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Arithmetic of Châtelet surfaces under extensions of base fields

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Abstract
For Châtelet surfaces defined over number fields, we study two arithmetic properties, the Hasse principle and weak approximation, when passing to an extension of the base field. Generalizing a construction of Y. Liang, we show that for an arbitrary extension of number fields L/K, there is a Châtelet surface over K which does not satisfy weak approximation over any intermediate field of L/K, and a Châtelet surface over K which satisfies the Hasse principle over an intermediate field L' if and only if $[L' : K]$ is even.

Keywords Rational points · Hasse principle · Weak approximation · Brauer-Manin obstruction · Châtelet surfaces

Mathematics Subject Classification 11G35 · 14G12 · 14G25 · 14G05

1 Introduction

Throughout this paper, let K be a number field, and let Ω_K be the set of all nontrivial places of K. For each $v \in \Omega_K$, let K_v denote the completion of K at v. Let $S \subset \Omega_K$ be a finite subset. Let $\mathbb{A}_K$ (respectively, $\mathbb{A}_K^S$) be the ring of adèles (adèles without S components) of K. We always assume that a field L is a finite extension of K. Let $S_L \subset \Omega_L$ denote the set of places of L lying over places in S.

Let X be a proper algebraic variety over K. The set $X(K)$ of K-rational points of X can be viewed as a subset of the set $X(\mathbb{A}_K)$ of adelic points via the diagonal embedding. We say that X is a counterexample to the Hasse principle if $X(\mathbb{A}_K) \neq \emptyset$ whereas

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$X(K) = \emptyset$. By the properness of X, the set of adelic points $X(\mathbb{A}_K^S)$ can be identified with the product $\prod_{v \in \Omega_K \setminus S} X(K_v)$, and hence equipped with the product topology of v-adic topologies. We say that X satisfies weak approximation (respectively, weak approximation off S) if $X(K)$ is dense in $X(\mathbb{A}_K)$ (respectively,in $X(\mathbb{A}_K^S)$), cf. [14, Chapter 5.1].

In this paper, we focus on the case where X is a Châtelet surface over K, i.e., a smooth projective model of affine surface in $\mathbb{A}_K^3$ defined by the equation

$$y^2 - az^2 = P(x), \tag{1}$$

where $a \in K^\times$, and $P(x)$ is a separable degree-4 polynomial in $K[x]$. Over a fixed number field K, the Hasse principle and weak approximation for Châtelet surfaces (and many other varieties) have been studied in a lot of earlier papers (e.g., [4–6, 12], etc.). In [8], Liang pioneered the study of non-invariance of arithmetic properties under extensions of base fields and proved, among others, that for any number field K, there is a Châtelet surface V over K and a quadratic extension L/K such that $V(K) \neq \emptyset$, V satisfies weak approximation, but the base extension V_L does not satisfies weak approximation off all archimedean places of L. In this paper, we generalize Liang's construction and obtain further results that apply to an arbitrary extension L/K.

Our main results are the following:

Theorem 1.1 *(Theorem 4.1) Let L/K be any extension of number fields, and let $S \subset \Omega_K \setminus \{$all complex and 2-adic places$\}$ be a finite nonempty subset such that every place in S splits completely in L.*

Then, there exists a Châtelet surface V defined over K with $V(K) \neq \emptyset$, such that for every intermediate field $K \subset L' \subset L$ and every finite subset $T' \subset \Omega_{L'}$, the base extension $V_{L'}$ satisfies weak approximation off T' if and only if $T' \cap S_{L'} \neq \emptyset$. In particular, the surface $V_{L'}$ does not satisfy weak approximation for every $K \subset L' \subset L$.

Theorem 1.2 *(Theorem 5.1) For any extension of number fields L/K, there exists a Châtelet surface V over K with $V(\mathbb{A}_K) \neq \emptyset$, such that for every intermediate field $K \subset L' \subset L$,*

- *If the degree $[L' : K]$ is odd, then the surface $V_{L'}$ is a counterexample to the Hasse principle, i.e., $V(L') = \emptyset$. In particular, the surface V is a counterexample to the Hasse principle.*
- *If the degree $[L' : K]$ is even, then the surface $V_{L'}$ satisfies weak approximation. In particular, in this case, $V(L') \neq \emptyset$.*

To construct the Châtelet surfaces in these theorems, the parameters in the equation (1), i.e.,the element $a \in K$ and the coefficients of the polynomial $P(x)$, need to be chosen carefully using approximation theorems for the affine line and Čebotarev's density theorem. To verify the statements about weak approximation and the Hasse principle, we shall analyze the Brauer-Manin obstruction and use the well-known theorem that for Châtelet surfaces this obstruction (to weak approximation or the Hasse principle) is the only one ([4, 5]).

2 Notation and preliminaries

2.1 Notation

Given a number field K, let $\mathcal{O}_K$ be the ring of its integers. Let $\infty_K \subset \Omega_K$ be the subset of all archimedean places, and let $2_K \subset \Omega_K$ be the subset of all 2-adic places. Let $\infty^r_K \subset \infty_K$ be the subset of all real places, and let $\infty^c_K \subset \infty_K$ be the subset of all complex places. Let $\Omega^f_K = \Omega_K \backslash \infty_K$ be the set of all finite places of K. For $v \in \infty_K$, let $\tau_v \colon K \hookrightarrow K_v$ be the embedding of K into its completion. For $v \in \Omega^f_K$, let $\mathcal{O}_{K_v}$ be its valuation ring. Let K^2 denote the set of square elements of K. Let $\mathcal{O}_S = \bigcap_{v\in\Omega^f_K\backslash S}(K \cap \mathcal{O}_{K_v})$ be the ring of S-integers. A strong approximation theorem [1, Chapter II §15] states that K is dense in $\mathring{A}^S_K$ for any nonempty S. In this paper, we only use the following special case:

Lemma 2.1 *The set K is dense in $\mathring{A}^{2_K}_K$.*

We will use the following version of Čebotarev's density theorem, cf. [10, Chapter VII §13].

Theorem 2.2 *(Čebotarev) The set of places of K splitting completely in L has positive density.*

2.2 Hilbert symbol

We use the Hilbert symbol $(a, b)_v \in \{\pm 1\}$, for $a, b \in K^\times_v$ and $v \in \Omega_K$. By definition, $(a, b)_v = 1$ if and only if $x_0^2 - ax_1^2 - bx_2^2 = 0$ has a K_v-solution in $\mathbb{P}^2$ with homogeneous coordinates $(x_0 : x_1 : x_2)$, which equivalently means that the curve defined over K_v by the equation $x_0^2 - ax_1^2 - bx_2^2 = 0$ in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. The Hilbert symbol gives a symmetric bilinear form on $K^\times_v / K^{\times 2}_v$ with value in $\mathbb{Z}/2\mathbb{Z}$, cf. [13, Chapter XIV, Proposition 7]. And this bilinear form is nondegenerate, cf. [13, Chapter XIV, Corollary 7].

2.3 Preparation lemmas

We state the following lemmas for later use. Lemmas 2.3–2.8 have already been given in the paper [15]. We give their statements here for the convenience of the reader.

