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On sign changes of Fourier coefficients of Hermitian cusp forms of degree two

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Abstract

We prove a quantitative result for the number of sign changes of the Fourier coefficients of a Hermitian cusp form of degree 2. In addition, we prove a quantitative result for the number of sign changes of the primitive Fourier coefficients. We give an explicit upper bound for the first sign change of the Fourier coefficients of a Hermitian cusp form of degree 2 over certain imaginary quadratic extensions.

Keywords Hermitian modular forms · Hermitian Jacobi forms · Jacobi forms with matrix index · Sturm bound · Sign changes

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1 Introduction and statement of the main results

The distribution of signs of the Fourier coefficients of a non-zero elliptic cusp form has been a subject of study for several mathematicians over the past years. One aspect of this problem is the study of number of sign changes of the Fourier coefficients. Knopp, Kohnen, and Pribitkin in [14] proved that the Fourier coefficients of a non-zero elliptic cusp form f on a congruence subgroup of the full modular group $SL_2(\mathbb{Z})$ have infinitely many sign changes. They use the Landau's theorem on Dirichlet series

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with non-negative coefficients and the finiteness of the Hecke L-function attached to the elliptic cusp form f to prove their result. In addition, one can see that [16] and [18] are devoted to the study of sign changes of the Fourier coefficients of an elliptic Hecke eigenform. A more subtle problem is to give an explicit upper bound for the first sign change. This has been studied for elliptic cusp forms of square-free level by Choie and Kohnen [2]. Later, their result has been improved by He and Zhao [11]. For elliptic Hecke eigenforms of level N the problem has been dealt in [13, 16].

The theory of elliptic modular forms has been generalized to several variables. Hermitian modular forms over an imaginary quadratic field K are one of those generalizations. In this article, we give a quantitative result for the number of sign changes of the Fourier coefficients of a Hermitian cusp form F of degree 2. Moreover, we also give a quantitative result for the number of sign changes of the primitive Fourier coefficients. Note that Yamana [22] has established that F is determined by its primitive Fourier coefficients. Also, we provide an explicit upper bound for the first sign change of the Fourier coefficients of a Hermitian cusp form over certain imaginary quadratic fields. To the best of our knowledge, this is the first attempt to study the distribution of signs of the Fourier coefficients of a Hermitian cusp form. Now, we introduce the necessary notations to state our results.

Let $d > 0$ be a square free integer. Throughout the article, let $K = \mathbb{Q}(\sqrt{-d})$ be a fixed imaginary quadratic field. Let

$$D_K = \begin{cases} -4d & \text{if } -d \equiv 2, 3 \pmod 4, \\ -d & \text{if } -d \equiv 1 \pmod 4) \end{cases}$$

be the discriminant of K. Let $\mathcal{O}_K$ be the ring of integers of K and $\mathcal{O}_K^\# = \frac{i}{\sqrt{|D_K|}}\mathcal{O}_K$ be the inverse different of K over $\mathbb{Q}$. The Hermitian modular group of degree 2 over K is given by

$$U_2(\mathcal{O}_K) = \{M \in M_4(\mathcal{O}_K) \mid \overline{M}^t J_2 M = J_2\},$$

where $J_2 = \begin{pmatrix} \mathbf{0}_2 & -\boldsymbol{I}_2 \\ \boldsymbol{I}_2 & \mathbf{0}_2 \end{pmatrix}$, $\boldsymbol{I}_2$ and $\mathbf{0}_2$ are the 2×2 identity matrix and zero matrix respectively. The subgroup

$$SU_2(\mathcal{O}_K) = U_2(\mathcal{O}_K) \cap SL_4(\mathcal{O}_K)$$

coincides with the full modular group $U_2(\mathcal{O}_K)$ if $D_K \neq -3, -4$. We denote by $S_k(SU_2(\mathcal{O}_K))$ the space of Hermitian cusp forms of degree 2 on $SU_2(\mathcal{O}_K)$ (defined in Sect. 2.1). Any $F \in S_k(SU_2(\mathcal{O}_K))$ has a Fourier series expansion of the form:

$$F(Z) = \sum_{T \in \Delta_2^+} A_F(T) e(tr(TZ)) = \sum_{\substack{n,m \in \mathbb{Z}, r \in \mathcal{O}_K^\# \\ n,m,nm-N(r)>0}} A_F(n,r,m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m, \quad (1)$$

where

$$\Delta_2^+ = \left\{ T = \begin{pmatrix} n & r \\ \overline{r} & m \end{pmatrix} \middle| \, n, m \in \mathbb{Z}, \; r \in \mathcal{O}_K^{\#}, \; T > 0 \right\}, Z \in \begin{pmatrix} \tau & z_1 \\ z_2 & \tau' \end{pmatrix} \in \mathcal{H}_2,$$

$q = e(\tau), \zeta_1 = e(z_1), \zeta_2 = e(z_2), q' = e(\tau'), e(z) = e^{2\pi i z}$. The first result of this article gives a quantitative result for the sign changes of the Fourier coefficients of F.

Theorem 1.1 *Let $F \in S_k(SU_2(\mathcal{O}_K))$ be a non-zero Hermitian cusp form with real Fourier coefficients $A_F(T)$. Then $A_F(T)$ changes sign at least once for $|D_K|\det(T) \in (X, X + X^{3/5}]$ for $X \gg 1$.*

For any $T \in \Delta_2^+$, we define

$$\mu(T) = \max\{l \in \mathbb{N} \mid l^{-1}T \in \Delta_2^+\}.$$

We say that T is primitive if $\mu(T) = 1$. The Fourier coefficient of F at a primitive T is known as primitive Fourier coefficient. The second result of this article gives the following quantitative result on the number of sign changes of the primitive Fourier coefficients.

Theorem 1.2 *Let $F \in S_k(SU_2(\mathcal{O}_K))$ be non-zero with real Fourier coefficients $A_F(T)$. Then the primitive Fourier coefficients $A_F(T)$ changes sign at least once for $|D_K|\det(T) \in (X, X + X^{3/5}]$ for $X \gg 1$.*

Theorem 1.1 implies that there are infinitely many sign changes of the Fourier coefficients of $F \in S_k(SU_2(\mathcal{O}_K))$. Next, we focus our attention on establishing an explicit upper bound for the first sign of any $F \in S_k(SU_2(\mathcal{O}_K))$. To accomplish this, we first establish a Sturm bound for Hermitian modular forms of degree 2.

Theorem 1.3 *Let $K = \mathbb{Q}(\sqrt{-d})$ where $d \in \{1, 2, 3, 7, 11, 15\}$. Also, let*

$$F(\tau, z_1, z_2, \tau') = \sum_{\substack{n,m \in \mathbb{Z}, r \in \mathcal{O}_K^{\#} \\ n,m,nm-N(r) \geq 0}} A_F(n, r, m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m \in M_k(SU_2(\mathcal{O}_K)).$$

If $A_F(n, r, m) = 0$ for all $0 \leq n \leq \beta$ and $0 \leq m \leq \beta$, where

$$\beta = \begin{cases} \left[\frac{k}{2(5-d)}\right] & \text{if } d = 1, 2, \\ \\ \left[\frac{2k}{(19-d)}\right] & \text{if } d = 3, 7, 11, 15, \end{cases}$$

then

$$F = 0.$$

Finally, using Theorem 1.3, we give an explicit upper bound for the first sign change of the Fourier coefficients of a Hermitian cusp form of degree 2.

Theorem 1.4 *Let* $K = \mathbb{Q}(\sqrt{-d})$ *where* $d \in \{1, 2, 3, 7, 11, 15\}$. *Suppose* $F \in S_k(SU_2(\mathcal{O}_K))$ *is non-zero with real Fourier coefficients* $A_F(T)$. *Then there exist* $T_1, T_2 \in \Delta_2^+$ *with*

$$tr(T_1),\ tr(T_2) \ll (4c_d k)^{2+\epsilon},$$

for any real $\epsilon > 0$, *where*

$$c_d = \begin{cases} \frac{7+d}{5-d} & \text{if } d = 1, 2, \\ \frac{29+d}{19-d} & \text{if } d = 3, 7, 11, 15, \end{cases}$$

such that

$$A_F(T_1)A_F(T_2) < 0.$$

Remark 1.5 For $F \in S_k(SU_2(\mathcal{O}_K))$ with complex Fourier coefficients $A_F(T)$, the Fourier series with $Re(A_F(T))$ (respectively $Im(A_F(T))$) are again in $S_k(SU_2(\mathcal{O}_K))$. Therefore, there is an obvious reformulation of Theorems 1.1, 1.2 and 1.4 for arbitrary $F \in S_k(SU_2(\mathcal{O}_K))$ with $A_F(T)$ replaced by $Re(A_F(T))$ and $Im(A_F(T))$.

The article is organized as follows: In the next section we recall the definition of three concepts used in this paper; Hermitian modular forms of degree 2, Hermitian Jacobi forms and Jacobi forms with matrix index. We show that Hermitian Jacobi forms occur as the coefficients in the Fourier–Jacobi expansion of a Hermitian modular form of degree 2. In Sects. 3 and 4, we give the proof of Theorems 1.1 and 1.2 respectively. Section 5 is the largest, and contains the proof of Theorem 1.3. We prove Proposition 5.1, Theorems 5.2, and 5.3 in this section, which may be of interest on their own. Finally, in Sect. 6, we prove Theorem 1.4.

