The Ramanujan Journal (2023) 60:141–155
https://doi.org/10.1007/s11139-022-00656-3

Weighted sum formulas for symmetric multiple zeta values

Kento Fujita[1] · **Yasushi Komori**[2]

Received: 3 October 2021 / Accepted: 29 August 2022 / Published online: 23 October 2022

Abstract
By using iterated integrals, Hirose defined refined symmetric multiple zeta values as lifts of symmetric multiple zeta values. In this article, we prove sum formulas with four parameters among symmetric multiple zeta values by iterated integral expressions of refined symmetric multiple zeta values. Specializing the parameters in our result gives several weighted sum formulas for these values.

Keywords Symmetric multiple zeta value · Refined symmetric multiple zeta value

Mathematics Subject Classification 11M32

1 Introduction

An index is a sequence of positive integers including the empty sequence $\emptyset$. In particular, an index $\mathbf{k} = (k_1, \ldots, k_r)$ is said to be admissible if either $k_r > 1$ or $\mathbf{k} = \emptyset$. The number $|\mathbf{k}| := k_1 + \cdots + k_r$ is called the weight of $\mathbf{k} = (k_1, \ldots, k_r)$ and r, the depth. Let $I(k, r)$ be the set of all indices whose weight is k and depth is r. For an admissible index $\mathbf{k} = (k_1, \ldots, k_r)$, the multiple zeta value (MZV) is the real number defined by

This work was supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C) 17K05185.

✉ Kento Fujita
19lc001a@al.rikkyo.ac.jp

Yasushi Komori
komori@rikkyo.ac.jp

1 Division of Saitama, Nippon Telegraph and Telephone East Corporation, 5-8-17, Tokiwa, Urawa-ku, Saitama-shi, Saitama 330-0061, Japan

2 Department of Mathematics, Rikkyo University, Nishi-Ikebukuro, Toshima-ku, Tokyo 171-8501, Japan

$$\zeta(\mathbf{k})=\zeta(k_1,\ldots,k_r):=\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}.$$

Conventionally, $\zeta(\emptyset)$ is understood as 1. Let $\mathcal{Z}$ be the $\mathbb{Q}$-algebra generated by all multiple zeta values. Set

$$\zeta_{\mathcal{S}}^{ш}(\mathbf{k})=\zeta_{\mathcal{S}}^{ш}(k_1,\ldots,k_r):=\sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}}\zeta^{ш}(k_1,\ldots,k_i)\zeta^{ш}(k_r,\ldots,k_{i+1}),$$

where $\zeta^{ш}(\mathbf{k})$ is the constant term of the shuffle regularized polynomial. Then the symmetric multiple zeta value (SMZV) is an element of $\mathcal{Z}/\zeta(2)\mathcal{Z}$ defined as

$$\zeta_{\mathcal{S}}(\mathbf{k}):=\zeta_{\mathcal{S}}^{ш}(\mathbf{k})\bmod\zeta(2)\mathcal{Z}.$$

SMZVs are introduced by Kaneko and Zagier in [5] as real counterparts of finite multiple zeta values (FMZVs). Moreover, Kaneko and Zagier conjectured that there exists a one-to-one correspondence between SMZVs and FMZVs, which is called the Kaneko–Zagier conjecture. This conjecture implies that any relations for SMZVs take the same form as the corresponding ones for FMZVs, and vice versa.

In [6], for an index $\mathbf{k}=(k_1,\ldots,k_r)$, Hirose defined the refined symmetric multiple zeta values $\zeta_{RS}(\mathbf{k})$ (RSMZVs) as an element of $\mathcal{Z}[2\pi i]$ in terms of iterated integrals and gives the following expression:

$$\zeta_{RS}(k_1,\ldots,k_r)=\sum_{\substack{0\le a\le b\le r\\ k_j=1\ (a<\forall j\le b)}}\frac{(-2\pi i)^{b-a}}{(b-a+1)!}(-1)^{k_r+\cdots+k_{b+1}}\zeta^{ш}(k_1,\ldots,k_a)\times\zeta^{ш}(k_r,\ldots,k_{b+1}).$$

By this expression, we easily see that $\zeta_{RS}(\mathbf{k})$ is a lift of $\zeta_{\mathcal{S}}(\mathbf{k})$:

$$\zeta_{RS}(k_1,\ldots,k_r)\equiv\zeta_{\mathcal{S}}^{ш}(k_1,\ldots,k_r)\pmod{2\pi i\mathcal{Z}[2\pi i]}.\tag{1.1}$$

The definition of RSMZVs, (1.1) and the method of the iterated integrals discussed in [2, 3] together lead to weighted sum formulas for SMZVs, which is our main purpose in this article.