Lemma 2.3 *Let v be an odd place of K. Let $a, b \in K^\times_v$ such that $v(a)$, $v(b)$ are even. Then $(a, b)_v = 1$.*

Lemma 2.4 *Let v be an odd place of K. Let $a, b, c \in K^\times_v$ such that $v(b) < v(c)$. Then $(a, b + c)_v = (a, b)_v$.*

Lemma 2.5 *The set $K^{\times 2}_v$ is an open subgroup of $K^\times_v$. If $v \in \Omega^f_K$, then $\mathcal{O}^\times_{K_v}$ is also an open subgroup of $K^\times_v$. So, they are nonempty open subset of K_v.*

Lemma 2.6 *Let $v \in \Omega^f_K$. For any $n \in \mathbb{Z}$, the set $\{x \in K_v | v(x) = n\}$ is a nonempty open subset of K_v.*

Lemma 2.7 *Let $v \in \Omega_K^f$. For any $a \in K_v^\times$, the sets $\{x \in K_v^\times | (a,x)_v = 1\}$, $\{x \in K_v^\times | (a,x)_v = 1\} \cap \mathcal{O}_{K_v}$ and $\{x \in \mathcal{O}_{K_v}^\times | (a,x)_v = 1\}$ are nonempty open subsets of K_v.*

Lemma 2.8 *Let $v \in \Omega_K^f$. For any $a \in K_v^\times$, the sets $\{x \in K_v^\times | (a,x)_v = -1\}$ and $\{x \in K_v^\times | (a,x)_v = -1\} \cap \mathcal{O}_{K_v}$ are open subsets of K_v. Furthermore, if $a \notin K_v^{\times 2}$, then they are nonempty.*

Lemma 2.9 *Let $v \in \Omega_K^f$. For any $a \in K_v^\times$ with $v(a)$ odd, the set $\{x \in \mathcal{O}_{K_v}^\times | (a,x)_v = -1\}$ is a nonempty open subset of K_v.*

Proof of Lemma 2.9 By Lemmas 2.5 and 2.8, the set is open in K_v. We need to show that it is nonempty. Since $a \notin K_v^2$, by the nondegeneracy of the bilinear form given by the Hilbert symbol, there exists an element $b \in K_v^\times$ such that $(a,b)_v = -1$. If $v(b)$ is odd, let $b' = -ab$. Then $(a,b')_v = (a,-ab)_v = (a,-a)_v(a,b)_v = -1$. Replacing b by b' if necessary, we can assume that $v(b)$ is even. Choose a prime element $\pi_v \in K_v$. Then $\pi_v^{-v(b)} \in K_v^{\times 2}$, so the element $b\pi_v^{-v(b)}$ is in this set.

2.4 Brauer-Manin obstruction

Cohomological obstructions have been used to explain failures of the Hasse principle and nondensity of $X(K)$ in $X(\mathbb{A}_K^S)$. Let $\mathrm{Br}(X) = H_{et}^2(X, \mathbb{G}_m)$ be the Brauer group of X. Let $\mathrm{inv}_v : \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z}$ be the local invariant map. The Brauer-Manin pairing

$$X(\mathbb{A}_K) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z},$$

suggested by Manin [9], between $X(\mathbb{A}_K)$ and $\mathrm{Br}(X)$, is provided by local class field theory. The left kernel of this pairing is denoted by $X(\mathbb{A}_K)^{\mathrm{Br}}$, which is a closed subset of $X(\mathbb{A}_K)$. By the global reciprocity in class field theory, there is an exact sequence:

$$0 \to \mathrm{Br}(K) \to \bigoplus_{v \in \Omega_K} \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

which induces an inclusion: $X(K) \subset pr^S(X(\mathbb{A}_K)^{\mathrm{Br}})$.

Remark 2.10 For any smooth, proper and rationally connected variety X defined over a number field K, it is conjectured by Colliot-Thélène [2] that the K-rational points set $X(K)$ is dense in $X(\mathbb{A}_K)^{\mathrm{Br}}$. Colliot-Thélène's conjecture holds for Châtelet surfaces, cf. [4, 5].

3 Châtelet surfaces

Let K be a number field. Given an equation (1), let V^0 be the affine surface in $\mathbb{A}_K^3$ defined by this equation. Let V be the natural smooth compactification of V^0 given in [14, Section 7.1], which is called the Châtelet surface given by this equation, cf. [12, Section 5]. By the Lang-Nishimura theorem ([7] and [11]) and the implicit function theorem, all smooth projective models of a given equation (1) are the same as to the discussion of the Hasse principle and weak approximation.

Remark 3.1 For any local field K_v, if $a \in K_v^{\times 2}$, then V is birationally equivalent to $\mathbb{P}^2$ over K_v. By the implicit function theorem, there exists a K_v-point on V.

Remark 3.2 For any local field K_v, by smoothness of V, the implicit function theorem implies that the nonemptiness of $V^0(K_v)$ is equivalent to the nonemptiness of $V(K_v)$, and that $V^0(K_v)$ is open dense in $V(K_v)$ with the v-adic topology. Given an element $A \in \mathrm{Br}(V)$, since $\mathrm{Br}(V)$ is torsion, the set of all possible values of the evaluation of A on $V(K_v)$ is finite. Indeed, by [14, Proposition 7.1.2], there exist only two possible values. They are determined by the evaluation of A on $V^0(K_v)$. In particular, if the evaluation of A on $V^0(K_v)$ is constant, then it is constant on $V(K_v)$.

In the following two sections, we will construct two kinds of Châtelet surfaces.

Choice of the parameter *a* for the equation (1)

Given an extension of number fields L/K, and a finite subset $S \subset \Omega_K \backslash (\infty_K^c \cup 2_K)$, we always use the following way to choose an element $a \in \mathcal{O}_K \backslash K^2$ for the parameter a in the equation (1).

By Theorem 2.2, we can take a place $v_0 \in \Omega_K^f \backslash 2_K$ splitting completely in L. If $S \neq \emptyset$, let $S_0 = S$, otherwise, let $S_0 = \{v_0\}$. Then $S_0 \neq \emptyset$.

For $v \in \Omega_K$, by Lemma 2.5, the set $K_v^{\times 2}$ is a nonempty open subset of K_v. For $v \in \Omega_K^f$, by Lemma 2.6, the set $\{a \in K_v | v(a) \text{ is odd}\}$ is a nonempty open subset of K_v. Using weak approximation for the affine line $\mathring{\mathrm{A}}^1$, we can choose an element $a \in K^\times$ satisfying the following conditions:

- $\tau_v(a) < 0$ for all $v \in S_0 \cap \infty_K$,
- $a \in K_v^{\times 2}$ for all $v \in 2_K$,
- $v(a)$ is odd for all $v \in S_0 \backslash \infty_K$.

These conditions do not change by multiplying an element in $K^{\times 2}$, so we can assume $a \in \mathcal{O}_K$. The conditions that $v(a)$ is odd for all $v \in S_0 \backslash \infty_K$, and that $\tau_v(a) < 0$ for all $v \in S_0 \cap \infty_K$, imply $a \in \mathcal{O}_K \backslash K_v^2$ for all $v \in S_0$. So $a \in \mathcal{O}_K \backslash K^2$.