Notation For any ring $R \subset \mathbb{C}$, we write by $R^n = \{(\alpha_1, \cdots, \alpha_n) \mid \alpha_i \in R\}$ the set of row matrices of size $1 \times n$ with entries in R. We denote by $M_n(R)$ the set of all $n \times n$ matrices with entries in R. Let $GL_n(R)$ be the group of matrices in $M_n(R)$ with non-zero determinant and let $SL_n(R)$ be the group of matrices with determinant 1. For any $M \in M_n(R)$, we write by $\overline{M}$ the complex conjugate of M and by M^t the transpose of matrix M. We denote by $\det(M)$ and $tr(M)$ the determinant and trace of the matrix M respectively. Also let $A[B]$ denote the matrix $\overline{B}^t AB$ for two complex matrices A and B of appropriate sizes. For $\alpha \in \mathbb{C}$, we write $e(\alpha) := e^{2\pi i\alpha}$ and $N(\alpha) := \alpha\overline{\alpha}$. We denote by $\mathcal{O}_K^\times$, the group of units in $\mathcal{O}_K$.

2 Preliminaries

2.1 Hermitian modular forms of degree two

The Hermitian upper-half space of degree 2 is defined by

$$\mathcal{H}_2 = \left\{ Z = \begin{pmatrix} \tau & z_1 \\ z_2 & \tau' \end{pmatrix} \in M_2(\mathbb{C}) \mid \frac{1}{2i}(Z - \overline{Z}^t) > 0 \right\}.$$

The Hermitian modular group $U_2(\mathcal{O}_K)$ acts on $\mathcal{H}_2$ by

$$M \cdot Z = (AZ + B)(CZ + D)^{-1} \quad \text{where } Z \in \mathcal{H}_2, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U_2(\mathcal{O}_K).$$

For any non-negative integer k, we define the action of $U_2(\mathcal{O}_K)$ on the set of functions from $\mathcal{H}_2$ to $\mathbb{C}$ by

$$(F \mid_k M)(Z) = (\det(CZ + D))^{-k} F(M \cdot Z).$$

For a positive integer N, we define the congruence subgroup $\Gamma_0^{(2)}(N)$ of $SU_2(\mathcal{O}_K)$ by

$$\Gamma_0^{(2)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU_2(\mathcal{O}_K) \mid C \equiv \mathbf{0}_2 \pmod{N\mathcal{O}_K} \right\}.$$

Note that if $N = 1$ then $\Gamma_0^{(2)}(1) = SU_2(\mathcal{O}_K)$.

Definition 2.1 A holomorphic function $F : \mathcal{H}_2 \to \mathbb{C}$ is called a Hermitian modular form of weight k on $\Gamma_0^{(2)}(N)$ if it satisfies

$$F \mid_k M = F \tag{2}$$

for all $M \in \Gamma_0^{(2)}(N)$.

We denote by $M_k(\Gamma_0^{(2)}(N))$ the space of Hermitian modular forms of degree 2 on the group $\Gamma_0^{(2)}(N)$. Any $F \in M_k(\Gamma_0^{(2)}(N))$ possesses a Fourier series expansion of the form:

$$\begin{aligned} F(Z) = F(\tau, z_1, z_2, \tau') &= \sum_{T \in \Delta_2} A_F(T) e(tr(TZ)) \\ &= \sum_{\substack{n,m \in \mathbb{Z}, r \in \mathcal{O}_K^\# \\ n,m,nm - N(r) \geq 0}} A_F(n, r, m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m, \end{aligned} \tag{3}$$

where $q = e(\tau)$, $\zeta_1 = e(z_1)$, $\zeta_2 = e(z_2)$, $q' = e(\tau')$ and

$$\Delta_2 = \left\{ T = \begin{pmatrix} n & r \\ \overline{r} & m \end{pmatrix} \middle|\, n, m \in \mathbb{Z},\ r \in \mathcal{O}_K^{\#},\ T \geq 0 \right\}.$$

Moreover, F is called a cusp form if $A_F(T) = 0$ whenever $\det(T) = 0$. We denote by $S_k(\Gamma_0^{(2)}(N))$ the space of cusp form in $M_k(\Gamma_0^{(2)}(N))$. We note down the following result by Yamana [22] which characterizes a Hermitian cusp form by its primitive Fourier coefficients.

Theorem 2.2 *Suppose $F \in S_k(SU_2(\mathcal{O}_K))$ is non-zero with Fourier coefficients $A_F(T)$. Then there exists a primitive matrix $T_0 \in \Delta_2^+$ such that $A_F(T_0) \neq 0$.*

The group $GL_2(\mathcal{O}_K)$ acts on Δ_2^+ by $T \mapsto \overline{g}^t T g$, where $g \in GL_2(\mathcal{O}_K)$. We have the following lemma.

Lemma 2.3 *Let $T \in \Delta_2^+$ be a primitive matrix. Then there exists $g \in SL_2(\mathcal{O}_K)$ such that $\overline{g}^t T g = \begin{pmatrix} * & * \\ * & p \end{pmatrix}$ for some odd prime p.*

Proof By [1, Lemma 3.1] there exists a $g \in GL_2(\mathcal{O}_K)$ such that $\overline{g}^t T g = \begin{pmatrix} * & * \\ * & p \end{pmatrix}$. Let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathcal{O}_K)$ and $T = \begin{pmatrix} n & r \\ \overline{r} & m \end{pmatrix}$. Then we have

$$\overline{g}^t T g = \begin{pmatrix} * & * \\ * & p = N(\beta)n + \delta r \overline{\beta} + \beta \overline{r} \overline{\delta} + N(\delta)m \end{pmatrix}.$$

We know that $\det(g) = \epsilon$, where $\epsilon \in \mathcal{O}_K^\times$. Therefore, we can take

$$g_1 = \begin{pmatrix} \alpha/\epsilon & \beta \\ \gamma/\epsilon & \delta \end{pmatrix} \in SL_2(\mathcal{O}_K)$$

such that $\overline{g}_1^t T g_1 = \begin{pmatrix} * & * \\ * & p \end{pmatrix}$. □

For any $g \in SL_2(\mathcal{O}_K)$, we have $\begin{pmatrix} (\overline{g}^t)^{-1} & \mathbf{0}_2 \\ \mathbf{0}_2 & g \end{pmatrix} \in SU_2(\mathcal{O}_K)$. Applying the transformation (2) on $F \in S_k(SU_2(\mathcal{O}_K))$, we get the following relation on the Fourier coefficients of F

$$A_F(gT\overline{g}^t) = A_F(T), \quad \text{for all } g \in SL_2(\mathcal{O}_K), T \in \Delta_2^+.$$

Now using Theorem 2.2 and Lemma 2.3 we get the following.

Lemma 2.4 *Suppose $F \in S_k(SU_2(\mathcal{O}_K))$ is non-zero with Fourier coefficients $A_F(T)$. Then, for some odd prime, p there exists a primitive $T_0 = \begin{pmatrix} * & * \\ * & p \end{pmatrix} \in \Delta_2^+$ such that $A_F(T_0) \neq 0$.*

2.2 Hermitian Jacobi forms

Let $G = SL_2(\mathbb{Z}) \ltimes \mathcal{O}_K^2$ be the Hermitian Jacobi group over $\mathcal{O}_K$. The Jacobi group G acts on $\mathcal{H} \times \mathbb{C}^2$ as follows:

$$(g, (\lambda, \mu)) \cdot (\tau, z_1, z_2) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z_1 + \lambda\tau + \mu}{c\tau + d}, \frac{z_2 + \overline{\lambda}\tau + \overline{\mu}}{c\tau + d} \right),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \tau \in \mathcal{H}, \lambda, \mu \in \mathcal{O}_K, z_1, z_2 \in \mathbb{C}$. For any positive integer N, let

$$\Gamma_0^{(1)}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N\mathbb{Z}} \right\}.$$

Definition 2.5 A holomorphic function $\phi : \mathcal{H} \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ is a Hermitian Jacobi form of weight k and index m on $\Gamma_0^{(1)}(N)$ if for each $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(1)}(N)$, and $\lambda, \mu \in \mathcal{O}_K$, we have

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z_1}{c\tau + d}, \frac{z_2}{c\tau + d} \right) = (c\tau + d)^k e^{\frac{2\pi i m c z_1 z_2}{c\tau + d}} \phi(\tau, z_1, z_2), \tag{4}$$

$$\phi\left(\tau, z_1 + \lambda\tau + \mu, z_2 + \overline{\lambda}\tau + \overline{\mu}\right) = e^{-2\pi i m(N(\lambda)\tau + \overline{\lambda} z_1 + \lambda z_2)} \phi(\tau, z_1, z_2) \tag{5}$$

and ϕ has a Fourier series expansion of the form

$$\phi = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}_K^{\#} \\ nm - N(r) \geq 0}} c(n, r) q^n \zeta_1^r \zeta_2^{\overline{r}},$$

where $q = e(\tau), \zeta_1 = e(z_1), \zeta_2 = e(z_2)$.

We denote by $J_{k,m}(\Gamma_0^{(1)}(N))$ the vector space of all Hermitian Jacobi forms of weight k and index m on $\Gamma_0^{(1)}(N)$.