Main Theorem *(Weighted sum formulas, cf.* [3]*) Let $\lambda_1,\lambda_2,\xi_1$, and ξ_2 be indeterminates. For $r,s\in\mathbb{Z}_{\ge0}$, we have*

$$\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}}\left((-1)^{i_2+j_2}\lambda_1^{i_1}\xi_1^{j_1}\lambda_2^{i_2}\xi_2^{j_2}+(\lambda_1^{i_1}\xi_1^{j_1}+\lambda_2^{i_1}\xi_2^{j_1})(\lambda_1+\lambda_2)^{i_2}(\xi_1+\xi_2)^{j_2}\right)\times\sum_{\substack{\mathbf{k}\in I(i_1+j_1+1,i_1+1)\\ \mathbf{l}\in I(i_2+j_2+1,i_2+1)}}\zeta_{\mathcal{S}}^{ш}(\mathbf{k},\mathbf{l})=0.\tag{1.2}$$

We remark that (1.2) holds without modulo $\zeta(2)\mathcal{Z}$. The weighted sum formulas for FMZVs are proved by Kamano in [3]. According to [3], by setting

$$w(\mathbf{k}) := \begin{cases} 0 & (k_1 > 1), \\ m & (k_1 = \cdots = k_m = 1, k_{m+1} > 1), \end{cases}$$

for $\mathbf{k} = (k_1, \ldots, k_r)$ and specializing the parameter $(\lambda_1, \lambda_2, \xi_1, \xi_2) = (1, 0, 0, 1)$ in (1.2), we have the following corollary.

Corollary 1.1 *For $k \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geq 0}$, we have*

$$\sum_{\mathbf{k} \in I(k+r,k)} w(\mathbf{k}) \zeta_{\mathcal{S}}(\mathbf{k}) = (-1)^{r-1} \zeta_{\mathcal{S}}(\overbrace{1, \ldots, 1}^{k-1}, r+1). \tag{1.3}$$

By substituting $(\lambda_1, \lambda_2, \xi_1, \xi_2) = (1, 1, -1, 1)$ in (1.2), the following corollary holds.

Corollary 1.2 *For $k \in \mathbb{Z}_{>0}$ and an even integer $r \geq 2$, we have*

$$\sum_{\mathbf{k} \in I(k+r,k)} 2^{w(\mathbf{k})} \zeta_{\mathcal{S}}(\mathbf{k}) = 0. \tag{1.4}$$

The proofs of (1.3) and (1.4) are exactly the same as for FMZVs. Thus for the details of Corollary 1.1–1.2, see [3].

Remark Either (1.3) or (1.4) does not hold without modulo $\zeta(2)\mathcal{Z}$ because we need to use the symmetric sum formula [7].

We denote by $\mathfrak{S}_n$ the symmetric group of degree n. For $p, q \in \mathbb{Z}_{\geq 0}$, Kamano introduced in [3]

$$\begin{aligned} W_{p,q} := \; &\{\sigma \in \mathfrak{S}_{p+q+1} \mid \sigma(1) < \cdots < \sigma(p), \sigma(p+1) < \cdots < \sigma(p+q), \\ &\sigma(p+q+1) = p+q+1\}, \end{aligned}$$

and for $\sigma \in W_{p,q}$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)$ and $\boldsymbol{\lambda}' = (\lambda_{p+1}, \ldots, \lambda_{p+q})$, he defined $P_i^{\sigma}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ $(1 \leq i \leq p+q)$ such that

$$\begin{aligned} &\sum_{i=1}^{p-1} \lambda_i \int_{t_i}^{t_{i+1}} \frac{dt'}{t'} + \lambda_p \int_{t_p}^{p+q+1} \frac{dt'}{t'} + \sum_{j=p+1}^{p+q-1} \lambda_j \int_{t_j}^{t_{j+1}} \frac{dt'}{t'} + \lambda_{p+q} \int_{t_{p+q}}^{p+q+1} \frac{dt'}{t'} \\ &= \sum_{i=1}^{p+q} P_i^{\sigma}(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \int_{t_{\sigma^{-1}(i)}}^{t_{\sigma^{-1}(i+1)}} \frac{dt'}{t'}, \end{aligned}$$

which is uniquely determined. Then the argument in [3, Section 3] with the iterated integrals for RSMZVs works well, and we obtain the following theorem.

Theorem 1.1 *For indeterminates λ_m, ξ_m $(1 \le m \le p+q)$ and $r, s \in \mathbb{Z}_{\ge 0}$, we have*

$$\sum_{\substack{|\boldsymbol{i}|=r \\ |\boldsymbol{j}|=s}} (-1)^{q+\sum_{k=1}^{q}(i_{p+k}+j_{p+k})} \prod_{m=1}^{p+q} \lambda_m^{i_m} \xi_m^{j_m}$$
$$\times \sum_{\substack{\mathbf{k}_1 \in I(i_1+j_1+1, i_1+1) \\ \vdots \\ \mathbf{k}_{p+q} \in I(i_{p+q}+j_{p+q}+1, i_{p+q}+1)}} \zeta_{\mathcal{S}}^{ш}(\mathbf{k}_1, \dots, \mathbf{k}_p, \mathbf{k}_{p+q}, \dots, \mathbf{k}_{p+1})$$
$$= \sum_{\sigma \in W_{p+q}} \sum_{\substack{|\boldsymbol{i}|=r \\ |\boldsymbol{j}|=s}} \left(\prod_{m=1}^{p+q} P_m^{\sigma}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')^{i_m} P_m^{\sigma}(\boldsymbol{\xi}, \boldsymbol{\xi}')^{j_m} \right)$$
$$\times \sum_{\substack{\mathbf{k}_1 \in I(i_1+j_1+1, i_1+1) \\ \vdots \\ \mathbf{k}_{p+q} \in I(i_{p+q}+j_{p+q}+1, i_{p+q}+1)}} \zeta_{\mathcal{S}}^{ш}(\mathbf{k}_1, \dots, \mathbf{k}_p, \mathbf{k}_{p+1}, \dots, \mathbf{k}_{p+q}).$$

We state the structure of this article. In Sect. 2, we review the fundamental facts of iterated integral and the definition of RSMZVs. In Sect. 3, by using iterated integrals, we give the proof of our main result.