Remark 3.3 Let $S' = \{v \in \infty_K^r | \tau_v(a) < 0\} \cup \{v \in \Omega_K^f \backslash 2_K | v(a) \text{ is odd}\}$, then S' is a finite set. By the conditions that $\tau_v(a) < 0$ for all $v \in S \cap \infty_K$, and that $v(a)$ is odd for all $v \in S \backslash \infty_K$, we have $S' \supset S$. Then $S' \neq \emptyset$.

Lemma 3.4 *Given an extension of number fields L/K, and a finite subset $S \subset \Omega_K \backslash (\infty_K^c \cup 2_K)$, we choose an element $a \in \mathcal{O}_K$ as above. If there exists one place in S splitting completely in L or $S = \emptyset$, then $a \in \mathcal{O}_K \backslash L^2$.*

Proof We use the notion S_0 as above. By our assumption, we can take a place $v \in S_0$ splitting completely in L. Take a place $w \in \Omega_L$ such that $w | v$. Then $K_v = L_w$. By the choice of a, if $v \in S_0 \backslash \infty_K$, then $v(a)$ is odd; if $v \in S_0 \cap \infty_K$, then $\tau_v(a) < 0$. In both cases, we have $a \in \mathcal{O}_K \backslash K_v^2 = \mathcal{O}_K \backslash L_w^2$. Hence $a \in \mathcal{O}_K \backslash L^2$. □

4 Weak approximation under extensions of base fields

In the paper [8], Liang studied the non-invariance of weak approximation under extensions of base fields. More precisely, for any number field K, Liang [8, Proposition 3.4] proved that there exist a Châtelet surface V over K and a quadratic extension L/K such that $V(K) \neq \emptyset$, V satisfies weak approximation, but the base extension V_L does not satisfy weak approximation, even off ∞_L. In this section, we generalize Liang's construction and obtain further results that apply to an arbitrary extension L/K.

4.1 Choice of parameters for the equation (1)

With $a \in \mathcal{O}_K \backslash K^2$ chosen as in Section 3, we choose an element $b \in K^\times$ in the following way.

Let $S' = \{v \in \infty_K^r | \tau_v(a) < 0\} \cup \{v \in \Omega_K^f \backslash 2_K | v(a) \text{ is odd}\}$ be as in Remark 3.3, then $S' \supset S$ is a finite set. By Lemma 2.6, for $v \in S\backslash\infty_K$, the set $\{b \in K_v | v(b) = -v(a)\}$ is a nonempty open subset of K_v; for $v \in S'\backslash(S\cup\infty_K)$, the set $\{b \in K_v | v(b) = v(a)\}$ is a nonempty open subset of $\mathcal{O}_{K_v}$. By Lemma 2.1, we can choose a nonzero element $b \in \mathcal{O}_S[1/2]$ satisfying the following conditions:

- $v(b) = -v(a)$ for all $v \in S\backslash\infty_K$,
- $v(b) = v(a)$ for all $v \in S'\backslash(S \cup \infty_K)$.

We choose an element $c \in K^\times$ with respect to the chosen a, b in the following way.

Let $S'' = \{v \in \Omega_K^f \backslash 2_K | v(b) \neq 0\}$, then S'' is a finite set and $S'\backslash\infty_K \subset S''$. By Theorem 2.2, we can take two different finite places $v_1, v_2 \in \Omega_K^f \backslash S''$ splitting completely in L. If $v \in S\backslash\infty_K$, then $v(a)$ is odd. In this case, by Lemma 2.9, the set $\{c \in \mathcal{O}_{K_v}^\times | (a, c)_v = -1\}$ is a nonempty open subset of $\mathcal{O}_{K_v}$. If $v \in \{v_1, v_2\}$, then $b \in \mathcal{O}_{K_v}^\times$. In this case, by Lemma 2.6, the sets $\{c \in K_v | v(c) = 1\}$ and $\{c \in K_v | v(1 + cb^2) = 1\}$ are nonempty open subsets of $\mathcal{O}_{K_v}$. Also by Lemma 2.1, we can choose a nonzero element $c \in \mathcal{O}_K[1/2]$ satisfying the following conditions:

- $\tau_v(1 + cb^2) < 0$ for all $v \in S \cap \infty_K$,
- $\tau_v(c) > 0$ for all $v \in (S'\backslash S) \cap \infty_K$,
- $(a, c)_v = -1$ and $v(c) = 0$ for all $v \in S\backslash\infty_K$,
- $v_1(c) = 1$ and $v_2(1 + cb^2) = 1$ for the chosen v_1, v_2 above.

Let $P(x) = (cx^2 + 1)((1 + cb^2)x^2 + b^2)$, and let V_1 be the Châtelet surface given by $y^2 - az^2 = (cx^2 + 1)((1 + cb^2)x^2 + b^2)$.

Proposition 4.1 *For any extension of number fields L/K, and any finite subset $S \subset \Omega_K \backslash (\infty_K^c \cup 2_K)$ splitting completely in L, there exists a Châtelet surface V_1 defined over K, which has the following properties.*

- *The Brauer group $\mathrm{Br}(V_1)/\mathrm{Br}(K) \cong \mathrm{Br}(V_{1L})/\mathrm{Br}(L) \cong \mathbb{Z}/2\mathbb{Z}$, is generated by an element $A \in \mathrm{Br}(V_1)$. The subset $V_1(K) \subset V_1(L)$ is nonempty.*
- *For any $v \in S$, there exist P_v and Q_v in $V_1(K_v)$ such that the local invariants $\mathrm{inv}_v(A(P_v)) = 0$ and $\mathrm{inv}_v(A(Q_v)) = 1/2$. For any $v \notin S$, and any $P_v \in V_1(K_v)$, the local invariant $\mathrm{inv}_v(A(P_v)) = 0$.*

- *For any $v' \in S_L$, there exist $P_{v'}$ and $Q_{v'}$ in $V_1(L_{v'})$ such that the local invariants* $\mathrm{inv}_{v'}(A(P_{v'})) = 0$ *and* $\mathrm{inv}_{v'}(A(Q_{v'})) = 1/2$. *For any $v' \notin S_L$, and any $P_{v'} \in V_1(L_{v'})$, the local invariant* $\mathrm{inv}_{v'}(A(P_{v'})) = 0$.

Proof For the extension L/K, and the finite set S, we check that the Châtelet surface V_1 chosen as in Subsection 4.1 has the required properties.

