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On exceptional sets in the Waring–Goldbach problem for fifth powers

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Abstract
In this paper, we consider exceptional sets in the Waring–Goldbach problem for fifth powers. We obtain new estimates of $E_s(N)(12 \leq s \leq 20)$, which denote the number of integers $n \leq N$ such that $n \equiv s (\text{mod } 2)$ and n cannot be represented as the sum of s fifth powers of primes. For example, we prove that $E_{20}(N) \ll N^{1-\frac{1}{4}-\frac{27}{1600}+\epsilon}$ for any $\epsilon > 0$. This improves upon the result of Feng and Liu (Front Math China 16:49–58, 2021).

Keywords Waring–Goldbach problem · Exceptional sets · Circle method · Sieve method

Mathematics Subject Classification 11P32 · 11P55 · 11P05

1 Introduction

In 1937, Vinogradov [10] found a new method for estimating sums over primes, thus he proved that every sufficiently large odd integer can be represented as the sum of three prime numbers which is known as the three prime theorem. Vinogradov's proof provided a blueprint for the subsequent applications of the circle method to additive prime number theory. Shortly after that, Vinogradov [11], and Hua [3] turned to study Waring's problem with prime variables which is known as the Waring–Goldbach problem.

We focus on the Waring–Goldbach problem for fifth powers. In this topic, Kawada and Wooley [4] proved that all sufficiently large odd integer can be represented as the sums of 21 fifth powers of primes. We consider the exceptional sets related to the

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solvability of the equation

$$p_1^5 + p_2^5 + \cdots + p_s^5 = n, \tag{1.1}$$

where $p_1, p_2, \ldots, p_s$ are unknown primes. For the recent results on exceptional sets in the Waring–Goldbach problem for fifth powers, readers can refer to Kumchev [6], Liu [8], Liu [9] and Feng-Liu [2]. The main result of this paper is the following.

Theorem 1.1 *For* $12 \le s \le 20$, *let* $E_s(N)$ *be the number of integers* $n \le N$ *satisfying* $n \equiv s (mod\ 2)$ *for which* (1.1) *cannot be solved in primes* $p_1, p_2, \ldots, p_s$. *Let* $\theta = \frac{27}{3200}$. *Then, for arbitrary* $\epsilon > 0$, *one has*

$$E_{12}(N) \ll N^{1-\theta-\frac{1}{120}+\epsilon}, \quad E_{13}(N) \ll N^{1-5\theta+\epsilon},$$
$$E_s(N) \ll N^{1-\frac{s-11}{40}-\theta+\epsilon} \ \ for \ \ 14 \le s \le 18,$$
$$E_{19}(N) \ll N^{1-\frac{1}{5}-2\theta+\epsilon}, \quad E_{20}(N) \ll N^{1-\frac{1}{4}-2\theta+\epsilon}.$$

Our result can be compared with previous results. For example, our results show that

$$E_{12}(N) \ll N^{1-\frac{1}{120}-\frac{27}{3200}+\epsilon}, \ \ E_{14}(N) \ll N^{1-\frac{3}{40}-\frac{27}{3200}+\epsilon}, \ \ E_{20}(N) \ll N^{1-\frac{1}{4}-\frac{27}{1600}+\epsilon}.$$

This improves upon the results of Feng and Liu [2]

$$E_{12}(N) \ll N^{1-\frac{1}{120}-\frac{73}{9600}+\epsilon}, \ \ E_{14}(N) \ll N^{1-\frac{7}{120}-\frac{73}{9600}+\epsilon}, \ \ E_{20}(N) \ll N^{1-\frac{9}{40}-\frac{73}{9600}+\epsilon}.$$

In fact, Feng and Liu [2] obtained $E_{12}(N) \ll N^{1-\theta'-\frac{1}{120}+\epsilon}$ and $E_{13}(N) \ll N^{1-5\theta'+\epsilon}$ with $\theta' = \frac{73}{9600}$. The improvement on θ' comes from the application of the sieve method and one can refer to Kumchev [6] for such method. The improvement upon the bound of $E_{14}(N)$ of Feng and Liu [2] comes from a new mean value theorem involving the sixth moment of the complete Weyl sum over fifth powers (see Lemma 2.8 in Section 2), and consequently, one can obtain new estimates of $E_s(N)$ for $15 \le s \le 18$. In order to obtain the estimates of $E_{19}(N)$ and $E_{20}(N)$, we apply the method of Kawada and Wooley [5] to establish a relation between $E_s(N)$ and $E_{s-4}(N)$.

As usual, we abbreviate $e^{2\pi i\alpha}$ to $e(\alpha)$. And we write $(a, b) = \gcd(a, b)$ to denote the greatest common divisor of a and b. The letter p, with or without indices, is a prime number. The letter ϵ denotes a sufficiently small positive real number, and the value of ϵ may change from statement to statement. Let N be a sufficiently large real number in terms of ϵ. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, while the implied constant may depend on ϵ. And $f \asymp g$ means $f \ll g \ll f$. We use $m \sim M$ as an abbreviation for the condition $M < m \le 2M$.