2 Preparation for proof

To prove main result, let us introduce some notions concerning regularized iterated integrals. Our basic references are [1, 6]. We write a pair of a point $p \in \mathbb{C}$ and a nonzero tangential vector $v \in T_p\mathbb{C} = \mathbb{C}$ as v_p. Fix $a_1, \dots, a_n \in \{0, 1\}$. Let $\gamma : [0, 1] \to \mathbb{C}$ be a cuspidal path, that is, a continuous piecewise smooth map, from v_p to w_q with $\gamma\big((0, 1)\big) \subset \mathbb{C} \setminus \{0, 1\}$, $\gamma(0) = p$, $\gamma'(0) = v$, $\gamma(1) = q$, and $\gamma'(1) = -w$. For the path γ, set a function $F_\gamma : (0, \frac{1}{2}) \to \mathbb{C}$ by

$$F_\gamma(\epsilon) := \int_{\epsilon < t_1 < \cdots < t_n < 1-\epsilon} \prod_{i=1}^{n} \frac{d\gamma(t_i)}{\gamma(t_i) - a_i}.$$

It is known [1, 6] that $F_\gamma(\epsilon)$ has an asymptotic development: there exist complex numbers $c_0, c_1, \dots, c_n \in \mathbb{C}$ such that

$$F_\gamma(\epsilon) = c_0 + \sum_{k=1}^{n} c_k (\log \epsilon)^k + O(\epsilon \log^{n+1} \epsilon)$$

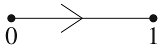

Fig. 1 the path dch

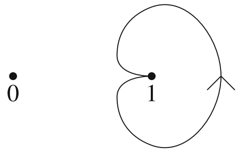

Fig. 2 the path α

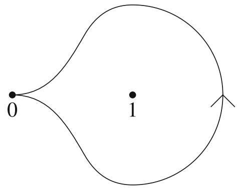

Fig. 3 the path β

as $\epsilon \to 0$. Then the regularized iterated integral $I_\gamma(v_p; a_1, \ldots, a_n; w_q)$ is defined by

$$I_\gamma(v_p; a_1, \ldots, a_n; w_q) := c_0.$$

Note that in a homotopy class, $c_0, c_1, \ldots, c_n$ are independent of the choice of a representative γ.

Hereafter, two tangential base points $0'$ and $1'$ are understood as

$$0' = 1_0, \qquad 1' = (-1)_1.$$

Let dch denote the straight path from $0'$ to $1'$ [1]. By using this path, $\zeta^{Ш}(k_1, \ldots, k_r)$ is given by

$$\zeta^{Ш}(k_1, \ldots, k_r) = (-1)^r I_{\mathrm{dch}}(0'; 1, \{0\}^{k_1-1}, \ldots, 1, \{0\}^{k_r-1}; 1'),$$

where $\{0\}^k$ means $\overbrace{0, \ldots, 0}^{k}$ for $k \in \mathbb{Z}_{\geq 0}$. We define the path α from $1'$ to $1'$ which circles 1 once counterclockwise and the composition path $\beta := \mathrm{dch} \cdot \alpha \cdot \mathrm{dch}^{-1}$ (see Figure 1, 2, and 3).

Let $\alpha^n = \overbrace{\alpha \cdots \alpha}^{n}$ for $n \in \mathbb{Z}_{>0}$.

Lemma 2.1 *(*[1, Theorem 3.253, Example 3.261]*) For any sequence $a_1, \ldots, a_m \in \{0, 1\}$, we have*

$$I_{\mathrm{dch}}(0'; a_1, \ldots, a_m; 1') = (-1)^m I_{\mathrm{dch}^{-1}}(1'; a_m, \ldots, a_1; 0') \in \mathcal{Z},$$

$$I_{\alpha^n}(1';a_1,\ldots,a_m;1')=\begin{cases}\dfrac{(2\pi in)^m}{m!} & (a_1=\cdots=a_m=1),\\ 0 & (\textit{otherwise}).\end{cases}$$

Set $\beta_n = \mathrm{dch}\cdot\alpha^n\cdot\mathrm{dch}^{-1}$. According to [6, Proposition 4], by Lemma 2.1, the iterated integral $I_{\beta_n}(0';a_1,\ldots,a_m;0')$ with $(a_1,\ldots,a_m)=(1,\{0\}^{k_1-1},\ldots,1,\{0\}^{k_r-1},1)$ is written as