By the choice of the places v_1, the polynomial x^2+c is an Eisenstein polynomial, so it is irreducible over K_{v_1}. Since $v_1(a)$ is even, we have $K(\sqrt{a})K_{v_1} \not\cong K_{v_1}[x]/(cx^2+1)$. So $K(\sqrt{a}) \not\cong K[x]/(cx^2+1)$. The same argument holds for the place v_2 and the polynomial $(1+cb^2)x^2+b^2$. Since all places of S split completely in L, by Lemma 3.4, we have $a \in \mathcal{O}_K \backslash L^2$. By the splitting condition of v_1, v_2, we have $L(\sqrt{a}) \not\cong L[x]/(cx^2+1)$ and $L(\sqrt{a}) \not\cong L[x]/((1+cb^2)x^2+b^2)$. So $P(x) = (cx^2+1)((1+cb^2)x^2+b^2)$ is separable and a product of two degree-2 irreducible factors over K and L. According to [14, Proposition 7.1.1], the Brauer group $\mathrm{Br}(V_1)/\mathrm{Br}(K) \cong \mathrm{Br}(V_{1L})/\mathrm{Br}(L) \cong \mathbb{Z}/2\mathbb{Z}$. Furthermore, by Proposition 7.1.2 in loc. cit, we take the quaternion algebra $A = (a, cx^2+1) \in \mathrm{Br}(V_1)$ as a generator element of this group. Then we have the equality $A = (a, cx^2+1) = (a, (1+cb^2)x^2+b^2)$ in $\mathrm{Br}(V_1)$.

Since $(x, y, z) = (0, b, 0)$ is a rational point on V_1^0, the set $V_1(K)$ is nonempty. We denote this rational point by Q_0.

We need to compute the evaluation of A on $V_1(K_v)$ for all $v \in \Omega_K$.

For any $v \in \Omega_K$, the local invariant $\mathrm{inv}_v(A(Q_0)) = 0$. By Remark 3.2, it suffices to compute the local invariant $\mathrm{inv}_v(A(P_v))$ for all $P_v \in V_1^0(K_v)$.

(1) Suppose that $v \in (\infty_K \backslash S') \cup 2_K$. Then $a \in K_v^{\times 2}$, so $\mathrm{inv}_v(A(P_v)) = 0$ for all $P_v \in V_1(K_v)$.
(2) Suppose that $v \in (S' \backslash S) \cap \infty_K$. For any $x \in K$, by the choice of c, we have $\tau_v(cx^2+1) > 0$. Then $(a, cx^2+1)_v = 1$, so $\mathrm{inv}_v(A(P_v)) = 0$ for all $P_v \in V_1(K_v)$.
(3) Suppose that $v \in S' \backslash (S \cup \infty_K)$. Take an arbitrary $P_v \in V_1^0(K_v)$. If $\mathrm{inv}_v(A(P_v)) = 1/2$, then $(a, cx^2+1)_v = -1 = (a, (1+cb^2)x^2+b^2)_v$ at P_v. By Lemma 2.4, the first equality implies $v(x) \le 0$. Since $v(a) = v(b) > 0$ and $v(c) \ge 0$, by Lemma 2.4, we have $(a, (1+cb^2)x^2+b^2)_v = (a, x^2)_v = 1$, which is a contradiction. So $\mathrm{inv}_v(A(P_v)) = 0$.
(4) Suppose that $v \in \Omega_K^f \backslash (S' \cup 2_K)$. Take an arbitrary $P_v \in V_1^0(K_v)$. If $\mathrm{inv}_v(A(P_v)) = 1/2$, then $(a, cx^2+1)_v = -1 = (a, (1+cb^2)x^2+b^2)_v$ at P_v. Since $v(a)$ is even, by Lemma 2.3, the first equality implies that $v(cx^2+1)$ is odd. Since $c \in \mathcal{O}_K[1/2]$, we have $v(x) \le 0$. So $v(c+x^{-2})$ is odd and positive. Since $v(b) \ge 0$, by Hensel's lemma, we have $1+b^2(c+x^{-2}) \in K_v^{\times 2}$. So $(a, (1+cb^2)x^2+b^2)_v = (a, x^2)_v(a, 1+b^2(c+x^{-2}))_v = 1$, which is a contradiction. So $\mathrm{inv}_v(A(P_v)) = 0$.
(5) Suppose that $v \in S \cap \infty_K$. Take $P_v = Q_0$, then $\mathrm{inv}_v(A(P_v)) = 0$. By the choice of b, c, we have $\tau_v(\frac{b^2}{-cb^2-1}) > \tau_v(\frac{1}{-c}) > 0$. Take $x_0 \in K$ such that $\tau_v(x_0) > \sqrt{\tau_v(\frac{b^2}{-cb^2-1})}$, then $\tau_v((cx_0^2+1)((1+cb^2)x_0^2+b^2)) > 0$ and $\tau_v(cx_0^2+1) < 0$. So there exists a $Q_v \in V_1^0(K_v)$ with $x = x_0$. Then $\mathrm{inv}_v(A(Q_v)) = 1/2$.
(6) Suppose that $v \in S \backslash \infty_K$. Take $P_v = Q_0$, then $\mathrm{inv}_v(A(P_v)) = 0$. Take $x_0 \in K_v$ such that $v(x_0) < 0$. Since $v(b) = -v(a) < 0$ and $v(c) = 0$, by Lemma 2.4, we have $(a, cx_0^2+1)_v = (a, cx_0^2)_v = (a, c)_v$ and $(a, (1+cb^2)x_0^2+b^2)_v =$

$(a, cb^2x_0^2)_v = (a, c)_v$. So $(a, (cx_0^2+1)((1+cb^2)x_0^2+b^2))_v = (a, c)_v(a, c)_v = 1$. Hence, there exists a $Q_v \in V_1^0(K_v)$ with $x = x_0$. Since $(a, c)_v = -1$, we have $\mathrm{inv}_v(A(Q_v)) = 1/2$.

Finally, we need to compute the evaluation of A on $V_1(L_{v'})$ for all $v' \in \Omega_L$. For any $v' \in \Omega_L$, the local invariant $\mathrm{inv}_{v'}(A(Q_0)) = 0$.

(1) Suppose that $v' \in S_L$. Let $v \in \Omega_K$ be the restriction of v' on K. By the assumption that v splits completely in L, we have $K_v = L_{v'}$. So $V_1(K_v) = V_1(L_{v'})$. By the argument already shown, there exist $P_v, Q_v \in V_1(K_v)$ such that $\mathrm{inv}_v(A(P_v)) = 0$ and $\mathrm{inv}_v(A(Q_v)) = 1/2$. View P_v, Q_v as elements in $V_1(L_{v'})$, and let $P_{v'} = P_v$ and $Q_{v'} = Q_v$. Then $\mathrm{inv}_{v'}(A(P_{v'})) = \mathrm{inv}_v(A(P_v)) = 0$ and $\mathrm{inv}_{v'}(A(Q_{v'})) = \mathrm{inv}_v(A(Q_v)) = 1/2$.

(2) Suppose that $v' \in \Omega_L \backslash S_L$. This local computation is the same as the case $v \in \Omega_K \backslash S$. □

Remark 4.2 For any $v \in S$, and any $P_v \in V_1(K_v)$, the local invariant of the evaluation of A on P_v is 0 or 1/2. Let $U_1 = \{P_v \in V_1(K_v) | \mathrm{inv}_v(A(P_v)) = 0\}$ and $U_2 = \{P_v \in V_1(K_v) | \mathrm{inv}_v(A(P_v)) = 1/2\}$. Then U_1 and U_2 are nonempty disjoint open subsets of $V_1(K_v)$, and $V_1(K_v) = U_1 \bigsqcup U_2$.