2 Preliminaries

Let

$$\psi(n,z)=\begin{cases}1, & \text{if } (n,\mathcal{P}(z))=1,\\ 0, & \text{otherwise,}\end{cases} \tag{2.1}$$

where

$$\mathcal{P}(z)=\prod_{p\le z}p.$$

We define $w_k(q)$ by

$$w_k(p^{uk+v})=\begin{cases}kp^{-u-\frac{1}{2}} & u\ge 0 \text{ and } v=1,\\ p^{-u-1} & u\ge 0 \text{ and } 2\le v\le k.\end{cases} \tag{2.2}$$

The following two lemmas are from Kumchev [7].

Lemma 2.1 *Let $k\ge 3$ and $0<\rho<(2^k+2)^{-1}$. Suppose that $\alpha\in\mathbb{R}$ and that there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ such that*

$$1\le q\le Q,\quad (a,q)=1,\quad |q\alpha-a|<Q^{-1} \tag{2.3}$$

holds with Q subject to

$$P^{4k\rho}\le Q\le P^{k-2k\rho}.$$

Let $M\ge N\ge 2$, $|\varepsilon_m|\le 1$, $|\eta_n|\le q$. Then,

$$\sum_{\substack{m\sim M\\ mn\sim P}}\sum_{n\sim N}\varepsilon_m\eta_n e(\alpha(mn)^k)\ll P^{1-\rho+\epsilon}+\frac{w_k(q)^{1/2}P^{1+\epsilon}}{(1+P^k|\alpha-a/q|)^{1/2}},$$

provided that

$$\max\left(P^{2^k\rho},P^{(k-1+4k\rho)/(2k+1)}\right)\le M\le P^{1-2\rho}.$$

Proof This is [7, Lemma 3.1]. □

Lemma 2.2 *Let $k\ge 3$ and $0<\rho<(2^k+2)^{-1}$. Suppose that $\alpha\in\mathbb{R}$ and that there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ such that (2.1) holds with Q given by*

$$Q=P^{(k^2-2k\rho)/(2k-1)}. \tag{2.4}$$

Let $M\ge N\ge 2$, $|\varepsilon_m|\le 1$, and let $\psi(n,z)$ be defined by (1.3), Then,

$$\sum_{\substack{m\sim M \\ mn\sim P}} \sum_{n\sim N} \varepsilon_m \psi(n,z) e(\alpha(mn)^k) \ll P^{1-\rho+\epsilon} + \frac{w_k(q)^{1/2} P^{1+\epsilon}}{(1+P^k|\alpha - a/q|)^{1/2}},$$

provided that

$$z \leq \min\left(P^{(k-(8k-2)\rho)/(2k-1)}, P^{1-(2^k+2)\rho}\right) \tag{2.5}$$

and

$$M \leq \min\left(P^{(k-(2k+1)\rho)/(2k-1)}, P^{1-(2^{k-1}+2)\rho}\right).$$

Proof This is [7, Lemma 3.3]. □

We shall apply Buchstab's combinatorial identity in the form

$$\psi(m, z_1') = \psi(m, z_2') - \sum_{\substack{z_2' < p \leq z_1' \\ pj = m}} \psi(j, p) \quad (2 \leq z_2' < z_1'). \tag{2.6}$$

Let

$$z_0 = P^{1-34\rho}.$$

Note that when $k = 5$ and $\rho \geq 1/67$, z_0 is the right hand side of the inequality in (2.5). Let

$$z_1 = (2P)^{1/3} \text{ and } z_2 = (2P)^{1/\alpha},$$

where

$$\alpha = \frac{1}{1-32\rho}.$$

Note that $\alpha \geq 3$ if $\rho \geq \frac{1}{48}$. In fact, we shall choose $\rho = \frac{1}{40}$.

Suppose that $m \leq 2P$. Applying (2.6), we obtain

$$\psi(m, \sqrt{2P}) = \psi(m, z_0) - \sum_{\substack{z_0 < p \leq \sqrt{2P} \\ jp = m}} \psi(j, p). \tag{2.7}$$

Splitting the summation in (2.7) into three parts, we have

$$\psi(m,\sqrt{2P}) = \psi(m,z_0) - \sum_{\substack{z_0 < p \le z_2 \\ jp=m}} \psi(j,p) - \sum_{\substack{z_2 < p \le z_1 \\ jp=m}} \psi(j,p) - \sum_{\substack{z_1 < p \le \sqrt{2P} \\ jp=m}} \psi(j,p) =: \kappa_1(m) - \kappa_2(m) - \kappa_3(m) - \kappa_4(m). \tag{2.8}$$

Applying (2.6), we obtain

$$\kappa_3(m) = \sum_{\substack{z_2 < p \le z_1 \\ jp=m}} \psi(j,z_0) - \sum_{\substack{z_0 < p_2 \le p_1 \\ z_2 < p_1 \le z_1 \\ jp_1p_2=m}} \psi(j,p_2), \tag{2.9}$$

and by splitting the second summation in (2.9) into two parts, we conclude that

$$\kappa_3(m) = \sum_{\substack{z_2 < p \le z_1 \\ jp=m}} \psi(j,z_0) - \sum_{\substack{z_0 < p_2 \le z_2 \\ z_2 < p_1 \le z_1 \\ jp_1p_2=m}} \psi(j,p_2) - \sum_{\substack{z_2 < p_2 \le p_1 \\ z_2 < p_1 \le z_1 \\ jp_1p_2=m}} \psi(j,p_2) =: \kappa_5(m) - \kappa_6(m) - \kappa_7(m). \tag{2.10}$$