2.2.1 Theta decomposition

The invariance of ϕ under the action of $(\lambda, 0)$ in (5) yields that the Fourier coefficient $c(n, r)$ is completely determined by $r \pmod{m\mathcal{O}_K}$ and $nm - N(r)$. We define

$$c_s(L) = \begin{cases} c(n, r) & \text{if } r \equiv s \pmod{m\mathcal{O}_K} \text{ and } L = |D_K|(nm - N(r)), \\ 0 & \text{otherwise.} \end{cases}$$

The theta decomposition of $\phi \in J_{k,m}(\Gamma_0^{(1)}(N))$ is given by

$$\phi_m(\tau, z_1, z_2) = \sum_{s \in \mathcal{O}_K^{\#}/m\mathcal{O}_K} h_s \theta_{m,s},$$

where

$$h_s(\tau) = \sum_{\substack{L=0 \\ L \equiv -N(s)|D_K| \pmod{m|D_K|\mathbb{Z}}}}^{\infty} c_s(L) e\left(\frac{L}{|D_K|m}\tau\right),$$
$$\theta_{m,s} = \sum_{r \equiv s \pmod{m\mathcal{O}_K}} e\left(\frac{N(r)}{m}\tau + rz_1 + \overline{r}z_2\right).$$

The theta components h_s of ϕ are elliptic modular forms on the prinicipal congruence subgroup $\Gamma^{(1)}(|D_K|Nm)$ (see [9, 10]).

2.3 Fourier–Jacobi expansion

Let $F \in S_k(\Gamma_0^{(2)}(N))$ has Fourier series expansion of the form (3). We write the Fourier series expansion of F as

$$F(\tau, z_1, z_2, \tau') = \sum_{m=1}^{\infty} \phi_m e(m\tau'),$$
$$\text{where } \phi_m = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}_K^{\#} \\ nm - N(r) \geq 0}} A_F(n, r, m) e(n\tau + rz_1 + \overline{r}z_2). \tag{6}$$

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(1)}(N)$ and $(\lambda, \mu) \in \mathcal{O}_K^2$, the matrices

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & \mu \\ \overline{\lambda} & 1 & \overline{\mu} & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are in $\Gamma_0^{(2)}(N)$. These matrices act on $\mathcal{H}_2$ by

$$(\tau, z_1, z_2, \tau') \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{z_1}{c\tau + d}, \frac{z_2}{c\tau + d}, \tau' - \frac{cz_1z_2}{c\tau + d}\right),$$
$$(\tau, z_1, z_2, \tau') \mapsto (\tau, z_1 + \lambda\tau + \mu, z_2 + \lambda\tau + \mu, \tau' + \lambda z_1 + \lambda z_2 + \lambda^2\tau + \lambda\mu)$$

respectively. Because F satisfies the transformation law (2), we can deduce the two transformation laws of Hermitian Jacobi forms for ϕ_m, and therefore,

$\phi_m \in J_{k,m}(\Gamma_0^{(1)}(N))$. We call (6) the Fourier–Jacobi expansion of F and ϕ_m's the Fourier–Jacobi coefficients of F.

2.4 Jacobi form with matrix index

The Jacobi group $\Gamma^\ell = SL_2(\mathbb{Z}) \ltimes (\mathbb{Z}^\ell \times \mathbb{Z}^\ell)$ acts on $\mathcal{H} \times \mathbb{C}^\ell$ as follows:

$$(g, (\lambda, \mu)) \cdot (\tau, z_1, \cdots, z_\ell) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z_1 + \lambda_1\tau + \mu_1}{c\tau + d}, \cdots, \frac{z_\ell + \lambda_\ell\tau + \mu_\ell}{c\tau + d}\right),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $\tau \in \mathcal{H}$, $\lambda = (\lambda_1, \cdots, \lambda_\ell)$, $\mu = (\mu_1, \cdots, \mu_\ell) \in \mathbb{Z}^\ell$ and $z = (z_1, \cdots, z_\ell) \in \mathbb{C}^\ell$.

Definition 2.6 Let M be a symmetric, positive definite, half-integral $\ell \times \ell$ matrix with integral diagonal entries. A holomorphic function $\psi : \mathcal{H} \times \mathbb{C}^\ell \longrightarrow \mathbb{C}$ is a Jacobi form of weight k and index M on $\Gamma_0^{(1)}(N)$ if for each $g \in \Gamma_0^{(1)}(N)$ and $\lambda, \mu \in \mathbb{Z}^\ell$, we have

$$\psi\left(\frac{a\tau + b}{c\tau + d}, \frac{z_1}{c\tau + d}, \cdots, \frac{z_\ell}{c\tau + d}\right) = (c\tau + d)^k e^{2\pi i \frac{cM[z^t]}{c\tau+d}} \psi(\tau, z_1, \cdots, z_\ell), \quad (7)$$

$$\psi(\tau, z_1 + \lambda_1\tau + \mu_1, \cdots, z_\ell + \lambda_\ell\tau + \mu_\ell) = e^{-2\pi i(\tau M[\lambda^t] + 2\lambda M z^t)} \psi(\tau, z_1, \cdots, z_\ell) \quad (8)$$

and ψ has a Fourier series expansion of the form

$$\psi(\tau, z_1, \cdots, z_\ell) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathbb{Z}^\ell \\ 4\det(M)n - M^\#[r^t] \geq 0}} c(n, r) q^n \zeta^r, \quad (9)$$

where $\tau \in \mathcal{H}$, $z = (z_1, \cdots, z_\ell) \in \mathbb{C}^\ell$, $q = e^{2\pi i\tau}$, $\zeta^r = e^{2\pi i r z^t}$ and $M^\#$ is the adjugate of M.

3 Proof of Theorem 1.1

Since $F \neq 0$, there exists a m_0 such that the Fourier–Jacobi coefficient $\phi_{m_0} \neq 0$ in the Fourier–Jacobi expansion of F. Therefore, there exists $s_0 \in \mathcal{O}_K^\# / m_0\mathcal{O}_K$ such that the theta component $h_{s_0} \neq 0$ in the theta decomposition of ϕ_{m_0}. The Fourier series expansion of h_{s_0} is given by

$$h_{s_0}(\tau) = \sum_{\substack{n=1 \\ n \equiv -N(s_0)|D_K| \pmod{m_0|D_K|\mathbb{Z}}}}^{\infty} a(n) e\left(\frac{n}{|D_K| m_0}\tau\right),$$

where

$$a(n)=\begin{cases}A_F(T) & \text{if } T=\begin{pmatrix}\frac{n+N(s_0)|D_K|}{|D_K|m_0} & s_0\\ \overline{s_0} & m_0\end{pmatrix}\in\Delta_2^+,\\ 0 & \text{otherwise.}\end{cases}$$

We have $h_{s_0}\in S_{k-1}(\Gamma^{(1)}(|D_K|m_0))$ and hence $h_{s_0}(|D_K|m_0\tau)\in S_{k-1}(\Gamma_1^{(1)}(|D_K|^2m_0^2))$. We know that

$$S_{k-1}(\Gamma_1^{(1)}(|D_K|^2m_0^2))=\bigoplus_{\psi}S_{k-1}(\Gamma_0^{(1)}(|D_K|^2m_0^2),\psi),$$

where the direct sum is over all Dirichlet characters modulo $|D_K|^2m_0^2$. For each Dirichlet character ψ mod $|D_K|^2m_0^2$, let $f_\psi\in S_{k-1}(\Gamma_0^{(1)}(|D_K|^2m_0^2),\psi)$ be such that

$$h_{s_0}(|D_K|m_0\tau)=\sum_{\psi}f_\psi(\tau).$$

Suppose the Fourier series expansion of f_ψ is given by $f_\psi=\sum_{n\geq 1}a_\psi(n)e(n\tau)$. Then from the above equation we have

$$\sum_{n\geq 1}a(n)e(n\tau)=\sum_{\psi}\sum_{n\geq 1}a_\psi(n)e(n\tau). \tag{10}$$

Let $\lambda=k-1$ and

$$\hat{a}(n)=\frac{a(n)}{n^{(\lambda-1)/2}}\quad\text{and}\quad\hat{a}_\psi(n)=\frac{a_\psi(n)}{n^{(\lambda-1)/2}}.$$

Putting these values in (10), we get

$$\sum_{n\geq 1}\hat{a}(n)n^{(\lambda-1)/2}e(n\tau)=\sum_{\psi}\sum_{n\geq 1}\hat{a}_\psi(n)n^{(\lambda-1)/2}e(n\tau).$$

From the above we also have

$$\hat{a}(n)=\sum_{\psi}\hat{a}_\psi(n). \tag{11}$$

Now using the bounds for $\hat{a}_\psi(n)$ from [12, Theorem 3.4, Corollary 3.5] and applying (11) we achieve the following two estimates for $\hat{a}(n)$

$$\hat{a}(n)\ll n^{\epsilon},$$
$$\sum_{n\leq X}\hat{a}(n)\ll X^{1/3+\epsilon}.$$

Also, applying Rankin–Selberg method [19, p. 357, Theorem 1], [20, Eq. 1.14] to $h_{s_0}(|D_K|m_0\tau)$ and following similar steps as we have in [12, Corollary 3.2], we get that

$$\sum_{n\leq X}\hat{a}^2(n) = cX + O(X^{3/5+\epsilon}),$$

where c is a constant depending on $h_{s_0}(|D_K|m_0\tau)$ and ϵ is any real number greater than 0. Now applying [12, Theorem 2.1] we get that $\hat{a}(n)$ changes sign at least once for $n\in(X, X+X^{3/5}]$ for $X\gg 1$. This implies that $a(n)$ and hence $A_F(T)$ where $T=\begin{pmatrix}\frac{n+N(s_0)|D_K|}{|D_K|m_0} & s_0\\ \overline{s_0} & m_0\end{pmatrix}$, change sign atleast once for $|D_K|\det(T)\in(X, X+X^{3/5}]$ for $X\gg 1$.