Applying the global reciprocity law, the surface V_1 in Proposition 4.1 has the following weak approximation properties.

Proposition 4.3 *Given an extension of number fields L/K, and a finite subset $S \subset \Omega_K \backslash (\infty_K^c \cup 2_K)$ splitting completely in L, let V_1 be a Châtelet surface satisfying those properties of Proposition 4.1.*

(1) If $S = \emptyset$, then V_1 and $V_{1\,L}$ satisfy weak approximation.
(2) If $S \neq \emptyset$, then V_1 satisfies weak approximation off S' for a finite subset $S' \subset \Omega_K$ if and only if $S' \cap S \neq \emptyset$.
(3) If $S \neq \emptyset$, the surface V_{1L} satisfies weak approximation off T for a finite subset $T \subset \Omega_L$ if and only if $T \cap S_L \neq \emptyset$.

Proof According to [4, 5, Theorem B], the Brauer-Manin obstruction to the Hasse principle and weak approximation is the only one for Châtelet surfaces, so $V_1(K)$ is dense in $V_1(\mathbb{A}_K)^{\mathrm{Br}}$.

(1) Suppose that $S = \emptyset$, then for any $(P_v)_{v\in\Omega_K} \in V_1(\mathbb{A}_K)$, by Proposition 4.1, the sum $\sum_{v\in\Omega_K} \mathrm{inv}_v(A(P_v)) = 0$. Since $\mathrm{Br}(V_1)/\mathrm{Br}(K)$ is generated by the element A, we have $V_1(\mathbb{A}_K)^{\mathrm{Br}} = V_1(\mathbb{A}_K)$. So $V_1(K)$ is dense in $V_1(\mathbb{A}_K)^{\mathrm{Br}} = V_1(\mathbb{A}_K)$, i.e. the surface V_1 satisfies weak approximation.

(2) (a) Suppose that $S' \cap S \neq \emptyset$. Take $v_0 \in S' \cap S$. For any finite subset $R \subset \Omega_K \backslash \{v_0\}$, take a nonempty open subset $M = V_1(K_{v_0}) \times \prod_{v\in R} U_v \times \prod_{v \notin R\cup\{v_0\}} V_1(K_v) \subset V_1(\mathbb{A}_K)$. Take an element $(P_v)_{v\in\Omega_K} \in M$ with $\mathrm{inv}_{v_0} A(P_{v_0}) = 0$. By Proposition 4.1 and $v_0 \in S$, we can take an element $P'_{v_0} \in V_1(K_{v_0})$ such that $\mathrm{inv}_{v_0} A(P'_{v_0}) = 1/2$. By Proposition 4.1, the sum $\sum_{v\in\Omega_K\backslash\{v_0\}} \mathrm{inv}_v(A(P_v))$ is 0 or 1/2 in $\mathbb{Q}/\mathbb{Z}$. If it is 1/2, then we replace P_{v_0} by P'_{v_0}. In this way, we get a new element $(P_v)_{v\in\Omega_K} \in M$. And the sum $\sum_{v\in\Omega_K} \mathrm{inv}_v(A(P_v)) = 0$ in $\mathbb{Q}/\mathbb{Z}$.

So $(P_v)_{v\in\Omega_K} \in V_1(\mathring{A}_K)^{\mathrm{Br}} \cap M$. Since $V_1(K)$ is dense in $V_1(\mathring{A}_K)^{\mathrm{Br}}$, the set $V_1(K) \cap M \neq \emptyset$, which implies that V_1 satisfies weak approximation off $\{v_0\}$. So V_1 satisfies weak approximation off S'.

(b) Suppose that $S \neq \emptyset$ and $S' \cap S = \emptyset$. Take $v_0 \in S$, and let $U_{v_0} = \{P_{v_0} \in V_1(K_{v_0}) | \mathrm{inv}_{v_0}(A(P_{v_0})) = 1/2\}$. For $v \in S\backslash\{v_0\}$, let $U_v = \{P_v \in V_1(K_v) | \mathrm{inv}_v(A(P_v)) = 0\}$. For any $v \in S$, by Remark 4.2, the set U_v is a nonempty open subset of $V_1(K_v)$. Let $M = \prod_{v\in S} U_v \times \prod_{v\notin S} V_1(K_v)$. It is a nonempty open subset of $V_1(\mathring{A}_K)$. For any $(P_v)_{v\in\Omega_K} \in M$, by Proposition 4.1 and the choice of U_v, the sum $\sum_{v\in\Omega_K} \mathrm{inv}_v(A(P_v)) = 1/2$ is nonzero in $\mathbb{Q}/\mathbb{Z}$. So $V_1(\mathring{A}_K)^{\mathrm{Br}} \cap M = \emptyset$, which implies $V_1(K) \cap M = \emptyset$. Hence V_1 does not satisfy weak approximation off S'.

(3) The same argument applies to V_{1L}. □

From the discussion in the proof of Proposition 4.3, we have the following weak approximation properties for Châtelet surfaces.

Theorem 4.1 *For any extension of number fields L/K, and any finite nonempty subset $S \subset \Omega_K\backslash(\infty_K^c \cup 2_K)$ splitting completely in L, there exists a Châtelet surface V over K with $V(K) \neq \emptyset$, such that for every intermediate field $K \subset L' \subset L$, $\mathrm{Br}(V)/\mathrm{Br}(K) \cong \mathrm{Br}(V_{L'})/\mathrm{Br}(L') \cong \mathbb{Z}/2\mathbb{Z}$, and for every finite subset $T' \subset \Omega_{L'}$, the base extension $V_{L'}$ satisfies weak approximation off T' if and only if $T' \cap S_{L'} \neq \emptyset$. In particular, the surface $V_{L'}$ does not satisfy weak approximation for every $K \subset L' \subset L$.*

Proof For the extension L/K and the set S, let V be the Châtelet surface chosen as in Sect. 4.1. Applying the same argument about the field L to its subfield L', the properties that we list are just what we have explained in Proposition 4.1 and Proposition 4.3. □

Using the construction method in Sect. 4.1, we have the following example, which is a special case of Proposition 4.1.

Example 4.4 For $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{3})$, and let $S = \{73\} \subset \Omega_K$. The prime numbers $11, 23, 73$ split completely in L. Using the construction method in Sect. 4.1, we choose data: $S = S' = S'' = 73,\ v_1 = 11,\ v_2 = 23,\ a = 73,\ b = 1/73,\ c = 99$ and $P(x) = (99x^2 + 1)(5428x^2/5329 + 1/5329)$. Then the Châtelet surface given by $y^2 - 73z^2 = P(x)$, has the properties of Propositions 4.1 and 4.3.