To deal with $\kappa_4(m)$, we observe that for $m \le 2P$,

$$\kappa_4(m) = \sum_{\substack{z_1 < p \le \sqrt{2P} \\ jp=m}} \psi(j,z_1),$$

and by (2.6), we obtain

$$\kappa_4(m) = \sum_{\substack{z_1 < p \le \sqrt{2P} \\ jp=m}} \psi(j,z_0) - \sum_{\substack{z_0 < p_2 \le z_1 \\ z_1 < p_1 \le \sqrt{2P} \\ jp_1p_2=m}} \psi(j,p_2). \tag{2.11}$$

Dividing the second summation in (2.11) into two parts, we get

$$\begin{aligned}\kappa_4(m) = & \sum_{\substack{z_1 < p \leq \sqrt{2P} \\ jp = m}} \psi(j, z_0) - \sum_{\substack{z_0 < p_2 \leq z_2 \\ z_1 < p_1 \leq \sqrt{2P} \\ jp_1p_2 = m}} \psi(j, p_2) \\ & - \sum_{\substack{z_2 < p_2 \leq z_1 \\ z_1 < p_1 \leq \sqrt{2P} \\ jp_1p_2 = m}} \psi(j, p_2) \\ =: & \, \kappa_8(m) - \kappa_9(m) - \kappa_{10}(m).\end{aligned} \tag{2.12}$$

Now we introduce

$$\psi_{\mathrm{g}}(m) = \kappa_1(m) - \kappa_2(m) - \kappa_5(m) + \kappa_6(m) - \kappa_8(m) + \kappa_9(m), \tag{2.13}$$

and

$$\psi_{\mathrm{b}}(m) = \kappa_7(m) + \kappa_{10}(m). \tag{2.14}$$

Let $\omega(u)$ be the continuous solution of the differential equation

$$\begin{cases} \big(u\omega(u)\big)' = \omega(u-1), & \text{if } u > 2 \\ \omega(u) = \frac{1}{u}, & \text{if } 1 < u \leq 2. \end{cases}$$

We introduce

$$C' = \int_{1/\alpha}^{1/3} \int_{1/\alpha}^{\beta} \omega\Big(\frac{1-\beta-\gamma}{\gamma}\Big) \frac{1}{\gamma^2 \beta} \mathrm{d}\gamma \mathrm{d}\beta$$

and

$$C'' = \int_{1/3}^{1/2} \int_{1/\alpha}^{(1-\beta)/2} \omega\Big(\frac{1-\beta-\gamma}{\gamma}\Big) \frac{1}{\gamma^2 \beta} \mathrm{d}\gamma \mathrm{d}\beta.$$

In fact, one has $\sum_{m \sim P} \kappa_7(m) \sim C' P (\log P)^{-1}$ and $\sum_{m \sim P} \kappa_{10}(m) \sim C'' P (\log P)^{-1}$. Let

$$C_{\mathrm{b}} = C' + C'' \ \text{ and } C_{\mathrm{g}} = 1 - C_{\mathrm{b}}.$$

We have the following conclusion.

Lemma 2.3 *Let $P \le m \le 2P$. Suppose that $\frac{1}{48} \le \rho < \frac{1}{34}$. We have*

$$\psi(m, \sqrt{2P}) = \psi_{\mathrm{g}}(m) + \psi_{\mathrm{b}}(m) \tag{2.15}$$

and

$$\psi(m, \sqrt{2P}) \ge \psi_{\mathrm{g}}(m). \tag{2.16}$$

Moreover, we have

$$\sum_{m \sim P} \psi_{\mathrm{b}}(m) = C_{\mathrm{b}} \frac{P}{\log P} + O\left(\frac{P}{\log^2 P}\right), \tag{2.17}$$

and

$$\sum_{m \sim P} \psi_{\mathrm{g}}(m) = C_{\mathrm{g}} \frac{P}{\log P} + O\left(\frac{P}{\log^2 P}\right). \tag{2.18}$$

Proof Note that (2.15) follows from (2.8), (2.10), (2.12), (2.13), and (2.14). Then, (2.16) follows by observing that $\psi_{\mathrm{b}}(m) \ge 0$. The asymptotic formulas (2.17) and (2.18) can be proved by the standard argument in prime number theory (see also Lemma 7.1 in [6]). □

We point out that if $\rho = \frac{1}{40}$ then

$$C_{\mathrm{g}} > 0 \text{ and } C_{\mathrm{b}} < 1.$$

In fact, the value $\frac{1}{40}$ can be further improved.