$$\begin{aligned}
&I_{\beta_n}(0';a_1,\ldots,a_m;0')\\
&=\sum_{0\le k\le l\le m} I_{\mathrm{dch}}(0';a_1,\ldots,a_k;1')I_{\alpha^n}(1';a_{k+1},\ldots,a_l;1')\\
&\quad\times I_{\mathrm{dch}^{-1}}(1';a_{l+1},\ldots,a_m;0')\\
&=\sum_{\substack{0\le k<l\le m\\ a_j=1\ (k<\forall j\le l)}}\frac{(2\pi in)^{k-l}}{(k-l)!}I_{\mathrm{dch}}(0';a_1,\ldots,a_k;1')I_{\mathrm{dch}^{-1}}(1';a_{l+1},\ldots,a_m;0')\\
&=\sum_{\substack{0\le a\le b\le r\\ k_j=1\ (a<\forall j\le b)}}\frac{(2\pi in)^{b-a+1}}{(b-a+1)!}I_{\mathrm{dch}}(0';1,\{0\}^{k_1-1},\ldots,1,\{0\}^{k_a-1};1')\\
&\quad\times I_{\mathrm{dch}^{-1}}(1';\{0\}^{k_{b+1}-1},1,\ldots,\{0\}^{k_r-1},1;0')\\
&=(-1)^r 2\pi in\sum_{\substack{0\le a\le b\le r\\ k_j=1\ (a<\forall j\le b)}}\frac{(-2\pi in)^{b-a}}{(b-a+1)!}(-1)^{k_r+\cdots+k_{b+1}}\zeta^{\text{ш}}(k_1,\ldots,k_a)\\
&\quad\times\zeta^{\text{ш}}(k_r,\ldots,k_{b+1})\in 2\pi in\mathcal{Z}[2\pi in].
\end{aligned}$$

Since the equation above holds for all $n\in\mathbb{Z}_{>0}$, we replace $2\pi in$ with T as follows. For $a_1,\ldots,a_m\in\{0,1\}$, there uniquely exists $L(a_1,\ldots,a_m;T)\in T\mathcal{Z}[T]$ such that

$$I_{\beta_n}(0';a_1,\ldots,a_m;0')=L(a_1,\ldots,a_m;2\pi in).$$

Thus we put

$$\zeta_{RS}(k_1,\ldots,k_r;T):=\frac{(-1)^r}{T}L(1,\{0\}^{k_1-1},\ldots,1,\{0\}^{k_r-1},1;T)$$

and get

$$\zeta_{RS}(k_1,\ldots,k_r;0)=\zeta_{\mathcal{S}}^{\text{ш}}(k_1,\ldots,k_r),$$

which plays an important role in the proof of the main result. Note that $L(a_1,\ldots,a_m;T)$ is an abbreviation of the symbol $L(e_{a_1}\cdots e_{a_m};T)$ introduced in [6].

Lemma 2.2 *(cf. [2, Lemma 2.1])*

$$\left(\int_t^X \frac{d\beta_n(t')}{\beta_n(t')-1}\right)^i = i! \int\limits_{t<p_1<\cdots<p_i<X} \prod_{a=1}^{i} \frac{d\beta_n(p_a)}{\beta_n(p_a)-1},$$

$$\left(\int_t^X \frac{d\beta_n(t')}{\beta_n(t')}\right)^j = j! \int\limits_{t<q_1<\cdots<q_j<X} \prod_{b=1}^{j} \frac{d\beta_n(q_b)}{\beta_n(q_b)}.$$

Proof We have

$$\begin{aligned}
\left(\int_t^X \frac{d\beta_n(t')}{\beta_n(t')-1}\right)^i &= \int\limits_{\substack{t<p_1<X \\ \vdots \\ t<p_i<X}} \prod_{m=1}^{i} \frac{d\beta_n(p_m)}{\beta_n(p_m)-1} \\
&= \sum_{\sigma\in\mathfrak{S}_i} \int\limits_{t<p_{\sigma(1)}<\cdots<p_{\sigma(i)}<X} \prod_{m=1}^{i} \frac{d\beta_n(p_m)}{\beta_n(p_m)-1} \\
&= i! \int\limits_{t<p_1<\cdots<p_i<X} \prod_{m=1}^{i} \frac{d\beta_n(p_m)}{\beta_n(p_m)-1},
\end{aligned}$$

which implies the first equality. By the same method, we easily get the second equality, and complete the proof. □

The following lemma is a key of the proof of our main result.

Lemma 2.3 *Fix sufficient small $\epsilon > 0$. For $X \in [\epsilon, 1-\epsilon]$ and a continuous function $f : \beta_n((\epsilon, 1-\epsilon))^m \to \mathbb{C}$, we have*

$$\begin{aligned}
&\int_{\beta_n^{-1}(1-X)}^{\beta_n^{-1}(1-\epsilon)} dz_m \int_{\beta_n^{-1}(1-X)}^{z_m} dz_{m-1} \cdots \int_{\beta_n^{-1}(1-X)}^{z_2} dz_1 f(z_1,\ldots,z_m) \\
&= \sum_{i=0}^{m} \left(\int_{\beta_n^{-1}(\epsilon)}^{\beta_n^{-1}(1-\epsilon)} dz_m \int_{\beta_n^{-1}(\epsilon)}^{z_n} dz_{m-1} \cdots \int_{\beta_n^{-1}(\epsilon)}^{z_{i+2}} dz_{i+1} \right) \\
&\quad \times \left(\int_{\beta_n(X)}^{\beta_n(1-\epsilon)} dz_i \int_{\beta_n(X)}^{z_i} dz_{i-1} \cdots \int_{\beta_n(X)}^{z_2} dz_1 \right) f(z_1,\ldots,z_m).
\end{aligned}$$

Proof The key idea is a replacement of the path from $\beta_n^{-1}(1-X)$ to $z \in \beta_n((\epsilon, 1-\epsilon))$ by two paths: one starts at $\beta_n^{-1}(1-X) = \beta_n(X)$ and goes backward to $\beta_n(1-\epsilon)$, and the other starts at $\beta_n(1-\epsilon) = \beta_n^{-1}(\epsilon)$ and goes forward to z. See Figure 4 in the case $n = 1$.