4 Proof of Theorem 1.2

The ring of integers $\mathcal{O}_K$ of K is $\mathbb{Z}+\omega\mathbb{Z}$, where

$$\omega=\begin{cases}\sqrt{-d} & \text{if } \;-d\equiv 2,3 \pmod 4,\\ \frac{1+\sqrt{-d}}{2} & \text{if } \;-d\equiv 1 \pmod 4.\end{cases}$$

We define the following set

$$\mathcal{J}=\left\{\begin{pmatrix}x & s\\ \bar{s} & y\end{pmatrix} \mid x,y\in\mathbb{Z}, s=\alpha+\omega\beta\in\mathcal{O}_K,\ 0\leq\alpha,\beta,x,y<p\right\}.$$

We first prove the following proposition which will be required to prove Theorem 1.2.

Proposition 4.1 *Let $F=\sum_{T\in\Delta_2^+}A_F(T)e(tr(TZ))\in S_k(SU_2(\mathcal{O}_K))$. For any prime p there exists a $G_p\in S_k(\Gamma_0^{(2)}(p^2))$ such that the Fourier coefficients of G_p is given by*

$$G_p=\sum_{\substack{T\in\Delta_2^+\\ p^{-1}T\in\Delta_2^+}}A_F(T)e(tr(TZ)).$$

Proof Let

$$G:=\frac{1}{p^4}\sum_{Y\in\mathcal{J}}F\mid_k\begin{pmatrix}\boldsymbol{I}_2 & p^{-1}Y\\ \boldsymbol{0}_2 & \boldsymbol{I}_2\end{pmatrix}.$$

We claim that $G\in S_k(\Gamma_0^{(2)}(p^2))$. It is enough to show that for any $Y\in\mathcal{J}$, we have $G'=F\mid_k\begin{pmatrix}\boldsymbol{I}_2 & p^{-1}Y\\ \boldsymbol{0}_2 & \boldsymbol{I}_2\end{pmatrix}\in S_k(\Gamma_0^{(2)}(p^2))$. Let $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_0^{(2)}(p^2)$. It is easy to check that

$$\begin{pmatrix} \boldsymbol{I}_2 & p^{-1}Y \\ \boldsymbol{0}_2 & \boldsymbol{I}_2 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{I}_2 & p^{-1}Y \\ \boldsymbol{0}_2 & \boldsymbol{I}_2 \end{pmatrix}^{-1} \in \Gamma_0^{(2)}(p^2).$$

This implies that $G' \mid_k M = G'$, which asserts our claim. Now the Fourier series expansion of G is given by

$$\begin{aligned} G(Z) &= \frac{1}{p^4} \sum_{Y \in \mathcal{H}} F(Z + p^{-1}Y) \\ &= \frac{1}{p^4} \sum_{Y \in \mathcal{J}} \sum_{T \in \Delta_2^+} A_F(T) e(tr(TZ + Tp^{-1}Y)) \\ &= \frac{1}{p^4} \sum_{T \in \Delta_2^+} A_F(T) e(tr(TZ)) \sum_{Y \in \mathcal{J}} e(tr(p^{-1}TY)). \end{aligned}$$

Now for any $T \in \Delta_2^+$, we have

$$\sum_{Y \in \mathcal{J}} e(tr(p^{-1}TY)) = \begin{cases} p^4 & \text{if } p^{-1}T \in \Delta_2^+, \\ 0 & \text{Otherwise.} \end{cases}$$

Therefore, the Fourier series expansion of G is given by

$$G = \sum_{\substack{T \in \Delta_2^+ \\ p^{-1}T \in \Delta_2^+}} A_F(T) e(tr(TZ)).$$

Thus, we get the required G_p. □

4.1 Proof of Theorem 1.2

Since $F \neq 0$, by Lemma 2.4 there exists a primitive $T_0 = \begin{pmatrix} n_0 & r_0 \\ \bar{r}_0 & p \end{pmatrix} \in \Delta_2^+$ for some odd prime p such that $A_F(T_0) \neq 0$. Applying Proposition 4.1, we construct G_p from F such that $G_p \in S_k(\Gamma_0^{(2)}(p^2))$ and the Fourier series expansion of G_p is given by

$$G_p = \sum_{\substack{T \in \Delta_2^+ \\ p^{-1}T \in \Delta_2^+}} A_F(T) e(tr(TZ)).$$

Let $H = F - G_p$. We observe that $H \in S_k(\Gamma_0^{(2)}(p^2))$ and the Fourier series expansion of H is given by

$$H = \sum_{\substack{T \in \Delta_2^+ \\ p^{-1}T \notin \Delta_2^+}} A_F(T) e(tr(TZ)).$$

Since T_0 is primitive $H \neq 0$. We consider the Fourier–Jacobi coefficient ϕ_p in the Fourier–Jacobi expansion of H whose Fourier series expansion is given by

$$\phi_p(\tau, z_1, z_2) = \sum_{\substack{T=\begin{pmatrix} n & r \\ \overline{r} & p \end{pmatrix} \in \Delta_2^+ \\ p^{-1}T \notin \Delta_2^+}} A_F(T) e(n\tau + rz_1 + \overline{r}z_2).$$

We have $\phi_p \in J_{k,p}(\Gamma_0^{(1)}(p^2))$. Let $s_0 \in \mathcal{O}_K^{\#}/p\mathcal{O}_K$ be such that $s_0 \equiv r_0 \pmod{p\mathcal{O}_K}$. We consider the theta component $h_{s_0} \neq 0$ in the theta decomposition of ϕ_p. The Fourier series expansion of h_{s_0} is given by

$$h_{s_0}(\tau) = \sum_{\substack{n \geq 1 \\ n \equiv -N(s_0)|D_K| \pmod{p|D_K|\mathbb{Z}}}} a(n) e\left(\frac{n\tau}{|D_K|p}\right),$$

where

$$a(n) = \begin{cases} A_F(T) & \text{if} \quad T = \begin{pmatrix} \frac{n+N(s_0)|D_K|}{|D_K|p} & s_0 \\ \overline{s_0} & p \end{pmatrix} \in \Delta_2^+ \text{and } p^{-1}T \notin \Delta_2^+, \\ 0 & \text{Otherwise.} \end{cases}$$

We have $h_{s_0} \in S_{k-1}(\Gamma^{(1)}(|D_K|p^3))$. Now doing the similar calculation as we have done in the proof of Theorem 1.1, we get that $A_F(T)$ where $T = \begin{pmatrix} \frac{n+N(s_0)|D_K|}{|D_K|p} & s_0 \\ \overline{s_0} & p \end{pmatrix}$, $p^{-1}T \notin \Delta_2^+$, changes sign atleast once for $|D_K|\det(T) \in (X, X + X^{3/5}]$ for $X \gg 1$.

5 Sturm bound

Sturm [21] proved that an elliptic modular form is determined by its first few Fourier series coefficients. The number of these first few Fourier coefficients is known as Sturm bound. Sturm's result has had a significant impact on the study of elliptic modular forms. In this section we first develop a Sturm bound for Jacobi form with matrix index. Following this we establish a relation between Jacobi form with matrix index and Hermitian Jacobi forms. We use this relation to derive a Sturm bound for Hermitian

Jacobi forms. Finally we prove Theorem 1.3 using the Fourier–Jacobi expansion of Hermitian modular form and Sturm bound of Hermitian Jacobi forms..

5.1 Sturm bound for Jacobi form with matrix index

Let

$$\phi = \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^l \\ 4\det(M)n - M^{\#}[r^t]\geq 0}} a(n,r)q^n\zeta^r \in J_{k,M}(\Gamma_0^{(1)}(N)).$$

Let

$$M = \begin{pmatrix} \alpha_{11} & \alpha_{12}/2 & \cdots & \alpha_{1\ell}/2 \\ \alpha_{12}/2 & \alpha_{22} & \cdots & \alpha_{2\ell}/2 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{1\ell}/2 & \cdots & \cdots & \alpha_{\ell\ell} \end{pmatrix},$$

where $\alpha_{ij} \in \mathbb{Z}$ for $i < j$ and $\alpha_{ii} \geq 0$. We consider the Taylor series expansion of ϕ at $z_1 = z_2 = \cdots = z_\ell = 0$, with Taylor coefficients $X_{v_1,\cdots,v_\ell}(\tau)$,

$$\phi = \sum_{v_1,\cdots,v_\ell\geq 0} X_{v_1,\cdots,v_\ell}(\tau) z_1^{v_1}\cdots z_\ell^{v_\ell}. \tag{12}$$

For each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(1)}(N)$, using the transformation property (7) of ϕ and above equation we get

$$\begin{aligned}
&\sum_{v_1,\cdots,v_\ell\geq 0} X_{v_1,\cdots,v_\ell}\left(\frac{a\tau+b}{c\tau+d}\right) z_1^{v_1}\cdots z_\ell^{v_\ell} \\
&= (c\tau+d)^{k+v_1+\cdots+v_\ell}\, e\left(\frac{\sum_{1\leq i\leq j\leq \ell} c\alpha_{ij}z_iz_j}{c\tau+d}\right) \sum_{v_1,\cdots v_\ell\geq 0} X_{v_1,\cdots,v_\ell}(\tau) z_1^{v_1}\cdots z_\ell^{v_\ell} \\
&= (c\tau+d)^{k+v_1+\cdots+v_\ell}\left(\sum_{1\leq i\leq j\leq \ell}\ \sum_{t_{ij}\geq 0} \frac{1}{t_{ij}!}\left(\frac{2\pi i\alpha_{ij}c}{c\tau+d}\right)^{t_{ij}} (z_iz_j)^{t_{ij}}\right) \\
&\times \sum_{v_1,\cdots,v_\ell\geq 0} X_{v_1,\cdots,v_\ell}(\tau) z_1^{v_1}\cdots z_\ell^{v_\ell}.
\end{aligned}$$