We view that $\psi_g(m)$ is *good*, since the corresponding exponential summation can be handed by Lemmas 2.1 and 2.2. Let

$$f(\alpha) = \sum_{m \sim P} \psi_g(m) e(\alpha m^5).$$

Lemma 2.4 *Suppose that $\frac{1}{48} \le \rho < \frac{1}{34}$. Suppose that $\alpha \in \mathbb{R}$ and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that (2.4) holds with Q given by (2.4). Then, for any fixed $\epsilon > 0$, one has*

$$f(\alpha) \ll P^{1-\rho+\epsilon} + \frac{w_5^{1/2}(q) P^{1+\epsilon}}{(1 + P^5|\alpha - a/q|)^{1/2}}.$$

Let

$$v_j = \left(\frac{33}{40}\right)^{j-1}, \quad j = 1, 2, \ldots, 6,$$

$$v_7 = \left(\frac{33}{40}\right)^5 \frac{136}{163}, \quad v_8 = \left(\frac{33}{40}\right)^5 \frac{576}{815}, \quad v_9 = v_{10} = \left(\frac{33}{40}\right)^5 \frac{512}{815}.$$

We write

$$v = \sum_{i=1}^{10} v_i. \tag{2.19}$$

Note that

$$v = 4.9817431213.$$

Let

$$P = (N/20)^{1/5}.$$

We define

$$r_s(n) = \sum_{\substack{p_1^5+\cdots+p_s^5=n \\ p_i \sim P^{v_i}\ (1\le i\le 10) \\ p_i\sim P\ (11\le i\le s)}} \prod_{i=1}^{s-1} \log p_i.$$

Note that $r_s(n)$ is the (weighted) number of solutions to $p_1^5 + p_2^5 + \cdots + p_s^5 = n$ in prime variables. We also define

$$R_s(n) = \sum_{\substack{p_1^5+\cdots+p_{s-1}^5+m^5=n \\ p_i \sim P^{v_i}\ (1\le i\le 10) \\ p_i\sim P\ (11\le i\le s-1),\ m\sim P}} \psi_g(m) \prod_{i=1}^{s-1} \log p_i.$$

By (2.16), we have

$$r_s(n) \ge R_s(n). \tag{2.20}$$

For $1 \le j \le 10$, we define

$$g_j(\alpha) = \sum_{p\sim P^{v_j}} (\log p) e(\alpha p^5). \tag{2.21}$$

Then, we write

$$G(\alpha) = f(\alpha) \prod_{i=2}^{10} g_i(\alpha). \tag{2.22}$$

For $\mathcal{X} \in [0, 1)$, we put

$$R_s(n, \mathcal{X}) = \int_{\mathcal{X}} g_1^{s-10}(\alpha) G(\alpha) e(-n\alpha) \mathrm{d}\alpha.$$

Note that

$$R_s(n) = R_s(n, [0, 1)).$$

Let

$$P_0 = P^{2v_9/5}.$$

Then, we define

$$\mathfrak{M} = \bigcup_{1 \le q \le P_0} \bigcup_{\substack{1 \le a \le q \\ (a, q) = 1}} \mathfrak{M}(q, a), \quad \mathfrak{m} = [0, 1) \backslash \mathfrak{M},$$

where

$$\mathfrak{M}(q, a) = \left\{ \alpha : \left| \alpha - \frac{a}{q} \right| \le \frac{P_0}{qN} \right\}.$$

The singular series is defined by

$$\mathfrak{S}_s(n) = \sum_{q=1}^{\infty} \frac{1}{\phi(q)^s} \sum_{\substack{1 \le a \le q \\ (q, a) = 1}} S^*(q, a)^s e\left(-\frac{an}{q}\right),$$

where

$$S^*(q, a) = \sum_{\substack{1 \le x \le q \\ (x, q) = 1}} e\left(\frac{ax^5}{q}\right).$$

And the singular integral is defined by

$$\mathfrak{J}_s(n) = \int_{-\infty}^{\infty} u_1^{s-11}(\beta)u^*(\beta) \prod_{1\le i\le 10} u_i(\beta)\mathrm{d}\beta,$$

where

$$u_i(\beta) = \int_{P^{v_i}}^{2P^{v_i}} e(x^5\beta)\mathrm{d}x, \quad u^*(\beta) = \int_P^{2P} \frac{e(x^5\beta)}{\log x}\mathrm{d}x.$$

The following result can be proved by the standard method of dealing with the major arcs.

Lemma 2.5 *Let n be an integer satisfying $N < n \le 2N$ and $n \equiv s(\bmod\ 2)$. One has*

$$R_s(n, \mathfrak{M}) = \left(C_{\mathrm{g}}\mathfrak{S}_s(n) + O(\frac{1}{\log P})\right)\mathfrak{J}_s(n).$$

Moreover, one has

$$\mathfrak{S}_s(n) \asymp 1 \quad \textit{and} \quad \mathfrak{J}_s(n) \asymp \frac{P^{s-15+v}}{\log P},$$

where v is given in (2.19).

Considering the underlying Diophantine equations and applying [4, Lemma 6.2], one has the following result.