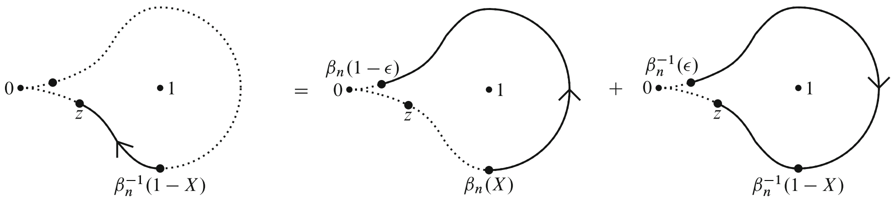

Fig. 4 replacement of the path β_n^{-1}

We repeatedly apply this separating method to each iterated integral as follows:

$$\begin{aligned}
&\int_{\beta_n^{-1}(1-X)}^{\beta_n^{-1}(1-\epsilon)} dz_m \int_{\beta_n^{-1}(1-X)}^{z_m} dz_{m-1} \cdots \int_{\beta_n^{-1}(1-X)}^{z_2} dz_1 f(z_1,\ldots,z_m)\\
&= \left(\int_{\beta_n(X)}^{\beta_n(1-\epsilon)} + \int_{\beta_n^{-1}(\epsilon)}^{\beta_n^{-1}(1-\epsilon)}\right) dz_m\\
&\quad \times \int_{\beta_n^{-1}(1-X)}^{z_m} dz_{m-1} \cdots \int_{\beta_n^{-1}(1-X)}^{z_2} dz_1 f(z_1,\ldots,z_m)\\
&= \int_{\beta_n(X)}^{\beta_n(1-\epsilon)} dz_m \int_{\beta_n(X)}^{z_m} dz_{m-1} \cdots \int_{\beta_n(X)}^{z_2} dz_1 f(z_1,\ldots,z_m)\\
&\quad + \int_{\beta_n^{-1}(\epsilon)}^{\beta_n^{-1}(1-\epsilon)} dz_m \left(\int_{\beta_n(X)}^{\beta_n(1-\epsilon)} + \int_{\beta_n^{-1}(\epsilon)}^{z_m}\right) dz_{m-1} \int_{\beta_n^{-1}(1-X)}^{z_{m-1}} dz_{m-2} \cdots\\
&\quad \times \int_{\beta_n^{-1}(1-X)}^{z_2} dz_1 f(z_1,\ldots,z_m)\\
&= \int_{\beta_n(X)}^{\beta_n(1-\epsilon)} dz_m \int_{\beta_n(X)}^{z_m} dz_{m-1} \cdots\\
&\quad \times \int_{\beta_n(X)}^{z_2} dz_1 f(z_1,\ldots,z_m)\\
&\quad + \int_{\beta_n^{-1}(\epsilon)}^{\beta_n^{-1}(1-\epsilon)} dz_m \int_{\beta_n(X)}^{\beta_n(1-\epsilon)} dz_{m-1} \int_{\beta_n(X)}^{z_{m-1}} dz_{m-2} \cdots \int_{\beta_n(X)}^{z_2} dz_1 f(z_1,\ldots,z_m)\\
&\quad + \int_{\beta_n^{-1}(\epsilon)}^{\beta_n^{-1}(1-\epsilon)} dz_m \int_{\beta_n^{-1}(\epsilon)}^{z_m} dz_{m-1} \left(\int_{\beta_n(X)}^{\beta_n(1-\epsilon)} + \int_{\beta_n^{-1}(\epsilon)}^{z_{m-1}}\right) dz_{m-2}\\
&\quad \times \int_{\beta_n^{-1}(1-X)}^{z_{m-2}} dz_{m-3} \cdots \int_{\beta_n^{-1}(1-X)}^{z_2} dz_1 f(z_1,\ldots,z_m)\\
&= \cdots = \sum_{i=0}^{m} \left(\int_{\beta_n^{-1}(\epsilon)}^{\beta_n^{-1}(1-\epsilon)} dz_m \int_{\beta_n^{-1}(\epsilon)}^{z_m} dz_{m-1} \cdots \int_{\beta_n^{-1}(\epsilon)}^{z_{i+2}} dz_{i+1}\right)\\
&\quad \times \left(\int_{\beta_n(X)}^{\beta_n(1-\epsilon)} dz_i \int_{\beta_n(X)}^{z_i} dz_{i-1} \cdots \int_{\beta_n(X)}^{z_2} dz_1\right) f(z_1,\ldots,z_m),
\end{aligned}$$

which completes the proof. □

Remark Although we showed Lemma 2.2 and 2.3 directly, they are consequences of general formulas:

for a cuspidal path $\gamma : [0,1] \to \mathbb{C}$ from a point x to a point y and $a, b \in (0,1)$, consider the possibly divergent integral

$$\int_{\gamma,a,b} w_{a_1} \cdots w_{a_m} := \int_{a<t_1<\cdots<t_m<b} \frac{d\gamma(t_1)}{\gamma(t_1)-a_1} \cdots \frac{d\gamma(t_m)}{\gamma(t_m)-a_m}$$