5 The Hasse principle under extensions of base fields

Iskovskikh [6] showed that the Châtelet surface over $\mathbb{Q}$ given by $y^2 + z^2 = (x^2 - 2)(-x^2 + 3)$ is a counterexample to the Hasse principle. A family of Châtelet surfaces over $\mathbb{Q}$ extending Iskovskikh's example was given in [3], (see also [14, Pages 145-146]). Poonen [12, Proposition 5.1] generalized their arguments to any number field. For an arbitrary number field K, he constructed a Châtelet surface defined over K, which is a counterexample to the Hasse principle. He used Čebotarev's density theorem for some ray class fields to choose the parameters for the equation (1). The Châtelet surface he constructed has the properties of [12, Lemma 5.5] (a special situation of the following Proposition 5.1: the case when $S = \{v_0\}$ for some place v_0 associated to some large prime element in $\mathcal{O}_K$), which is the main ingredient in the proof of [12, Proposition 5.1]. In this section, we generalize Poonen's construction and consider the Hasse principle of Châtelet surfaces under extensions of base fields.

5.1 Choice of parameters for the equation (1)

Given an extension of number fields L/K, and a finite subset $S \subset \Omega_K \backslash (\infty_K^c \cup 2_K)$, we choose $a \in \mathcal{O}_K \backslash K^2$ as in Section 3. Then we choose an element $b \in K^\times$ in the following way.

Let $S' = \{v \in \infty_K^r | \tau_v(a) < 0\} \cup \{v \in \Omega_K^f \backslash 2_K | v(a) \text{ is odd}\}$ be as in Remark 3.3, then $S' \supset S$ is a finite set. If $v \in S\backslash\infty_K$, then $v(a)$ is odd. Then by Lemma 2.9, the set $\{b \in \mathcal{O}_{K_v}^\times | (a,b)_v = -1\}$ is a nonempty open subset of $\mathcal{O}_{K_v}$. If $v \in S'\backslash(S \cup \infty_K)$, then by Lemma 2.7, the set $\{b \in \mathcal{O}_{K_v}^\times | (a,b)_v = 1\}$ is a nonempty open subset of $\mathcal{O}_{K_v}$. By Lemma 2.1, we can choose a nonzero element $b \in \mathcal{O}_K[1/2]$ satisfying the following conditions:

- $\tau_v(b) < 0$ for all $v \in S \cap \infty_K$,
- $\tau_v(b) > 0$ for all $v \in (S'\backslash S) \cap \infty_K$,
- $(a,b)_v = -1$ and $v(b) = 0$ for all $v \in S\backslash\infty_K$,
- $(a,b)_v = 1$ and $v(b) = 0$ for all $v \in S'\backslash(S \cup \infty_K)$.

We choose an element $c \in K^\times$ with respect to the chosen a, b in the following way.

Let $S'' = \{v \in \Omega_K^f \backslash 2_K | v(b) \neq 0\}$, then S'' is a finite set and $S' \cap S'' = \emptyset$. By Theorem 2.2, we can take two different finite places $v_1, v_2 \in \Omega_K^f \backslash (S' \cup S'' \cup 2_K)$ splitting completely in L. If $v \in (S'\backslash\infty_K) \cup \{v_1, v_2\}$, then $b \in \mathcal{O}_{K_v}^\times$. In this case, by Lemma 2.6, the sets $\{c \in K_v | v(bc+1) = v(a)+2\}$, $\{c \in K_v | v(c) = 1\}$ and $\{c \in K_v | v(bc+1) = 1\}$ are nonempty open subsets of $\mathcal{O}_{K_v}$.

If $v \in S''$, by Lemma 2.7, the set $\{c \in \mathcal{O}_{K_v}^\times | (a,c)_v = 1\}$ is a nonempty open subset of $\mathcal{O}_{K_v}$. Also by Lemma 2.1, we can choose a nonzero element $c \in \mathcal{O}_K[1/2]$ satisfying the following conditions:

- $0 < \tau_v(c) < -1/\tau_v(b)$ for all $v \in S \cap \infty_K$,
- $\tau_v(bc+1) < 0$ for all $v \in (S'\backslash S) \cap \infty_K$,
- $v(bc+1) = v(a) + 2$ for all $v \in S'\backslash\infty_K$,
- $(a,c)_v = 1$ for all $v \in S''$,
- $v_1(c) = 1$ and $v_2(bc+1) = 1$ for the chosen v_1, v_2 above.

Let $P(x) = (x^2 - c)(bx^2 - bc - 1)$, and let V_2 be the Châtelet surface given by $y^2 - az^2 = (x^2 - c)(bx^2 - bc - 1)$.

Proposition 5.1 *For any extension of number fields L/K, and any finite subset $S \subset \Omega_K \backslash (\infty_K^c \cup 2_K)$ splitting completely in L, there exists a Châtelet surface V_2 defined over K, which has the following properties.*

- *The Brauer group* $\mathrm{Br}(V_2)/\mathrm{Br}(K) \cong \mathrm{Br}(V_{2L})/\mathrm{Br}(L) \cong \mathbb{Z}/2\mathbb{Z}$, *is generated by an element* $A \in \mathrm{Br}(V_2)$. *The subset* $V_2(\mathring{\mathbb{A}}_K) \subset V_2(\mathring{\mathbb{A}}_L)$ *is nonempty.*

- *For any* $v \in \Omega_K$, *and any* $P_v \in V_2(K_v)$,

$$\mathrm{inv}_v(A(P_v)) = \begin{cases} 0 & if \quad v \notin S, \\ 1/2 & if \quad v \in S. \end{cases}$$

- *For any* $v' \in \Omega_L$, *and any* $P_{v'} \in V_2(L_{v'})$,

$$\mathrm{inv}_{v'}(A(P_{v'})) = \begin{cases} 0 & if \quad v' \notin S_L, \\ 1/2 & if \quad v' \in S_L. \end{cases}$$

Proof For the extension L/K and the finite set S, we check that the Châtelet surface V_2 chosen as in Sect. 5.1 has the required properties.

Firstly, we need to check that V_2 has an $\AA_K$-point.