This implies that

$$X_{v_1,\cdots,v_\ell}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+v_1+\cdots+v_\ell}$$

$$\times \sum_{\substack{1\le i\le j\le \ell,\\ t_{ij}\ge 0,\\ v_j-2t_{jj}-\sum_{s=1}^{j-1}t_{sj}-\sum_{s=j+1}^{\ell}t_{js}\ge 0}} \left(\frac{2\pi ic}{c\tau+d}\right)^{\sum_{1\le i\le j\le \ell}t_{ij}} \frac{\prod\limits_{1\le i\le j\le \ell}\alpha_{ij}^{t_{ij}}}{\prod\limits_{1\le i\le j\le \ell}t_{ij}!}$$
$$\times\ X_{v_1-2t_{11}-\sum_{s=1}^{\ell}t_{1s},\cdots,v_\ell-\sum_{s=1}^{\ell-1}t_{s\ell}-t_{\ell\ell}}(\tau).$$

Following Eichler and Zagier [6, p. 31], we define

$$\zeta_{v_1,\cdots,v_\ell}(\tau)=\sum_{\substack{1\le i\le j\le \ell,\ t_{ij}\ge 0\\ v_j-2t_{jj}-\sum_{s=1}^{j-1}t_{sj}-\sum_{s=j+1}^{\ell}t_{js}\ge 0}} (-2\pi i)^{(\sum_{1\le i\le j\le \ell}t_{ij})}$$
$$\times\ \frac{(k+\sum_{i=1}^{\ell}v_i-\sum_{1\le i\le j\le \ell}t_{ij}-2)!}{(k+\sum_{i=1}^{\ell}v_i-2)!}\frac{\prod\limits_{1\le i\le j\le \ell}\alpha_{ij}^{t_{ij}}}{\prod\limits_{1\le i\le j\le \ell}t_{ij}!}$$
$$\times\ X^{(\sum_{1\le i\le j\le \ell}t_{ij})}_{(v_1-2t_{11}-\sum_{j=2}^{\ell}t_{1j},\cdots,v_\ell-\sum_{i=1}^{\ell-1}t_{i\ell}-2t_{\ell\ell})}(\tau),$$

where $g^{(\nu)}(\tau)=\left(\frac{\partial}{\partial\tau}\right)^{\nu}g(\tau)$. It can be readily checked that $\zeta_{v_1,\cdots,v_\ell}(\tau)\in M_{k+v_1+\cdots+v_\ell}(\Gamma_0^{(1)}(N))$. The Fourier expansion of $\zeta_{v_1,\cdots,v_\ell}(\tau)$ is given by

$$\zeta_{v_1,\cdots,v_\ell}(\tau)=(2\pi i)^{v_1+\cdots+v_\ell}\sum_{n\ge 0}\left(\sum_{\substack{r=(r_1,\cdots,r_\ell)\in\mathbb{Z}^\ell\\ 4\det(M)n-M^{\#}[r^t]\ge 0}}\kappa\ a(n,r)\right)q^n,$$

where

$$\kappa=\sum_{\substack{1\le i\le j\le \ell,\\ t_{ij}\ge 0,\\ v_j-2t_{jj}-\sum_{s=1}^{j-1}t_{sj}-\sum_{s=j+1}^{\ell}t_{js}\ge 0}}\frac{(k+\sum_{i=1}^{\ell}v_i-\sum_{1\le i\le j\le \ell}t_{ij}-2)!}{(k+\sum_{i=1}^{\ell}v_i-2)!}$$
$$\times\frac{\prod_{1\le i\le j\le \ell}(-n\alpha)^{t_{ij}}\prod_{w=1}^{\ell}r_w^{v_w-\sum_{s=1}^{w-1}t_{sw}-\sum_{s=w+1}^{\ell}t_{ws}-2t_{ww}}}{\prod_{1\le i\le j\le \ell}t_{ij}!\ \prod_{w=1}^{\ell}\left(v_w-\sum_{s=1}^{w-1}t_{sw}-\sum_{s=w+1}^{\ell}t_{ws}-2t_{ww}\right)!}.$$

We further define

$$D_{v_1,\cdots,v_\ell}(\phi)=(2\pi i)^{-(v_1+\cdots+v_\ell)}\ \frac{(k+\sum_{i=1}^{n}v_i-2)!(\sum_{i=1}^{\ell}v_i)!}{(k+\beta-2)!}\ \zeta_{v_1,\cdots,v_\ell}(\tau),$$

where we take $\beta=\frac{\sum_{i=1}^{\ell}v_i}{2}$, if $\sum_{i=1}^{\ell}v_i$ is even and $\beta=\frac{1+\sum_{i=1}^{\ell}v_i}{2}$, if $\sum_{i=1}^{\ell}v_i$ is odd.

Proposition 5.1 *We define*

$$D : J_{k,M}(\Gamma_0^{(1)}(N)) \to \bigoplus_{\substack{(v_1,\cdots,v_\ell)\\ 0\le v_i\le 2\alpha_{ii}}} M_{k+v_1+\cdots+v_\ell}\,(\Gamma_0^{(1)}(N)),$$

where the map from $J_{k,M}(\Gamma_0^{(1)}(N))$ to $M_{k+v_1,\cdots,v_\ell}(\Gamma_0^{(1)}(N))$ is given by

$$\phi \mapsto D_{v_1,\cdots,v_\ell}(\phi).$$

Then the linear map D is injective.

Proof We will show that if $\phi \ne 0$ then $D(\phi) \ne 0$. Let us choose $a_\ell, a_{\ell-1}, \cdots, a_1$ in a minimal way such that Taylor coefficient $X_{a_1,\cdots,a_\ell}(\tau)$ of ϕ in (12) is non-zero and for all τ

$$X_{v_1,\cdots,v_\ell}(\tau) = 0 \quad (0 \le \forall v_\ell < a_\ell; \forall v_{\ell-1},\cdots,v_1 \ge 0),$$
$$X_{v_1,\cdots,v_{\ell-1},a_\ell}(\tau) = 0 \quad (0 \le \forall v_{\ell-1} < a_{\ell-1}; \forall v_{\ell-2},\cdots,v_1 \ge 0),\cdots,$$
$$X_{v_1,a_2\cdots,a_\ell}(\tau) = 0 \quad (0 \le \forall v_1 < a_1).$$

We claim that $a_i \le 2\alpha_{ii}$ for all $1 \le i \le \ell$. By [3, Lemma 3.1], we know that the function

$$f_1(\tau, z_1) = \sum_{v_1\ge 0} X_{v_1,a_2,\cdots,a_\ell}(\tau) z_1^{v_1}$$

is a non-zero classical Jacobi cusp form of weight $k + a_2 \cdots + a_\ell$ and index α_{11}. Therefore, by Eichler and Zagier result [6, Theorem 1.2, p. 10] we have $a_1 \le 2\alpha_{11}$. Now suppose $2 \le i \le \ell$. We choose $b_\ell, \cdots, b_{i-1}, b_{i+1}, \cdots, b_1, b_i$ in a minimal way such that $X_{b_1,b_2,\cdots,b_\ell}(\tau) \ne 0$ and for all τ

$$X_{v_1,\cdots,v_\ell}(\tau) = 0 \quad (0 \le \forall v_\ell < a_\ell; \forall v_{\ell-1},\cdots,v_1, v_i \ge 0),\cdots$$
$$X_{v_1,b_2,\cdots,b_{i-1},v_i,b_{i+1},\cdots,b_\ell}(\tau) = 0 \quad (0 \le \forall v_1 < b_1; \forall v_i \ge 0),$$
$$X_{b_1,\cdots,b_{i-1},v_i,b_{i+1}\cdots,b_\ell}(\tau) = 0 \quad (0 \le \forall v_i < b_i).$$

Again using minimality condition of $b_\ell, \cdots, b_{i-1}, b_{i+1}, \cdots, b_1, b_i$, we see that

$$f_i(\tau, z_i) = \sum_{v_i\ge 0} X_{b_1,\cdots,v_i,\cdots,b_\ell}(\tau) z_i^{v_i}$$

is a non-zero classical Jacobi form of weight $k + b_1 + \cdots + b_{i-1} + b_{i+1} + \cdots + b_\ell$ and index α_{ii}. Therefore, $b_i \le 2\alpha_{ii}$. Since both $a_\ell, a_{\ell-1}, \cdots, a_1$ and $b_\ell, \cdots, b_{i-1}, b_{i+1}, \cdots, b_1, b_i$ are minimal we must have

$$a_\ell = b_\ell, \cdots, a_{i-1} = b_{i-1}, \text{ and, } a_i \le b_i.$$

This implies that $a_i \le 2\alpha_{ii}$ for all $1 \le i \le n$. So we have proved that if $\phi \ne 0$ then there exists a Taylor coefficient $X_{a_1,\cdots,a_\ell}(\tau) \ne 0$ such that $a_i \le 2\alpha_{ii}$ for all $1 \le i \le \ell$. Again using the minimality of $a_\ell, a_{\ell-1}, \cdots, a_1$, we see that $D_{a_1,\cdots,a_\ell}(\phi) = \alpha X_{a_1,\cdots,a_\ell}(\tau)$ for some non-zero $\alpha \in \mathbb{C}$. Thus, $D(\phi) \ne 0$. □

In the following theorem we establish a Sturm bound for Jacobi form with matrix index.

Theorem 5.2 *Let* $\gamma = [SL_2(\mathbb{Z}) : \Gamma_0^{(1)}(N)]$. *Let*

$$\phi = \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^l \\ 4\det(M)n - M^{\#}[r^t] \ge 0}} a(n,r) q^n \zeta^r \in J_{k,M}(\Gamma_0^{(1)}(N)).$$

If $a(n,r) = 0$, *for all*

$$n \le \frac{1}{12}(k + 2tr(M))\gamma,$$

then $\phi = 0$.