with $a_1, \ldots, a_m \in (0,1)$. Then, Lemma 2.2 follows from the shuffle product

$$\int_{\gamma,a,b} w_{a_1} \cdots w_{a_m} \int_{\gamma,a,b} w_{b_1} \cdots w_{b_n} = \int_{\gamma,a,b} w_{a_1} \cdots w_{a_m} ш\, w_{b_1} \cdots w_{b_n},$$

where $w_{a_1} \cdots w_{a_m} ш\, w_{b_1} \cdots w_{b_n}$ with $b_1, \ldots, b_n \in (0,1)$ means that all permutations of $w_{a_1} \cdots w_{a_m}$ and $w_{b_1} \cdots w_{b_n}$ with its order kept. Moreover, Lemma 2.3 is essentially a combination of the path composition formula

$$\int_{\gamma,a,b} w_{a_1} \cdots w_{a_m} = \sum_{j=0}^{m} \int_{\gamma,a,c} w_{a_1} \cdots w_{a_j} \int_{\gamma,c,b} w_{a_{j+1}} \cdots w_{a_m}$$

and

$$\int_{\gamma,a,b} w_{a_1} \cdots w_{a_m} = \int_{\gamma^{-1},1-a,1-b} w_{a_1} \cdots w_{a_m}.$$

Remark Lemma 2.3 plays the same role as in [3, Lemma 2.1]. We will see below in (3.2) that only the term $i = m$ remains and the other terms vanish by modulo $2\pi i n \mathcal{Z}[2\pi i n]$.

3 Proof of main theorem

Proof (Proof of Main Theorem) Fix sufficiently small $\epsilon > 0$. For $X \in [\epsilon, 1-\epsilon]$, we define

$$\begin{aligned} I_{r,s}(X) &= I_{r,s}(\lambda_1, \lambda_2, \xi_1, \xi_2; X) \\ &:= \frac{1}{r!s!} \int_{\substack{\epsilon<t<X \\ \epsilon<u<X}} \left(\lambda_1 \int_t^X \frac{d\beta_n(t')}{\beta_n(t')-1} + \lambda_2 \int_u^X \frac{d\beta_n(u')}{\beta_n(u')-1} \right)^r \\ &\quad \times \left(\xi_1 \int_t^X \frac{d\beta_n(t')}{\beta_n(t')} + \xi_2 \int_u^X \frac{d\beta_n(u')}{\beta_n(u')} \right)^s \times \frac{d\beta_n(t) d\beta_n(u)}{(\beta_n(t)-1)(\beta_n(u)-1)}. \end{aligned}$$

We calculate $I_{r,s}(X)$ in two ways. By using the binomial expansion and Lemma 2.2, We have

$$\begin{aligned}
&I_{r,s}(X)\\
&=\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}}\frac{\lambda_1^{i_1}\xi_1^{j_1}\lambda_2^{i_2}\xi_2^{j_2}}{i_1!i_2!j_1!j_2!}\int_{\epsilon<t<X}\frac{d\beta_n(t)}{\beta_n(t)-1}\left(\int_t^X\frac{d\beta_n(t')}{\beta_n(t')-1}\right)^{i_1}\left(\int_t^X\frac{d\beta_n(t')}{\beta_n(t')}\right)^{j_1}\\
&\quad\times\int_{\epsilon<u<X}\frac{d\beta_n(u)}{\beta_n(u)-1}\left(\int_u^X\frac{d\beta_n(u')}{\beta_n(u')-1}\right)^{i_2}\left(\int_u^X\frac{d\beta_n(u')}{\beta_n(u')}\right)^{j_2}\\
&=\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}}\lambda_1^{i_1}\xi_1^{j_1}\lambda_2^{i_2}\xi_2^{j_2}\int_{\substack{\epsilon<t<p_1<\cdots<p_{i_1}<X\\ \epsilon<t<q_1<\cdots<q_{j_1}<X\\ \epsilon<u<x_1<\cdots<x_{i_2}<X\\ \epsilon<u<y_1<\cdots<y_{j_2}<X}}\frac{d\beta_n(t)}{\beta_n(t)-1}\prod_{a=1}^{i_1}\frac{d\beta_n(p_a)}{\beta_n(p_a)-1}\\
&\quad\times\prod_{b=1}^{j_1}\frac{d\beta_n(q_b)}{\beta_n(q_b)}\frac{d\beta_n(u)}{\beta_n(u)-1}\prod_{c=1}^{i_2}\frac{d\beta_n(x_c)}{\beta_n(x_c)-1}\prod_{d=1}^{j_2}\frac{d\beta_n(y_d)}{\beta_n(y_d)}\\
&=\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}}\lambda_1^{i_1}\xi_1^{j_1}\lambda_2^{i_2}\xi_2^{j_2}\sum_{\substack{a_1+\cdots+a_{i_1+j_1}=i_1\\ b_1+\cdots+b_{i_2+j_2}=i_2}}\int_{\substack{\epsilon<t<\mu_1<\cdots<\mu_{i_1+j_1}<X\\ \epsilon<u<\nu_1<\cdots<\nu_{i_2+j_2}<X}}\frac{d\beta_n(t)}{\beta_n(t)-1}\\
&\quad\times\prod_{k=1}^{i_1+j_1}\frac{d\beta_n(\mu_k)}{\beta_n(\mu_k)-a_k}\frac{d\beta_n(u)}{\beta_n(u)-1}\prod_{l=1}^{i_2+j_2}\frac{d\beta_n(\nu_l)}{\beta_n(\nu_l)-b_l},
\end{aligned}$$