(1) Suppose that $v \in (\infty_K \backslash S') \cup 2_K$. Then $a \in K_v^{\times 2}$. By Remark 3.1, the surface V_2 admits a K_v-point.
(2) Suppose that $v \in (S' \backslash S) \cap \infty_K$. Let $x_0 = 0$. Since $\tau_v(b) > 0$ and $\tau_v(bc+1) < 0$, we have $\tau_v(c) < 0$ and $\tau_v((x_0^2 - c)(bx_0^2 - bc - 1)) = \tau_v(c(bc+1)) > 0$, which implies that V_2^0 admits a K_v-point with $x = 0$.
(3) Suppose that $v \in S' \backslash (S \cup \infty_K)$. Take $x_0 \in K_v$ such that the valuation $v(x_0) < 0$. Since $b \in \mathcal{O}_{K_v}^\times$ and $c \in \mathcal{O}_K[1/2]$, by Lemma 2.4, we have $(a, x_0^2 - c)_v = (a, x_0^2)_v = 1$ and $(a, bx_0^2 - bc - 1)_v = (a, bx_0^2)_v = (a, b)_v$. By the choice of b, we have $(a, b)_v = 1$. Hence $(a, (x_0^2 - c)(bx_0^2 - bc - 1))_v = (a, b)_v = 1$, which implies that V_2^0 admits a K_v-point with $x = x_0$.
(4) Suppose that $v \in S''$. By the choice of a, b, c, we have $(a, c)_v = 1$, $bc+1 \in \mathcal{O}_{K_v}^\times$, and the valuation $v(a)$ is even. By Lemma 2.3, we have $(a, bc+1)_v = 1$. Let $x_0 = 0$. Then $(a, (x_0^2-c)(bx_0^2-bc-1))_v = (a, c(bc+1))_v = (a, c)_v(a, bc+1)_v = 1$, which implies that V_2^0 admits a K_v-point with $x = 0$.
(5) Suppose that $v \in \Omega_K^f \backslash (S' \cup S'' \cup 2_K)$. Then $v(b) = 0$. Take $x_0 \in K_v$ such that the valuation $v(x_0) < 0$. Since $b \in \mathcal{O}_{K_v}^\times$ and $c \in \mathcal{O}_K[1/2]$, by Lemma 2.4, we have $(a, x_0^2 - c)_v = (a, x_0^2)_v = 1$ and $(a, bx_0^2 - bc - 1)_v = (a, bx_0^2)_v = (a, b)_v$. Since $v(a)$ and $v(b)$ are both even, by Lemma 2.3, we have $(a, b)_v = 1$. So $(a, (x_0^2 - c)(bx_0^2 - bc - 1))_v = (a, b)_v = 1$, which implies that V_2^0 admits a K_v-point with $x = x_0$.
(6) Suppose that $v \in S \cap \infty_K$. Let $x_0 = 0$. Then by the choice of a, b, c, we have $\tau_v(a) < 0$, $\tau_v(c) > 0$ and $\tau_v(bc+1) > 0$. So $(a, (x_0^2 - c)(bx_0^2 - bc - 1))_v = (a, c(bc+1))_v = 1$, which implies that V_2^0 admits a K_v-point with $x = 0$.
(7) Suppose that $v \in S \backslash \infty_K$. Choose a prime element π_v and take $x_0 = \pi_v$. By the choice of a, b, c, we have $b, c \in \mathcal{O}_{K_v}^\times$, $v(bx_0^2) = 2$, and $v(bc+1) = v(a) + 2 \geq 3$. By Lemma 2.4, we have $(a, x_0^2 - c)_v = (a, -c)_v$ and $(a, bx_0^2 - bc - 1)_v = (a, bx_0^2)_v$. By Hensel's lemma, we have $-bc = 1 - (bc+1) \in K_v^{\times 2}$. So $(a, (x_0^2 - c)(bx_0^2 - bc - 1))_v = (a, -bcx_0^2)_v = 1$, which implies that V_2^0 admits a K_v-point with $x = \pi_v$.

Secondly, we need to prove the statement about the Brauer group, and find the element A in this proposition.

By the choice of the places v_1, the polynomial x^2-c is an Eisenstein polynomial, so it is irreducible over K_{v_1}. Since $v_1(a)$ is even, we have $K(\sqrt{a})K_{v_1} \ncong K_{v_1}[x]/(x^2-c)$. So $K(\sqrt{a}) \ncong K[x]/(x^2-c)$. The same argument holds for the place v_2 and the polynomial bx^2-bc-1. Since all places of S split completely in L, by lemma 3.4, we have $a \in \mathcal{O}_K \backslash L^2$. By the splitting condition of v_1, v_2, we have $L(\sqrt{a}) \ncong L[x]/(x^2-c)$ and $L(\sqrt{a}) \ncong L[x]/(bx^2-bc-1)$. So $P(x) = (x^2-c)(bx^2-bc-1)$ is separable and a product of two degree-2 irreducible factors over K and L. According to [14, Proposition 7.1.1], the Brauer group $\mathrm{Br}(V_2)/\mathrm{Br}(K) \cong \mathrm{Br}(V_{2L})/\mathrm{Br}(L) \cong \mathbb{Z}/2\mathbb{Z}$. Furthermore, by Proposition 7.1.2 in loc. cit, we take the quaternion algebra $A = (a, x^2-c) \in \mathrm{Br}(V_2)$ as a generator element of this group. Then we have the equality $A = (a, x^2-c) = (a, bx^2-bc-1)$ in $\mathrm{Br}(V_2)$.

Thirdly, We need to compute the evaluation of A on $V_2(K_v)$ for all $v \in \Omega_K$.

By Remark 3.2, it suffices to compute the local invariant $\mathrm{inv}_v(A(P_v))$ for all $P_v \in V_2^0(K_v)$ and all $v \in \Omega_K$.

(1) Suppose that $v \in (\infty_K \backslash S') \cup 2_K$. Then $a \in K_v^{\times 2}$, so $\mathrm{inv}_v(A(P_v)) = 0$ for all $P_v \in V_2(K_v)$.

(2) Suppose that $v \in (S' \backslash S) \cap \infty_K$. By the choice of b, c, we have $\tau_v(b) > 0$ and $\tau_v(bc+1) < 0$. So, for any $x \in K$, we have $(a, bx^2-bc-1)_v = 1$. Hence $\mathrm{inv}_v(A(P_v)) = 0$ for all $P_v \in V_2^0(K_v)$.

(3) Suppose that $v \in S' \backslash (S \cup \infty_K)$. By the choice of b, we have $(a, b)_v = 1$. Take an arbitrary $P_v \in V_2^0(K_v)$.
If $v(x) < 0$ at P_v, by Lemma 2.4, we have $(a, x^2-c)_v = (a, x^2)_v = 1$. If $v(x) > 0$ at P_v, since $b, c \in \mathcal{O}_{K_v}^{\times}$ and $v(bc+1) = v(a) + 2 \geq 3$, by Lemma 2.4, we have $(a, x^2-c)_v = (a, -c)_v$. By Hensel's lemma, we have $-bc = 1 - (bc+1) \in K_v^{\times 2}$. So $(a, x^2-c)_v = (a, -c)_v = (a, -bc)_v = 1$. If $v(x) = 0$ at P_v, since $b \in \mathcal{O}_{K_v}^{\times}$ and $v(bc+1) = v(a) + 2 \geq 3$, by Lemma 2.4, we have $(a, bx^2-bc-1)_v = (a, bx^2)_v = 1$. So $\mathrm{inv}_v(A(P_v)) = 0$.

(4) Suppose that $v \in \Omega_K^f \backslash (S' \cup 2_K)$. Take an arbitrary $P_v \in V_2^0(K_v)$. If $\mathrm{inv}_v(A(P_v)) = 1/2$, then $(a, bx^2-bc-1)_v = (a, x^2-c)_v = -1$ at P_v. Since $v(a)$ is even, by Lemma 2.3, the last equality implies that $v(x^2-c)$ is odd, so it is positive. So $v(bx^2-bc-1) = 0$. By Lemma 2.3, we have $(a, bx^2-bc-1)_v = 1$, which is a contradiction. So $\mathrm{inv}_v(A(P_v)) = 0$.

(5) Suppose that $v \in S \cap \infty_K$. Take an arbitrary $P_v \in V_2^0(K_v)$. If $A(P_v) = 0$, then $(a, bx^2-bc-1)_v = (a, x^2-c)_v = 1$ at P_v. The last equality implies that $\tau_v(x^2-c) > 0$. By the choice of b, we have $\tau_v(b) < 0$, so $\tau_v(bx^2-bc-1) < 0$, which contradicts $(a, bx^2-bc-1)_v = 1$. So $\mathrm{inv}_v(A(P_v)) = 1/2$.

