$, we get

$$\varrho = \frac{\sqrt{1+t-t^2}}{\sqrt{t}}, x = \frac{1}{2}\left(\frac{1}{t} - \frac{\sqrt{1+t-t^2}}{\sqrt{t}}\right), \quad y = \frac{1}{2}\left(\frac{1}{t} + \frac{\sqrt{1+t-t^2}}{\sqrt{t}}\right). \tag{31}$$

Employing (31) and (24) in (26) with $n = 15$, we deduce

$$\left(m - \frac{15}{m}\right)^2 = \frac{\left(1 + t - t^2\right)\left(1 + 5t + 5t^2 + 3t^3\right)^2}{t^7}, \tag{32}$$

and employing (31) in (28), we obtain

$$m - \frac{15}{m} = -\frac{\sqrt{1 + t - t^2}\left(1 + 5t + 5t^2 + 3t^3\right)}{t^{7/2}}. \tag{33}$$

We can verify that these two Eqs. (32) and (33) are equivalent. So far, we employed the method of parameterization to prove the modular equation of degree 15 which involves the multiplier m. □

Proof (ii). Substituting (27) into (28), we obtain

$$\begin{aligned} L &= (1 + 3x + 3x^2 + 3y + 6xy + 2x^2y + 3y^2 + 2xy^2) - 0 \\ &= L - (1 - x - x^2 + x^3 - y - 2xy - x^2y - y^2 - xy^2 + y^3) \\ &= (x + y)(4(x + y + xy) + 4 - x^2 - y^2), \end{aligned} \tag{34}$$

By using the Eq. (15), we can write

$$x + y + xy = \sqrt{\frac{1}{2}(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)})}, \tag{35}$$

Substituting (34) and (35) into (28), we arrive at

$$m - \frac{15}{m} = 2(x^2 - y^2)(4\sqrt{\frac{1}{2}(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)})} + 4 - x^2 - y^2). \tag{36}$$

which completes the proof. □

Proof (iii). If β over α is degree 15, we employ the identity (See [1],p. 433 eq. (20.6))

$$\begin{aligned} \sqrt{\frac{1}{2}(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)})} &= \frac{\sqrt{1 - \sqrt{1-\alpha}}}{\sqrt{2}} \frac{\sqrt{1 - \sqrt{1-\beta}}}{\sqrt{2}} \\ &+ \frac{\sqrt{1 + \sqrt{1-\alpha}}}{\sqrt{2}} \frac{\sqrt{1 + \sqrt{1-\beta}}}{\sqrt{2}}, \end{aligned} \tag{37}$$

and translate this identity in form of Ramanujan's theta functions (See [1], Entry 10 and 11, pp. 122–123)

$$\sqrt{\frac{1}{2}(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)})} = \frac{\varphi(q^2)\varphi(q^{30})}{\varphi(q)\varphi\left(q^{15}\right)} + 4q^8 \frac{\psi(q^4)\psi(q^{60})}{\varphi(q)\varphi\left(q^{15}\right)}, \tag{38}$$

and use (8), we get

$$(\alpha\beta)^{1/4} = 4q^4 \frac{\psi(q^2)\psi(q^{30})}{\varphi(q)\varphi(q^{15})}, \quad \{(1-\alpha)(1-\beta)\}^{1/4} = \frac{\varphi(-q)\varphi(-q^{15})}{\varphi(q)\varphi(q^{15})}. \tag{39}$$

Substituting (38) and (39) into (20), we completes the proof. □

Theorem 4 (New modular equations for degrees $(n_1, n_2) = (3, 5)$) *Let α and β have degrees $(n_1, n_2) = (3, 5)$, and $m = \frac{z_3}{z_5}$ is the multiplier for degree $n = \frac{5}{3}$. Then*

1. *The natural form*

$$\begin{aligned} m - \frac{5}{3m} &= \frac{2}{3}\left((\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}\right) \\ &\quad \times \Big[1 - 3\left((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8}\right) \\ &\quad + 3\left((\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right) + 2(\alpha\beta(1-\alpha)(1-\beta))^{1/8} \\ &\quad \times \left(3 - (\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}\right)\Big], \end{aligned} \tag{40}$$

2. *The emphatic form*

$$\begin{aligned} &m - \frac{5}{3m} = \frac{2}{3}\left((\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4}\right) \\ &\times \left[4\sqrt{\frac{1}{2}(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)})} - 4 + \left((\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right)\right], \end{aligned} \tag{41}$$

3. *Ramanujan's theta functions form*

$$\begin{aligned} \frac{\varphi^2(q^3)}{\varphi^2(q^5)} - \frac{5}{3}\frac{\varphi^2(q^5)}{\varphi^2(q^3)} &= \frac{2}{3}\left(4\frac{q^2\psi(q^6)\psi(q^{10})}{\varphi(q^3)\varphi\left(q^5\right)} - \frac{\varphi(-q^3)\varphi(-q^5)}{\varphi(q^3)\varphi\left(q^5\right)}\right) \\ &\quad \times \left[4\left(\frac{\varphi(q^6)\varphi(q^{10}))}{\varphi(q^3)\varphi\left(q^5\right)} + 4\frac{q^4\psi(q^{12}))\psi(q^{20}))}{\varphi(q^3)\varphi\left(q^5\right)}\right)\right. \\ &\quad \left. -4 + \left(4\frac{q^2\psi(q^6)\psi(q^{10})}{\varphi(q^3)\varphi\left(q^5\right)} + \frac{\varphi(-q^3)\varphi(-q^5)}{\varphi(q^3)\varphi\left(q^5\right)}\right)\right]. \end{aligned} \tag{42}$$

The proof of Theorem 4 is similar to the proof of the Theorem 3. Hence we omit the details.

Acknowledgements The authors would like to thank the anonymous referee for his helpful comments and invaluable suggestions.

Author Contributions CDZ contributed to the conception and proof, LY performed the validation and writing. They reviewed the manuscript.

References

1. Berndt, B.C.: Ramanujan's Notebooks: Part III. Springer, New York (1991). https://doi.org/10.1007/978-1-4612-0965-2
2. Vasuki, K.R., Veeresha, R.G.: On Ramanujan's modular equation of degree 7. J. Number Theory **153**, 304–308 (2015). https://doi.org/10.1016/j.jnt.2015.01.002
3. Berndt, B.C.: Ramanujan's Notebooks. Part V. Springer, New York (1998). https://doi.org/10.1007/978-1-4612-1624-7