where we put

$$a_k=\begin{cases}1 & \text{if } \mu_k \text{ is one of } p_1,\ldots,p_{i_1},\\ 0 & \text{if } \mu_k \text{ is one of } q_1,\ldots,q_{j_1},\end{cases}\qquad b_l=\begin{cases}1 & \text{if } \nu_l \text{ is one of } x_1,\ldots,x_{i_2},\\ 0 & \text{if } \nu_l \text{ is one of } y_1,\ldots,y_{j_2},\end{cases}$$

for $1\le k\le i_1+j_1$ and $1\le l\le i_2+j_2$. Since $\beta_n^{-1}(t)=\beta_n(1-t)$, we have

$$\begin{aligned}
&\int_{\epsilon<u<\nu_1<\cdots<\nu_{i_2+j_2}<X}\frac{d\beta_n(u)}{\beta_n(u)-1}\prod_{l=1}^{i_2+j_2}\frac{d\beta_n(\nu_l)}{\beta_n(\nu_l)-b_l}\\
&\quad=(-1)^{i_2+j_2+1}\int_{1-X<\nu_{i_2+j_2}<\cdots<\nu_1<u<1-\epsilon}\prod_{l=1}^{i_2+j_2}\frac{d\beta_n^{-1}(\nu_l)}{\beta_n^{-1}(\nu_l)-b_l}\frac{d\beta_n^{-1}(u)}{\beta_n^{-1}(u)-1}.
\end{aligned}$$

From Lemma 2.3, we obtain

$$(-1)^{i_2+j_2+1}\int\limits_{1-X<\nu_{i_2+j_2}<\cdots<\nu_1<u<1-\epsilon}\prod_{l=1}^{i_2+j_2}\frac{d\beta_n^{-1}(\nu_l)}{\beta_n^{-1}(\nu_l)-b_l}\frac{d\beta_n^{-1}(u)}{\beta_n^{-1}(u)-1}$$
$$=(-1)^{i_2+j_2+1}\Bigg(\int\limits_{X<\nu_{i_2+j_2}<\cdots<\nu_1<u<1-\epsilon}\prod_{l=1}^{i_2+j_2}\frac{d\beta_n(\nu_l)}{\beta_n(\nu_l)-b_l}\frac{d\beta_n(u)}{\beta_n(u)-1}$$
$$+\sum_{i=1}^{i_2+j_2+1}\int\limits_{X<\nu_{i_2+j_2}<\cdots<\nu_i<1-\epsilon}\prod_{l=i}^{i_2+j_2}\frac{d\beta_n(\nu_l)}{\beta_n(\nu_l)-b_l}$$
$$\times\int\limits_{\epsilon<\nu_{i-1}<\cdots<\nu_1<u<1-\epsilon}\prod_{m=1}^{i-1}\frac{d\beta_n^{-1}(\nu_m)}{\beta_n^{-1}(\nu_m)-b_m}\frac{d\beta_n^{-1}(u)}{\beta_n^{-1}(u)-1}\Bigg),$$

Thus we have

$$\frac{(-1)^{r+2}}{2\pi in}\int\limits_{\epsilon<X<1-\epsilon}I_{r,s}(X)\frac{d\beta_n(X)}{\beta_n(X)-1}$$
$$=\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}}(-1)^{i_2+j_2+1}\lambda_1^{i_1}\xi_1^{j_1}\lambda_2^{i_2}\xi_2^{j_2}\sum_{\substack{a_1+\cdots+a_{i_1+j_1}=i_1\\ b_1+\cdots+b_{i_2+j_2}=i_2}}\frac{(-1)^{k+2}}{2\pi in}$$
$$\times\int\limits_{\epsilon<t<\mu_1<\cdots<\mu_{i_1+j_1}<X<\nu_{i_2+j_2}<\cdots<\nu_1<u<1-\epsilon}\frac{d\beta_n(t)}{\beta_n(t)-1}$$
$$\times\prod_{k=1}^{i_1+j_1}\frac{d\beta_n(\mu_k)}{\beta_n(\mu_k)-a_k}\frac{d\beta_n(X)}{\beta_n(X)-1}$$
$$\times\prod_{l=1}^{i_2+j_2}\frac{d\beta_n(\nu_l)}{\beta_n(\nu_l)-b_l}\frac{d\beta_n(u)}{\beta_n(u)-1}$$
$$+P_{r,s}^{\epsilon}(2\pi in), \tag{3.1}$$

where $P_{r,s}^{\epsilon}(2\pi in)$ is given by

$$P_{r,s}^{\epsilon}(2\pi in)=\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}}(-1)^{i_2+j_2+1}\lambda_1^{i_1}\xi_1^{j_1}\lambda_2^{i_2}\xi_2^{j_2}\sum_{\substack{a_1+\cdots+a_{i_1+j_1}=i_1\\ b_1+\cdots+b_{i_2+j_2}=i_2}}\sum_{i=1}^{i_2+j_2+1}\frac{(-1)^{k+i+2}}{2\pi in}$$
$$\times\int\limits_{\substack{\epsilon<t<\mu_1<\cdots<\mu_{i_1+j_1}<X<\nu_{i_2+j_2}<\cdots<\nu_i<1-\epsilon\\ \epsilon<u<\nu_1<\cdots<\nu_{i-1}<1-\epsilon}}\frac{d\beta_n(t)}{\beta_n(t)-1}$$

$$\times \prod_{k=1}^{i_1+j_1} \frac{d\beta_n(\mu_k)}{\beta_n(\mu_k)-a_k} \frac{d\beta_n(X)}{\beta_n(X)-1} \prod_{l=i}^{i_2+j_2} \frac{d\beta_n(\nu_l)}{\beta_n(\nu_l)-b_l}$$
$$\times \frac{d\beta_n(u)}{\beta_n(u)-1} \prod_{m=1}^{i-1} \frac{d\beta_n(\nu_m)}{\beta_n(\nu_m)-b_m}.$$