Lemma 2.6 *Let $g_j(\alpha)$ be defined in (2.21) and let $G(\alpha)$ be defined in (2.22). Then, one has*

$$\int_0^1 |G^2(\alpha)|\mathrm{d}\alpha \ll P^{v+\epsilon}. \tag{2.23}$$

And for $1 \le j \le 10$, one has

$$\int_0^1 |G^2(\alpha)g_j^2(\alpha)|\mathrm{d}\alpha \ll P^{2v-5+2v_j+\epsilon}. \tag{2.24}$$

We define the exponential sum

$$h(\alpha) = \sum_{X/2\le x\le X} e(\alpha x^5). \tag{2.25}$$

For $\mathcal{Z} \subseteq [1, 2X^5] \cap \mathbb{Z}$, we introduce

$$K(\alpha) = \sum_{n\in\mathcal{Z}} e(n\alpha).$$

And we use Z to denote the cardinality of $\mathcal{Z}$. The following result can be found in [1].

Lemma 2.7 (Lemma 3.1 [1]) *We have*

$$\int_0^1 |h^2(\alpha)K^2(\alpha)|\mathrm{d}\alpha \ll P^{\epsilon}(P^{2-\frac{1}{8}}Z + Z^{1+\frac{2}{5}}) \tag{2.26}$$

and

$$\int_0^1 |h^4(\alpha)K^2(\alpha)|\mathrm{d}\alpha \ll P^{\epsilon}(P^{4-\frac{1}{4}}Z + Z^{1+\frac{4}{5}}). \tag{2.27}$$

We provide a similar result involving the sixth moment of $h(\alpha)$.

Lemma 2.8 *We have*

$$\int_0^1 |h^6(\alpha)K^2(\alpha)|\mathrm{d}\alpha \ll P^{\epsilon}(P^{6-\frac{3}{8}}Z + PZ^2). \tag{2.28}$$

Proof Let

$$\mathfrak{R} = \bigcup_{q \le P^{\frac{5}{16}}} \bigcup_{\substack{1 \le a \le q \\ (q,a)=1}} \mathfrak{R}(q,a), \tag{2.29}$$

where

$$\mathfrak{R}(q,a) = \left\{\alpha : \left|q\alpha - a\right| \le P^{\frac{5}{16}-5}\right\}.$$

Then, we define the function $\Psi : [0, 1) \to [0, \infty)$ as

$$\Psi(\alpha) = w_5(q)P\left(1 + P^5\left|\alpha - \frac{a}{q}\right|\right)^{-1}, \tag{2.30}$$

when $\alpha \in \mathfrak{R}(q,a) \subseteq \mathfrak{R}$, otherwise by taking $\Psi(\alpha) = 0$.

The following is well known (see for example (2.8) in [12])

$$h(\alpha) \ll \Psi(\alpha) + P^{1-\frac{1}{16}+\epsilon},$$

and therefore,

$$h(\alpha)^6 \ll \Psi(\alpha)^6 + P^{6-\frac{3}{8}+\epsilon}.$$

Then, we have

$$\int_0^1 |h^6(\alpha)K^2(\alpha)|\mathrm{d}\alpha \ll \int_0^1 \Psi(\alpha)^6|K(\alpha)|^2\mathrm{d}\alpha + P^{6-\frac{3}{8}+\epsilon}\left(\int_0^1 |K(\alpha)|^2\mathrm{d}\alpha\right) \tag{2.31}$$

Since $w_5(q) \le q^{-1/5}$, one has

$$\begin{aligned}
&\int_0^1 \Psi(\alpha)^6|K(\alpha)|^2\mathrm{d}\alpha \\
&\le P^6 \sum_{q\le P^{5/16}} q^{-6/5}\int_{-P^{5/16-5}}^{P^{5/16-5}}(1+P^5|\beta|)^{-6}\sum_{1\le a\le q}\left|K\left(\frac{a}{q}+\beta\right)\right|^2\mathrm{d}\beta \qquad (2.32)\\
&\ll ZP^{5/4} + Z^2P^{1+\epsilon}.
\end{aligned}$$

Now (2.28) follows from (2.31) and (2.32). This completes the proof. □

Lemma 2.9 *We have*

$$\int_0^1 |h^8(\alpha)K^2(\alpha)|\mathrm{d}\alpha \ll P^7Z + P^{\frac{11}{2}+\epsilon}Z^{\frac{3}{2}}. \tag{2.33}$$

Proof This follows from Lemma 6.1 of [5] (by choosing $k=5$ and $j=3$). □

3 Proof of Theorem 1.1

First, we estimate the contribution from the minor arcs $\mathfrak{m}$, which were defined in Sect. 2. Denote

$$\mathfrak{N} = \bigcup_{q\le P^{\frac{15}{16}}}\ \bigcup_{\substack{1\le a\le q\\(q,a)=1}} \mathfrak{N}(q,a),$$

where

$$\mathfrak{N}(q,a) = \left\{\alpha : \left|q\alpha - a\right| \le P^{\frac{15}{16}-5}\right\}.$$