Proof If $a(n,r) = 0$ for all $n \le \frac{1}{12}(k + 2tr(M))\gamma$ then using the Fourier expansion of $D_{v_1,\cdots,v_\ell}(\phi)$ and Sturm result for elliptic modular forms we get that

$$D_{v_1,\cdots,v_\ell}(\phi) = 0$$

for all $(v_1, \cdots, v_\ell)$ satisfying $0 \le v_i \le 2\alpha_{ii}$ for all $1 \le i \le \ell$. This implies that $D(\phi) = 0$ and hence $\phi = 0$ as D is injective. □

5.2 Relation between Hermitian Jacobi and Jacobi form with matrix index

In [17, Theorem 2.3], Meher and the second author proved a relation between Hermitian Jacobi form over $\mathbb{Q}(i)$ and Jacobi form with matrix index. In the following theorem, we generalize their result for an arbitrary imaginary quadratic field.

Theorem 5.3 *Let* $K = \mathbb{Q}(\sqrt{-d})$ *be an imaginary quadratic field. Suppose*

$$A = \begin{cases} \begin{pmatrix} m & 0 \\ 0 & md \end{pmatrix} & \text{if } -d \equiv 2, 3 \pmod 4, \\[2ex] \begin{pmatrix} m & \frac{m}{2} \\ \frac{m}{2} & m\left(\frac{1+d}{4}\right) \end{pmatrix} & \text{if } -d \equiv 1 \pmod 4. \end{cases}$$

Then the space $J_{k,m}(\Gamma_0^{(1)}(N))$ *is isomorphic to* $J_{k,A}(\Gamma_0^{(1)}(N))$ *as a vector space over* $\mathbb{C}$.

Proof First let us consider the case $-d \equiv 2, 3 \pmod 4$. We define a map

$$\eta_1 : J_{k,m}(\Gamma_0^{(1)}(N)) \rightarrow J_{k,A}(\Gamma_0^{(1)}(N))$$

by

$$\phi(\tau, z_1, z_2) \mapsto \phi\left(\tau, z_1 + i\sqrt{d}z_2, z_1 - i\sqrt{d}z_2\right).$$

Let $\hat{\phi}(\tau, z_1, z_2) = \phi\left(\tau, z_1 + i\sqrt{d}z_2, z_1 - i\sqrt{d}z_2\right)$. Using the transformation property of ϕ mentioned in (4) and (5) we can verify that $\hat{\phi}$ satisfies (7) and (8). Suppose ϕ has Fourier series expansion

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}_K^{\#} \\ nm-N(r)\geq 0}} c_\phi(n, r) q^n \zeta_1^r \zeta_2^{\bar{r}},$$

then

$$\hat{\phi}(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}_K^{\#} \\ nm-N(r)\geq 0}} c_\phi(n, r) e\left(n\tau + r\left(z_1 + i\sqrt{d}z_2\right) + \bar{r}\left(z_1 - i\sqrt{d}z_2\right)\right).$$

Any $r \in \mathcal{O}_K^{\#}$ can be written as $r = \frac{i}{2\sqrt{d}}\left(\alpha + i\sqrt{d}\beta\right)$, where $\alpha, \beta \in \mathbb{Z}$. We now consider an element $\rho \in \mathbb{Z}^2$, where $\rho = (-\beta, -\alpha)$. Then the correspondence $r \mapsto \rho$ is clearly a bijection from $\mathcal{O}_K^{\#}$ to $\mathbb{Z}^2$. Now the above Fourier series expansion of $\hat{\phi}$ can be expressed as

$$\begin{aligned}\hat{\phi}(\tau, z_1, z_2) &= \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}_K^{\#} \\ nm-N(r)\geq 0}} c_\phi(n, r) e(n\tau - \beta z_1 - \alpha z_2) \\ &= \sum_{\substack{n\in\mathbb{Z},\ \rho\in\mathbb{Z}^2 \\ 4\det(A)n-A^{\#}[\rho]\geq 0}} c_{\hat{\phi}}(n, \rho) e(n\tau + (-\beta)z_1 + (-\alpha)z_2),\end{aligned}$$

which is of the form given in (9). This implies that η_1 is a well defined linear map. In a similar manner, one can show that the map

$$\eta_2 : J_{k,A}(\Gamma_0^{(1)}(N)) \rightarrow J_{k,m}(\Gamma_0^{(1)}(N))$$

defined by

$$\psi(\tau, z_1, z_2) \mapsto \psi\left(\tau, \frac{z_1 + z_2}{2}, \frac{z_1 - z_2}{2i\sqrt{d}}\right)$$

is also a well-defined linear map. Now consider the composition map $\eta_2 \circ \eta_1$,

$$\begin{aligned}(\eta_2 \circ \eta_1)(\phi(\tau, z_1, z_2)) &= \eta_2\left(\phi\left(\tau, z_1 + i\sqrt{d}z_2, z_1 - i\sqrt{d}z_2\right)\right)\\ &= \phi\left(\tau, \frac{z_1 + i\sqrt{d}z_2 + z_1 - i\sqrt{d}z_2}{2}, \frac{z_1 + i\sqrt{d}z_2 - z_1 + i\sqrt{d}z_2}{2i\sqrt{d}}\right)\\ &= \phi(\tau, z_1, z_2).\end{aligned}$$

Similarly we can also check that $(\eta_1 \circ \eta_2)(\psi(\tau, z_1, z_2)) = \psi(\tau, z_1, z_2)$ and hence $\eta_2 \circ \eta_1 = I_1$, $\eta_1 \circ \eta_2 = I_2$, where I_1 and I_2 are identity maps on the vector spaces $J_{k,m}(\Gamma(\mathcal{O}_K))$ and $J_{k,A}(\Gamma^2)$ respectively.
Now consider the case $-d \equiv 1 \pmod 4$. Here we define the maps

$$\eta_1 : J_{k.m}(\Gamma_0^{(1)}(N)) \to J_{k,A}(\Gamma_0^{(1)}(N))$$

by

$$\phi(\tau, z_1, z_2) \mapsto \phi\left(\tau, \frac{2z_1 + z_2 + i\sqrt{d}z_2}{2}, \frac{2z_1 + z_2 - i\sqrt{d}z_2}{2}\right)$$

and

$$\eta_2 : J_{k,A}(\Gamma_0^{(1)}(N)) \to J_{k,m}(\Gamma_0^{(1)}(N))$$

by

$$\psi(\tau, z_1, z_2) \mapsto \psi\left(\tau, \frac{z_1 + z_2}{2} - \frac{z_1 - z_2}{2i\sqrt{d}}, \frac{z_1 - z_2}{i\sqrt{d}}\right).$$

Approaching as above we can easily verify that η_1 and η_2 are well defined linear maps and also they satisfy

$$\eta_2 \circ \eta_1 = I_1, \quad \eta_1 \circ \eta_2 = I_2.$$

□

5.3 Sturm bound for Hermitian Jacobi forms

Using Theorems 5.2 and 5.3, we establish a Sturm bound for Hermitian Jacobi forms.

Theorem 5.4 *Let* $\gamma = [SL_2(\mathbb{Z}) : \Gamma_0^{(1)}(N)]$. *Suppose* $K = \mathbb{Q}(\sqrt{-d})$ *and*

$$\phi = \sum_{\substack{n \geq 0, r \in \mathcal{O}_K \\ 4nm - N(r) \geq 0}} a_\phi(n, r) q^n \zeta_1^r \zeta_2^{\overline{r}} \in J_{k,m}(\Gamma_0^{(1)}(N)).$$

If $a_\phi(n,r)=0$ *for all* $n\le\beta$*, where*

$$\beta=\begin{cases}\frac{1}{12}(k+2m(1+d))\gamma & \text{if } -d\equiv 2,3 \pmod 4,\\ \frac{1}{12}\left(k+\frac{m(5+d)}{2}\right)\gamma & \text{if } -d\equiv 1 \pmod 4,\end{cases}$$

then $\phi=0$.

Proof We begin with the case $-d\equiv 2,3 \pmod 4$. The Fourier series expansion of $\hat{\phi}(\tau,z_1,z_2)$ in Theorem 5.3 is given by

$$\hat{\phi}(\tau,z_1,z_2)=\sum_{\substack{n\in\mathbb{Z},r\in\mathcal{O}_K^{\#}\\ nm-N(r)\ge 0}} a_\phi(n,r)e(n\tau-\beta z_1-\alpha z_2)$$

$$=\sum_{\substack{n\in\mathbb{Z},\ \rho\in\mathbb{Z}^2\\ 4\det(A)n-A^{\#}[\rho^t]\ge 0}} a_{\hat{\phi}}(n,\rho)e(n\tau+(-\beta)z_1+(-\alpha)z_2),$$

where $r=\frac{i}{2\sqrt{d}}\left(\alpha+i\sqrt{d}\beta\right)$, $\rho=(-\beta,-\alpha)$, $\alpha,\beta\in\mathbb{Z}$. Now, if $a_\phi(n,r)=0$ for all $n\le\beta$, then by Theorem 5.2, we see that $\hat{\phi}=0$. Since η is an isomorphism, we get $\phi=0$. This completes the proof when $-d\equiv 2,3 \pmod 4$. The case of $-d\equiv 1 \pmod 4$ follows similarly. □

If we put $N=1$ in the above we get the following.