(6) Suppose that $v \in S \backslash \infty_K$. By the choice of b, we have $(a, b)_v = -1$. Take an arbitrary $P_v \in V_2^0(K_v)$. If $v(x) \leq 0$ at P_v, because of $b \in \mathcal{O}_{K_v}^{\times}$ and $v(bc+1) = v(a) + 2 \geq 3$, by Lemma 2.4, we have $(a, bx^2-bc-1)_v = (a, bx^2)_v = -1$. If $v(x) > 0$ at P_v, since $b, c \in \mathcal{O}_{K_v}^{\times}$ and $v(bc+1) = v(a) + 2 \geq 3$, by Lemma 2.4, we have $(a, x^2-c)_v = (a, -c)_v$. By Hensel's lemma, we have $-bc = 1 - (bc+1) \in K_v^{\times 2}$.
So $(a, x^2-c)_v = (a, -c)_v = -(a, -bc)_v = -1$. So $\mathrm{inv}_v(A(P_v)) = 1/2$.

Finally, we need to compute the evaluation of A on $V_2(L_{v'})$ for all $v' \in \Omega_L$.

(1) Suppose that $v' \in S_L$. Let $v \in \Omega_K$ be the restriction of v' on K. By the assumption that v splits completely in L, we have $K_v = L_{v'}$. So $V_2(K_v) = V_2(L_{v'})$. Then for any $P_{v'} \in V_2(L_{v'})$, denote $P_{v'}$ in $V_2(K_v)$ by P_v. Then by the argument already shown, the local invariant $\mathrm{inv}_{v'}(A(P_{v'})) = \mathrm{inv}_v(A(P_v)) = 1/2$.
(2) Suppose that $v' \in \Omega_L \backslash S_L$. This local computation is the same as the case $v \in \Omega_K \backslash S$. □

Remark 5.2 If the surface V_2 has a K-rational point Q, then by the global reciprocity law, the sum $\sum_{v \in \Omega_K} \mathrm{inv}_v(A(Q)) = 0$ in $\mathbb{Q}/\mathbb{Z}$. If the number $\sharp S$ is odd, then from Proposition 5.1, this sum is $\sharp S/2$, which is nonzero in $\mathbb{Q}/\mathbb{Z}$. So, in this case, the surface V_2 has no K-rational point, which implies that the surface V_2 is a counterexample to the Hasse principle. If the number $\sharp S$ is even, then for any $(P_v)_{v \in \Omega_K} \in V_2(\mathbb{A}_K)$, by Proposition 5.1, the sum $\sum_{v \in \Omega_K} \mathrm{inv}_v(A(P_v)) = \sharp S/2$ is 0 in $\mathbb{Q}/\mathbb{Z}$. Since $\mathrm{Br}(V_2)/\mathrm{Br}(K)$ is generated by the element A, we have $V_2(\mathbb{A}_K)^{\mathrm{Br}} = V_2(\mathbb{A}_K) \neq \emptyset$. According to [4, 5, Theorem B], the Brauer-Manin obstruction to the Hasse principle and weak approximation is the only one for Châtelet surfaces. So, in this case, the set $V_2(K) \neq \emptyset$, and it is dense in $V_2(\mathbb{A}_K)^{\mathrm{Br}} = V_2(\mathbb{A}_K)$, i.e. the surface V_2 has a K-rational point and satisfies weak approximation. In particular, if the number $\sharp S = 0$, i.e. $S = \emptyset$, though the Brauer group $\mathrm{Br}(V_2)/\mathrm{Br}(K)$ is nontrivial, it gives no obstruction to weak approximation for V_2.

Combining the construction method in Sect. 5.1 with the global reciprocity law, we can relate the properties in Proposition 4.1 to the Hasse principle and weak approximation.

Theorem 5.1 *For any extension of number fields L/K, there exists a Châtelet surface V over K with $V(\mathbb{A}_K) \neq \emptyset$, such that for every intermediate field $K \subset L' \subset L$, $\mathrm{Br}(V)/\mathrm{Br}(K) \cong \mathrm{Br}(V_{L'})/\mathrm{Br}(L') \cong \mathbb{Z}/2\mathbb{Z}$, and that the surface $V_{L'}$ has the following properties.*

- *If the degree $[L' : K]$ is odd, then the surface $V_{L'}$ is a counterexample to the Hasse principle, i.e. $V(L') = \emptyset$. In particular, the surface V is a counterexample to the Hasse principle.*
- *If the degree $[L' : K]$ is even, then the surface $V_{L'}$ satisfies weak approximation. In particular, in this case, $V(L') \neq \emptyset$.*

Proof By Theorem 2.2, we can take a place $v_0 \in \Omega_K \backslash (\infty_K^c \cup 2_K)$ splitting completely in L. Let $S = \{v_0\}$. Using the construction method in Sect. 5.1, there exists a Châtelet surface V defined over K having the properties of Proposition 5.1. For any subfield $L' \subset L$ over K, by the same argument as in the proof of Proposition 5.1, we have $\mathrm{Br}(V)/\mathrm{Br}(K) \cong \mathrm{Br}(V_{L'})/\mathrm{Br}(L') \cong \mathbb{Z}/2\mathbb{Z}$. Since v_0 splits completely in L, it also does in L'. Since $\sharp S$ is odd, $[L' : K]$ is odd if and only if $\sharp S_{L'}$ is odd. Applying the same argument about the field L to its subfield L', the properties that we list are just what we have explained in Remark 5.2. □

Remark 5.3 The Brauer group $\mathrm{Br}(V)/\mathrm{Br}(K) \cong \mathrm{Br}(V_{L'})/\mathrm{Br}(L') \cong \mathbb{Z}/2\mathbb{Z}$ in Theorem 5.1 is nontrivial, and it gives an obstruction to the Hasse principle for V, and also for $V_{L'}$ if $[L' : K]$ is odd. But it no longer gives an obstruction to weak approximation for $V_{L'}$ if $[L' : K]$ is even.

Using the construction method in Sect. 5.1, we have the following example, which is a special case of Proposition 5.1.

Example 5.4 Let $K=\mathbb{Q}$, and let ζ_7 be a primitive 7-th root of unity. Let $\alpha=\zeta_7+\zeta_7^{-1}$ with the minimal polynomial x^3+x^2-2x-1. Let $L=\mathbb{Q}(\alpha)$. Then $[L:K]=3$. Let $S=\{13\}$. Since $13^2\equiv 1 \mod 7$, $41^2\equiv 1 \mod 7$, and $43\equiv 1 \mod 7$, the places $13, 41, 43$ split completely in L. Using the construction method in Sect. 5.1, we choose data: $S=\{13\},\ S'=\{13,29\},\ S''=\{5\},\ v_1=43,\ v_2=41,\ a=377,\ b=5,\ c=878755181$ and $P(x)=(x^2-878755181)(5x^2-4393775906)$. Then the Châtelet surface given by $y^2-377z^2=P(x)$ has the properties of Proposition 5.1.

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