Lemma 3.1 *One has*

$$\sup_{\alpha\in\mathfrak{m}\cap\mathfrak{N}} |g_1(\alpha)| \ll P^{1-\frac{1}{32}+\epsilon} \tag{3.1}$$

and for $j\in\{1,2\}$,

$$\sup_{\alpha\in\mathfrak{m}\setminus\mathfrak{N}} |g_j(\alpha)| \ll P^{v_j(1-\frac{1}{48})+\epsilon}. \tag{3.2}$$

Proof These estimates can be found on page 52 in [2]. □

In particular, by Lemma 3.1, we have

$$\sup_{\alpha\in\mathfrak{m}}|g_1(\alpha)| \ll P^{1-\frac{1}{48}+\epsilon}. \tag{3.3}$$

For $\mathfrak{n}\subseteq\mathfrak{m}$, we introduce

$$I(\mathfrak{n}) = \int_{\mathfrak{n}} |g_1^2(\alpha)G^2(\alpha)|\mathrm{d}\alpha. \tag{3.4}$$

Lemma 3.2 *One has*

$$I(\mathfrak{m}\cap\mathfrak{N}) \ll P^{2(1-\frac{1}{32})+v+\epsilon}. \tag{3.5}$$

Proof Note that (3.5) follows from (2.23) and (3.1). □

Lemma 3.3 *One has*

$$I(\mathfrak{m}\setminus\mathfrak{N}) \ll P^{2v-3-\frac{27}{640}+\epsilon},. \tag{3.6}$$

Proof Recalling $\mathfrak{R}$ defined in (2.29) and $\Psi(\alpha)$ defined in (2.30), we use $\mathfrak{B}$ to denote the set of ordered pairs $(\alpha,\beta)\in(\mathfrak{m}\backslash\mathfrak{N})^2$ for which $\alpha-\beta\in\mathfrak{R}(mod\ 1)$, and put $\mathfrak{b}=\mathfrak{m}^2\backslash\mathfrak{B}$.

We introduce

$$\Upsilon_1 = P^{2-\frac{1}{8}+\epsilon}\int_{\mathfrak{m}\backslash\mathfrak{N}}\int_{\mathfrak{m}\backslash\mathfrak{N}}|G^2(\alpha)G^2(\beta)|\mathrm{d}\alpha\,\mathrm{d}\beta,$$

and

$$\Upsilon_2 = \int\int_{\mathfrak{B}}\Psi^2(\alpha-\beta)|G^2(\alpha)G^2(\beta)|\mathrm{d}\alpha\,\mathrm{d}\beta.$$

The argument leading to (3) in [3] implies

$$I(\mathfrak{m}\setminus\mathfrak{N})^2 \ll P(\Upsilon_1+\Upsilon_2). \tag{3.7}$$

By (2.23), we have

$$\Upsilon_1 \ll P^{2-\frac{1}{8}+2v+\epsilon}. \tag{3.8}$$

Note that

$$\begin{aligned}\Upsilon_2 \ll \sup_{\beta\in\mathfrak{m}\backslash\mathfrak{N}}|f^2(\beta)g_2^2(\beta)g_5^2(\beta)\cdots g_{10}^2(\beta)|\int_0^1|g_3^2(\alpha)G^2(\alpha)|\\ \times\left(\int_0^1\Psi^2(\alpha-\beta)|g_4^2(\beta)|\mathrm{d}\beta\right)\mathrm{d}\alpha.\end{aligned} \tag{3.9}$$

By [12, Lemma 2.2], one has uniformly for $\alpha \in [0, 1)$ that

$$\int_0^1 \Psi^2(\alpha-\beta)|g_4^2(\beta)|\mathrm{d}\beta \ll P^{2v_4-3+\epsilon}. \tag{3.10}$$

By Lemma 2.4,

$$f(\alpha) \ll P^{1-\frac{1}{40}+\epsilon}. \tag{3.11}$$

Now conclude from (3.9)-(3.11) and (2.24) that

$$\Upsilon_2 \ll P^{4v-8-\frac{27}{320}+\epsilon}. \tag{3.12}$$

From (3.7), (3.8), and (3.12), we obtain

$$I(\mathfrak{m}\setminus\mathfrak{N}) \ll P^{2v-3-\frac{27}{640}+\epsilon}. \tag{3.13}$$

This completes the proof. □

Lemma 3.4 *One has*

$$I(\mathfrak{m}) \ll P^{2v-3-\frac{27}{640}+\epsilon}, \tag{3.14}$$

and

$$\int_{\mathfrak{m}} |g_1^4(\alpha)G^2(\alpha)|\mathrm{d}\alpha \ll P^{2v-1-\frac{27}{640}-\frac{1}{24}+\epsilon}. \tag{3.15}$$

Proof Note that (3.14) follows from (3.5) and (3.6). Combining (3.3) and (3.14), we have

$$\int_{\mathfrak{m}} |g_1^4(\alpha)G^2(\alpha)|\mathrm{d}\alpha \ll \left(\sup_{\alpha\in\mathfrak{m}} |g_1^2(\alpha)|\right)\int_{\mathfrak{m}} |g_1^2(\alpha)G^2(\alpha)|\mathrm{d}\alpha \ll P^{2v-1-\frac{27}{640}-\frac{1}{24}+\epsilon}.$$