Corollary 5.5 *Let* $K=\mathbb{Q}(\sqrt{-d})$*, where* $d>0$ *be square free. Let*

$$\phi=\sum_{\substack{n\ge 0,r\in\mathcal{O}_K\\ 4nm-N(r)\ge 0}} a(n,r)q^n\zeta_1^r\zeta_2^{\overline{r}}\in J_{k,m}(SL_2(\mathbb{Z})).$$

If $a(n,r)=0$ *for all* $n\le\beta$*, where*

$$\beta=\begin{cases}\frac{1}{12}(k+2m(1+d)) & \text{if } -d\equiv 2,3 \pmod 4,\\ \frac{1}{12}\left(k+\frac{m(5+d)}{2}\right) & \text{if } -d\equiv 1 \pmod 4,\end{cases}$$

then $\phi=0$.

We are now ready to prove Theorem 1.3. We first define some necessary terms. For

$$\phi=\sum_{\substack{n\ge 0,r\in\mathcal{O}_K\\ 4nm-N(r)\ge 0}} a(n,r)q^n\zeta_1^r\zeta_2^{\overline{r}}\in J_{k,m}(SL_2(\mathbb{Z})),$$

we define

$$\operatorname{ord}(\phi)=\min\{n\mid a(n,r)\neq 0\}.$$

From Corollary 5.5, Sturm bound for Hermtian Jacobi form when $-d \equiv 2, 3 \pmod 4$ is

$$\beta = \frac{1}{12}(k + 2m(1+d)).$$

We put $\beta = m = t$ in the above and see that it is possible to get a positive value of $t = \frac{k}{2(5-d)}$ if $d = 1, 2$. Similarly, when $-d \equiv 1 \pmod 4$, we will get a positive value of $t = \frac{2k}{19-d}$, if $d \in \{3, 7, 11, 15\}$. We will use the Fourier–Jacobi expansion and a transformation of Hermitian modular form F to show that if the Fourier coefficients $A_F(n, r, m) = 0$ for all $n \le t$ and $m \le t$ then $F = 0$.

5.4 Proof of Theorem 1.3

We will prove the result when $d \in \{1, 2\}$. The case $d \in \{3, 7, 11, 15\}$ will follow similarly. We consider the Fourier–Jacobi expansion of F

$$F = \sum_{m \ge 0} \phi_m(\tau, z_1, z_2) e(m\tau').$$

We will show that $\phi_m = 0$ for all $m \ge 0$. We first consider that $m \le \frac{k}{2(5-d)}$. Then

$$\mathrm{ord}(\phi_m) > \frac{k}{2(5-d)} = \frac{k}{2(5-d)} \left(\frac{1+d}{6} + \frac{5-d}{6} \right) = \frac{k}{12} + \frac{m(1+d)}{6}.$$

Therefore, by Corollary 5.5, we have $\phi_m = 0$. Assume that $m > \frac{k}{2(5-d)}$. We use induction on m to show that $\phi_m = 0$. Suppose that $\phi_{m'} = 0$ for all $m' < m$. Now we consider ϕ_m. For $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the matrix $M = \begin{pmatrix} (\overline{g}^t)^{-1} & \mathbf{0}_2 \\ \mathbf{0}_2 & g \end{pmatrix} \in SU_2(\mathcal{O}_K)$. Using the transformation property (2) of F for M, we get $F(\tau, z_1, z_2, \tau') = (-1)^k F(\tau', z_2, z_1, \tau)$, which shows that $A_F(n, r, m) = (-1)^k A_F(m, \overline{r}, n) = 0$ for all $n < m$. Therefore,

$$\mathrm{ord}(\phi_m) \ge m = m \left(\frac{1+d}{6} + \frac{5-d}{6} \right) > \frac{m(1+d)}{6} + \frac{k}{12}.$$

Again by Corollary 5.5, we have $\phi_m = 0$. This completes the proof.

Remark 5.6 (1) The space of cusp form $S_k(SU_2(\mathcal{O}_K)) = \{0\}$ if $k < 2(5-d)$ when $d = 1, 2$ and if $k < \frac{19-d}{2}$ when $d = 3, 7, 11, 15$.

(2) The bound in Theorem 1.3 is sharp for $K = \mathbb{Q}(i), \mathbb{Q}(\sqrt{2}i)$. We have explained this in the next Example 5.7.

Example 5.7 The Hermitian Eisenstien series of even weight $k > 4$ over a field $K = \mathbb{Q}(\sqrt{-d})$ is given by

$$E_k^{(K)} = \sum_{M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \backslash U_2(\mathcal{O}_K)} \det(M)^{-k/2} \det(CZ+D)^{-k}, \quad Z \in \mathcal{H}_2.$$

Moreover, Krieg [15] constructed weight 4 Eisenstein series by Maass lift. The Eisenstein series $E_k^{(K)}$ has rational Fourier coefficients for $k \geq 4$ [7, 8, 15]. There are Hermitian cusp forms over any imaginary quadratic field [4, Corollary 2], [8],

$$\begin{aligned} F_{10}^{(K)} &= E_{10}^{(K)} - E_4^{(K)} E_6^{(K)} \in S_{10}(SU_2(\mathcal{O}_K)), \\ F_{12}^{(K)} &= E_{12}^{(K)} - \frac{441}{691}\left(E_4^{(K)}\right)^3 - \frac{250}{691}\left(E_6^{(K)}\right)^2 \in S_{12}(SU_2(\mathcal{O}_K)). \end{aligned}$$

If $K = \mathbb{Q}(i)$ then

$$\chi_8 = -\frac{61}{230400}\left(E_8^{(K)} - \left(E_4^{(K)}\right)^2\right) \in S_8(SU_2(\mathcal{O}_K)).$$

Let $\beta(k) = [k/8]$ be the Sturm bound for $K = \mathbb{Q}(i)$ in Theorem 1.3. We define

$$H_k(Z) = \begin{cases} \left(E_4^{(K)}\right)^i \left(E_6^{(K)}\right)^j \chi_8^{\beta(k)} \ (i+j+\beta(k)=k,\ i,j=0,1) & \text{if } k \not\equiv 2 \pmod 8, \\ \chi_8^{\beta(k)-1} F_{10}^{(K)} & \text{if } k \equiv 2 \pmod 8. \end{cases}$$

Since χ_8 and $F_{10}^{(K)}$ are cusp forms, we have $A_{\chi_8}(n,r,m) = A_{F_{10}^{(K)}}(n,r,m) = 0$ whenever $n = 0$ or $m = 0$. We have

$$A_{\chi_8}\left(\begin{pmatrix} 1 & (1+i)/2 \\ (1-i)/2 & 1 \end{pmatrix}\right) = A_{F_{10}^{(K)}}\left(\begin{pmatrix} 1 & (1+i)/2 \\ (1-i)/2 & 1 \end{pmatrix}\right) = 1.$$

Therefore, we check that $A_{H_k}(n,r,m) = 0$ whenever $n \leq \beta(k)-1$ and $m \leq \beta(k)-1$ but $H_k \neq 0$. Hence the bound is sharp for $K = \mathbb{Q}(i)$. Similarly, one can check that the bound in Theorem 1.3 is sharp for $K = \mathbb{Q}(\sqrt{2}i)$ using $E_4^{(K)}$ and Hermitian cusp form ϕ_6 and ϕ_8 of weight 6 and 8 respectively constructed by Dern and Krieg [5].

6 Proof of Theorem 1.4

We will prove the result when $d \in \{1,2\}$. The case $d \in \{3,7,11,15\}$ will follow similarly. We have assumed that $F \neq 0$, therefore, by Theorem 1.3 there exists $T_0 =$

$\begin{pmatrix} n_0 & r_0 \\ \overline{r}_0 & m_0 \end{pmatrix}$ such that

$$tr(T_0) \leq \frac{k}{(5-d)} \tag{13}$$

and $A_F(T_0) \neq 0$. We consider the Fourier–Jacobi coefficient $\phi_{m_0} \neq 0$ in the Fourier–Jacobi expansion of F. The Fourier series expansion of ϕ_{m_0} is given by

$$\phi_{m_0}(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}_K^{\#} \\ nm_0 - N(r)\geq 0}} c(n,r)e(n\tau + rz_1 + \overline{r}z_2),$$
$$\text{where } c(n,r) = A_F\left(\begin{pmatrix} n & r \\ \overline{r} & m_0 \end{pmatrix}\right).$$

We define

$$\hat{\phi}(\tau, z_1, z_2) = \phi_{m_0}(\tau, z_1 + i\sqrt{d}z_2, z_1 - i\sqrt{d}z_2).$$

Using Theorem 5.3, we get that $\hat{\phi} \in J_{k,M_0}(SL_2(\mathbb{Z}))$, where $M_0 = \begin{pmatrix} m_0 & 0 \\ 0 & m_0 d \end{pmatrix}$. We consider

$$\varphi(\tau, z_1, z_2) = \prod_{\epsilon_1, \epsilon_2} \hat{\phi}(\tau, \epsilon_1 z_1, \epsilon_2 z_2),$$

where $\epsilon_1, \epsilon_2 \in \{1, -1\}$. We can also check that $\hat{\phi}(\tau, \epsilon_1 z_1, \epsilon_2 z_2) \in J_{k,M_0}(SL_2(\mathbb{Z}))$ for every $\epsilon_1, \epsilon_2 \in \{1, -1\}$. The Fourier series expansion of $\hat{\phi}(\tau, \epsilon_1 z_1, \epsilon_2 z_2)$ is given by

$$\hat{\phi}(\tau, \epsilon_1 z_1, \epsilon_2 z_2) = \sum_{\substack{n\in\mathbb{Z}, s=(a,b)\in\mathbb{Z}^2 \\ 4\det(M_0)n - M_0^{\#}[s^t]\geq 0}} c_{\epsilon_1,\epsilon_2}(n,s)e(n\tau + az_1 + bz_2),$$

where

$$c_{\epsilon_1,\epsilon_2}(n,s) = A_F\left(\begin{pmatrix} n & r \\ \overline{r} & m_0 \end{pmatrix}\right), \quad r = \frac{i}{2\sqrt{d}}(-\epsilon_2 b - i\sqrt{d}\epsilon_2 a) \in \mathcal{O}_K^{\#}.$$