This completes the proof. □

Proof of Theorem 1.1 For $12 \le s \le 20$, we introduce $\mathcal{E}_s(N)$ to denote the set of n satisfying $N/2 \le n \le N$, $n \equiv s \pmod 2$ and $R_s(n) = 0$. Then, we define

$$\mathcal{K}_s(\alpha) = \sum_{n\in\mathcal{E}_s(N)} e(-n\alpha).$$

By the definition of $\mathcal{E}_s(N)$, we have

$$\sum_{n\in\mathcal{E}_s(N)} R_s(n,\mathfrak{M}) = -\sum_{n\in\mathcal{E}_s(N)} R_s(n,\mathfrak{m}) = -\int_{\mathfrak{m}} g_1(\alpha)^{s-10}G(\alpha)\mathcal{K}_s(\alpha)\mathrm{d}\alpha.$$

Then, by Lemma 2.5, we obtain

$$\int_{\mathfrak{m}} |g_1(\alpha)^{s-10} G(\alpha)\mathcal{K}_s(\alpha)|\mathrm{d}\alpha \gg P^{v+s-15}(\log P)^{-1}|\mathcal{E}_s(N)|, \tag{3.16}$$

where $|\mathcal{E}_s(N)|$ denotes the cardinality of $\mathcal{E}_s(N)$.

By Schwarz's inequality, we have

$$\int_{\mathfrak{m}} |g_1(\alpha)^2 G(\alpha)\mathcal{K}_{12}(\alpha)|\mathrm{d}\alpha \ll \left(\int_{\mathfrak{m}} |g_1^4(\alpha)G^2(\alpha)|\mathrm{d}\alpha\right)^{1/2} \left(\int_0^1 |\mathcal{K}_{12}^2(\alpha)|\mathrm{d}\alpha\right)^{1/2}. \tag{3.17}$$

Then, by (3.15), (3.16), and (3.17), we obtain

$$P^{v-3}(\log P)^{-1}|\mathcal{E}_{12}(N)| \ll \left(P^{2v-1-\frac{27}{640}-\frac{1}{24}+\epsilon}\right)^{1/2} |\mathcal{E}_{12}(N)|^{1/2}.$$

Thus, we can get

$$|\mathcal{E}_{12}(N)| \ll N^{1-\theta-\frac{1}{120}+\epsilon}. \tag{3.18}$$

Next we deal with $s = 13$. By Schwarz's inequality, we have

$$\begin{aligned}&\int_{\mathfrak{m}} |g_1(\alpha)^3 G(\alpha)\mathcal{K}_{13}(\alpha)|\mathrm{d}\alpha \\ &\quad \ll \left(\int_{\mathfrak{m}} |g_1^2(\alpha)G^2(\alpha)|\mathrm{d}\alpha\right)^{1/2} \left(\int_0^1 |g_1^4(\alpha)\mathcal{K}_{13}^2(\alpha)|\mathrm{d}\alpha\right)^{1/2}.\end{aligned} \tag{3.19}$$

Then, by (2.27), (3.14), (3.16), and (3.19), we obtain

$$\begin{aligned}&P^{v-2}(\log P)^{-1}|\mathcal{E}_{13}(N)| \\ &\quad \ll P^{\epsilon}\left(P^{2v-3-\frac{27}{640}}\right)^{1/2}\left(P^{4-\frac{1}{4}}|\mathcal{E}_{13}(N)| + |\mathcal{E}_{13}(N)|^{1+\frac{4}{5}}\right)^{1/2}.\end{aligned}$$

Therefore, we have

$$|\mathcal{E}_{13}(N)| \ll N^{1-5\theta+\epsilon}. \tag{3.20}$$

Now we deal with $s = 14$. By Schwarz's inequality, we have

$$\begin{aligned}&\int_{\mathfrak{m}} |g_1(\alpha)^4 G(\alpha)\mathcal{K}_{14}(\alpha)|\mathrm{d}\alpha \\ &\quad \ll \left(\int_{\mathfrak{m}} |g_1^2(\alpha)G^2(\alpha)|\mathrm{d}\alpha\right)^{1/2} \left(\int_0^1 |g_1^6(\alpha)\mathcal{K}_{14}^2(\alpha)|\mathrm{d}\alpha\right)^{1/2}.\end{aligned} \tag{3.21}$$

We deduce from (2.28), (3.14), (3.16), and (3.21) that

$$P^{v-1}(\log P)^{-1}|\mathcal{E}_{14}(N)| \ll P^{\epsilon}\left(P^{2v-3-\frac{27}{640}}\right)^{1/2}\left(P^{6-\frac{3}{8}}|\mathcal{E}_{14}(N)|+P|\mathcal{E}_{14}(N)|^2\right)^{1/2}.$$

Therefore, we have

$$|\mathcal{E}_{14}(N)| \ll N^{1-\frac{3}{40}-\theta+\epsilon}. \tag{3.22}$$

Let $E_s(M, N)$ be the set of integers n satisfying $M \le n \le N$ and $n \equiv s (\bmod 2)$ for which (1.1) cannot be solved in primes $p_1, p_2, \ldots, p_s$. In view of (2.20), one has

$$|E_s(N/2, N)| \ll |\mathcal{E}_s(N)|,$$

and by the dyadic argument,

$$E_s(N) \ll (\log N) \sup_{2\le M\le N} |\mathcal{E}_s(M)|. \tag{3.23}$$

Now we can obtain the desired estimates of $E_s(N)$ for $12 \le s \le 14$ from (3.18), (3.20), (3.22), and (3.23).

To establish the upper bounds of $E_s(N)$ for $15 \le s \le 18$, we follow the proof in [2]. In fact, by Lemma 4 (a) in [2], one can prove

$$E_s(N) \ll (\log N) \sup_{2\le M\le N}\left(M^{\epsilon-\frac{1}{40}}E_{s-1}(3M)+M^{\epsilon-\frac{2}{3}}E_{s-1}(3M)^{5/3}\right). \tag{3.24}$$

Now we can obtain the desired estimates of $E_s(N)$ for $15 \le s \le 18$ by using (3.24) iteratively.

Let $X = (N/16)^{1/5}$. For $1 \le m \le N$, we introduce $\lambda(m)$ to denote the number of representations of m in the form $m = n - p_1^4 - p_2^4 - p_3^4 - p_4^4$, where $n \in E_s(N/2, N)$ and $X/2 \le p_1, p_2, p_4, p_4 \le X$. Note that if $\lambda(m) \ge 1$, then $m \in E_{s-4}(1, N)$. Then, by Cauchy's inequality,

$$\Big(\sum_{1\le m\le N}\lambda(m)\Big)^2 \le \Big(\sum_{\substack{1\le m\le N\\ \lambda(m)\ge 1}} 1\Big)\Big(\sum_{1\le m\le N}\lambda(m)^2\Big) \le |E_{s-4}(1, N)|\Big(\sum_{1\le m\le N}\lambda(m)^2\Big). \tag{3.25}$$

We use $\lambda^+(m)$ to denote the number of representations of m in the form $m = n - x_1^4 - x_2^4 - x_3^4 - x_4^4$, where $n \in E_s(N/2, N)$ and $X/2 \le x_1, x_2, x_4, x_4 \le X$. One has trivially $\lambda(m) \le \lambda^+(m)$, and then by (3.25),

$$\Big(\sum_{1\le m\le N}\lambda(m)\Big)^2 \le |E_{s-4}(1, N)|\Big(\sum_{1\le m\le N}\lambda^+(m)^2\Big). \tag{3.26}$$

We introduce

$$K(\alpha) = \sum_{n \in E_s(N/2,N)} e(n\alpha).$$

Then,

$$\sum_{1 \le m \le N} \lambda^+(m)^2 = \int_0^1 |K(\alpha)^2 h(\alpha)^8| \mathrm{d}\alpha, \tag{3.27}$$

where $h(\alpha)$ is defined in (2.25). We also have

$$\sum_{1 \le m \le N} \lambda(m) \gg X^4 (\log X)^{-4} |E_s(N/2, N)|. \tag{3.28}$$

We conclude from (3.26), (3.27), and (3.28) that

$$X^8 (\log X)^{-8} |E_s(N/2, N)|^2 \ll |E_{s-4}(1, N)| \int_0^1 |K(\alpha)^2 h(\alpha)^8| \mathrm{d}\alpha. \tag{3.29}$$

We remark that the proof of (3.29) is based on the method developed in [5]. We deduce from (2.33) and (3.29) that

$$\begin{aligned} & X^8 (\log X)^{-8} |E_s(N/2, N)|^2 \\ & \quad \ll |E_{s-4}(1, N)| \Big(X^7 |E_s(N/2, N)| + X^{\frac{11}{2}+\epsilon} |E_s(N/2, N)|^{\frac{3}{2}} \Big), \end{aligned}$$

and therefore,

$$|E_s(N/2, N)| \ll N^{-\frac{1}{5}+\epsilon} |E_{s-4}(1, N)| + N^{-1+\epsilon} |E_{s-4}(1, N)|^2. \tag{3.30}$$

On invoking the estimate $E_{15}(N) \ll N^{1-\frac{1}{10}-\theta+\epsilon}$ and $E_{16}(N) \ll N^{1-\frac{1}{8}-\theta+\epsilon}$, we deduce from (3.30) that

$$|E_{19}(N/2, N)| \ll N^{-\frac{1}{5}+\epsilon} E_{15}(N) + N^{-1+\epsilon} E_{15}(N)^2 \ll N^{1-\frac{1}{5}-2\theta+\epsilon}$$

and

$$|E_{20}(N/2, N)| \ll N^{-\frac{1}{5}+\epsilon} E_{16}(N) + N^{-1+\epsilon} E_{16}(N)^2 \ll N^{1-\frac{1}{4}-2\theta+\epsilon}.$$

Finally, the desired estimates of $E_{19}(N)$ and $E_{20}(N)$ follow from the dyadic argument.

This completes the proof of Theorem 1.1. □

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