Now $\varphi(\tau, z_1, z_2) \in J_{4k,4M_0}(SL_2(\mathbb{Z}))$. Also, by construction, $\varphi(\tau, z_1, z_2)$ is an even function in the variable z_1 and z_2. We consider the Taylor series expansion of $\varphi(\tau, z_1, z_2)$ around $z_1 = z_2 = 0$

$$\varphi(\tau, z_1, z_2) = \sum_{\alpha\geq 0, \beta\geq 0} X_{v_1,v_2}(\tau) z_1^{v_1} z_2^{v_2}.$$

Since $\varphi \neq 0$ there exists a non-zero Taylor coefficient in the above equation. We choose a_2, a_1 in a minimal way such that $X_{a_1,a_2} \neq 0$ and

$$X_{v_1,v_2}(\tau) = 0 \quad (0 \leq \forall v_2 < a_2; \forall v_1 \geq 0),$$
$$X_{v_1,a_2}(\tau) = 0 \quad (0 \leq \forall v_1 < a_1).$$

Then from the proof of Proposition 5.1 we get that $a_1 \leq 8m_0$ and $a_2 \leq 8m_0 d$. Also $D_{a_1,a_2}(\varphi) = \alpha X_{a_1,a_2}(\tau)$ in Proposition 5.1, for some non-zero $\alpha \in \mathbb{C}$. This implies that $X_{a_1,a_2}(\tau)$ is a non-zero elliptic modular form of weight $k_1 = 4k + a_1 + a_2$. Therefore, we have

$$k_1 \leq 4(k + 2m_0(1 + d)). \tag{14}$$

Suppose the Fourier series expansion of φ is given by

$$\varphi(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, s=(\alpha,\beta)\in\mathbb{Z}^2 \\ 4\det(4M_0)n - 4M_0^{\#}[s^t]\geq 0}} d(n, s)e(n\tau + \alpha z_1 + \beta z_2).$$

Let $f = \frac{a_1!\, a_2!}{(2\pi i)^{a_1+a_2}} X_{a_1,a_2}(\tau)$. It can be easily checked that

$$f = \frac{1}{(2\pi i)^{a_1+a_2}} \left(\partial_{z_1}^{a_1} \partial_{z_2}^{a_2} \varphi(\tau, z_1, z_2)\right)_{z_1=0, z_2=0}.$$

If the Fourier series expansion of f is given by $\sum_{n\geq 1} d(n)e(n\tau)$ then we have

$$d(n) = \sum_{\substack{s=(\alpha,\beta)\in\mathbb{Z}^2 \\ 4\det(4M_0)n - 4M_0^{\#}[s^t]\geq 0}} d(n, s)\alpha^{a_1}\beta^{a_2}.$$

Now by [11, Theorem 2] there exists $n_1 \geq 1$ such that

$$n_1 \ll k_1^{2+\epsilon}, \quad d(n_1) < 0.$$

Therefore, by (14), we have

$$n_1 \ll (4(k + 2m_0(1 + d)))^{2+\epsilon}. \tag{15}$$

We have

$$d(n_1) = \sum_{\substack{s=(\alpha,\beta)\in\mathbb{Z}^2 \\ 4\det(4M_0)n_0 - M_0^{\#}[s^t]\geq 0}} d(n_1, s)\alpha^{a_1}\beta^{a_2} < 0.$$

Since $\varphi(\tau, z_1, z_2)$ is an even function in each variable z_1, z_2, the integers a_1, a_2 are even. Therefore, there exists $s_0 = (\alpha_0, \beta_0)$ such that $d(n_1, s_0) < 0$. Also $d(n_1, s_0)$ is a finite sum of product of Fourier coefficients of the form $A_F\left(\begin{pmatrix} n_{\epsilon_1,\epsilon_2} & * \\ * & m_0 \end{pmatrix}\right)$ with $\sum n_{\epsilon_1,\epsilon_2} = n_1$. Therefore, atleast one of the Fourier coefficient

$$A_F\,(T_1) < 0, \qquad T_1 = \begin{pmatrix} n_{\epsilon_1,\epsilon_2} & * \\ * & m_0 \end{pmatrix}.$$

We have $tr\,(T_1) = n_{\epsilon_1,\epsilon_2} + m_0 < n_1 + tr(T_0)$. Using (15) and (13) we have

$$\begin{aligned} tr(T_1) &\ll (4(k + 2(1+d)tr(T_0)))^{2+\epsilon} + tr(T_0) \\ &\ll \left(4\left(\frac{7+d}{5-d}\right)k\right)^{2+\epsilon}. \end{aligned}$$

Now replacing F by $-F$ and proceeding as above, we get a matrix T_2 such that

$$tr(T_2) \ll \left(4\left(\frac{7+d}{5-d}\right)k\right)^{2+\epsilon}$$

and $-A_F(T_2) < 0$. This proves the result.

Remark 6.1 We note that Theorem 1.1 is true for Hermitian modular forms on the congruence subgroup $\Gamma_0^{(2)}(N)$ of $SU_2(\mathcal{O}_K)$ with Dirichlet character $\chi \pmod N$, where χ acts on $\Gamma_0^{(2)}(N)$ by

$$\chi(M) = \det(D), \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(2)}(N).$$

Moreover, if the conductor of χ is N then Theorem 1.2 also holds.

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References

1. Anamby, P., Das, S.: Distinguishing Hermitian cusp forms of degree 2 by a certain subset of all Fourier coefficients. Publ. Mat. **63**, 307–341 (2019)

2. Choie, Y., Kohnen, W.: The first sign change of Fourier coefficients of cusp forms. Am. J. Math. **131**, 517–543 (2009)
3. Choie, Y., Gun, S., Kohnen, W.: An explicit bound for the first sign change of the Fourier coefficients of a Siegel cusp form. Int. Math. Res. Not. **2014**, 3782–3792 (2015)
4. Dern, T., Krieg, A.: Graded rings of Hermitian modular forms of degree 2. Manuscr. Math. **110**, 251–272 (2003)
5. Dern, T., Krieg, A.: The graded ring of Hermitian modular forms of degree 2 over $\mathbb{Q}(\sqrt{-2})$. J. Number Theory **107**, 241–265 (2004)
6. Eichler, M., Zagier, D.: The theory of Jacobi forms. Progress in mathematics, vol. 55, p. 148. Birkhäuser Boston Inc, Boston (1985)
7. Hauffe-Waschbüsch, A., Krieg, A.: On Hecke theory for Hermitian modular forms. In: Modular forms and related topics in number theory. Springer Proc. Math. Stat., vol. 73–88, p. 340. Springer, Singapore (2020)
8. Hauffe-Waschbüsch, A., Krieg, A., Williams, B.: On Hermitian Eisenstein series of degree 2. arXiv:2205.12492
9. Haverkamp, K.: Hermitesche Jacobiformen. In: Schriftenreihe des Mathematischen Instituts der Universität Münster, 3. Serie., vol. 15. Univ. Münster, Math. Inst., Münster, p. 105 (1995)
10. Haverkamp, K.: Hermitian Jacobi forms. Results Math. **29**, 78–89 (1996)
11. He, X., Zhao, L.: On the first sign change of Fourier coefficients of cusp forms. J. Number Theory **190**, 212–228 (2018)
12. Hulse, T.A., Kuan, C.I., Lowry-Duda, D., Walker, A.: Sign changes of coefficients and sums of coefficients of L-functions. J. Number Theory **177**, 112–135 (2017)
13. Iwaniec, H., Kohnen, W., Sengupta, J.: The first negative Hecke eigenvalue. Int. J. Number Theory **3**, 355–363 (2007)
14. Knopp, M., Kohnen, W., Pribitkin, W.A.: On the signs of Fourier coefficients of cusp forms. Ramanujan J. **7**, 269–277 (2003)
15. Krieg, A.: The Maass spaces on the Hermitian half-space of degree 2. Math. Ann. **289**, 663–681 (1991)
16. Matomaki, K., Radziwill, M.: Sign changes of Hecke eigenvalues. Geom. Funct. Anal. **25**, 1937–1955 (2019)
17. Meher, J., Singh, S.K.: Congruences in Hermitian Jacobi and Hermitian modular forms. Forum Math. **32**, 501–523 (2020)
18. Murty, M.R.: Oscillation of Fourier coefficients of modular forms. Math. Ann. **262**, 431–446 (1983)
19. Rankin, R.A.: Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetic functions, II. The order of the Fourier coefficients of integral modular forms. Proc. Camb. Philos. Soc. **35**, 357–372 (1939)
20. Selberg, A.: On the estimation of Fourier coefficients of modular forms. In: Proc. Sympos. Pure Math., vol. VIII, pp. 1–15. Amer. Math. Soc., Providence, R.I. (1965)
21. Sturm, J.: On the congruence of modular forms. In: Number theory (New York, 1984–1985). Lecture Notes in Math, pp. 275–280. Springer, Berlin (1987)
22. Yamana, S.: Determination of holomorphic modular forms by primitive Fourier coefficients. Math. Ann. **344**, 853–862 (2009)