Hence the regularization of (3.1) with the replacement $2\pi i n$ by T is equal to

$$\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} (-1)^{i_2+j_2+1} \lambda_1^{i_1} \xi_1^{j_1} \lambda_2^{i_2} \xi_2^{j_2}$$
$$\times \sum_{\substack{\mathbf{k}\in I(i_1+j_1+1,i_1+1)\\ \mathbf{l}\in I(i_2+j_2+1,i_2+1)}} \Biggl(\zeta_{RS}(\mathbf{k},\mathbf{l};T) + \sum_{i=0}^{i_2} \sum_{m=0}^{l_i-1} \frac{(-1)^{k+i+2}}{T}$$
$$\times L(1,\{0\}^{k_0-1},\ldots,1,\{0\}^{k_{i_1}-1},1,\{0\}^{l_{i_2}-1},1,\ldots,\{0\}^{l_{i+1}-1},1,\{0\}^{m+1};T)$$
$$\times L(1,\{0\}^{l_0-1},\ldots,1,\{0\}^{l_i-m};T)\Biggr). \tag{3.2}$$

Next, we give another expression of $I_{r,s}(X)$. By dividing the region of the integral, we have

$$I_{r,s}(X) = I_{r,s}^{(1)}(X) + I_{r,s}^{(2)}(X),$$

where we set

$$I_{r,s}^{(1)}(X) := \frac{1}{r!s!} \int\limits_{\epsilon<t<u<X} \cdots,$$
$$I_{r,s}^{(2)}(X) := \frac{1}{r!s!} \int\limits_{\epsilon<u<t<X} \cdots.$$

Using the binomial expansion and Lemma 2.2, We have

$$I_{r,s}^{(1)}(X)$$
$$= \frac{1}{r!s!} \int\limits_{\epsilon<t<u<X} \left(\lambda_1 \int_t^u \frac{d\beta_n(t')}{\beta_n(t')-1} + (\lambda_1+\lambda_2) \int_u^X \frac{d\beta_n(u')}{\beta_n(u')-1}\right)^r$$
$$\times \left(\xi_1 \int_t^u \frac{d\beta_n(t')}{\beta_n(t')} + (\xi_1+\xi_2) \int_u^X \frac{d\beta_n(u')}{\beta_n(u')}\right)^s \frac{d\beta_n(t)d\beta_n(u)}{(\beta_n(t)-1)(\beta_n(u)-1)}$$
$$= \sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} \frac{\lambda_1^{i_1}(\lambda_1+\lambda_2)^{i_2}\xi_1^{j_1}(\xi_1+\xi_2)^{j_2}}{i_1!i_2!j_1!j_2!}$$

$$\times \int_{\epsilon<t<u<X} \frac{d\beta_n(t)}{\beta_n(t)-1}\left(\int_t^u \frac{d\beta_n(t')}{\beta_n(t')-1}\right)^{i_1}\left(\int_t^u \frac{d\beta_n(t')}{\beta_n(t')}\right)^{j_1}$$
$$\times\left(\int_u^X \frac{d\beta_n(u')}{\beta_n(u')-1}\right)^{i_2}\left(\int_u^X \frac{d\beta_n(u')}{\beta_n(u')}\right)^{j_2}\frac{d\beta_n(u)}{\beta_n(u)-1}$$
$$= \sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} \lambda_1^{i_1}(\lambda_1+\lambda_2)^{i_2}\xi_1^{j_1}(\xi_1+\xi_2)^{j_2}$$
$$\times \int_{\substack{\epsilon<t<p_1<\cdots<p_{i_1}<u<x_1<\cdots<x_{i_2}<X\\ \epsilon<t<q_1<\cdots<q_{j_1}<u<y_1<\cdots<y_{j_2}<X}} \frac{d\beta_n(t)}{\beta_n(t)-1}\prod_{a=1}^{i_1}\frac{d\beta_n(p_a)}{\beta_n(p_a)-1}$$
$$\times \prod_{b=1}^{j_1}\frac{d\beta_n(q_b)}{\beta_n(q_b)}\frac{d\beta_n(u)}{\beta_n(u)-1}\prod_{c=1}^{i_2}\frac{d\beta_n(x_c)}{\beta_n(x_c)-1}\prod_{d=1}^{j_2}\frac{d\beta_n(y_d)}{\beta_n(y_d)}$$
$$= \sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} \lambda_1^{i_1}(\lambda_1+\lambda_2)^{i_2}\xi_1^{j_1}(\xi_1+\xi_2)^{j_2}$$
$$\times \sum_{\substack{a_1+\cdots+a_{i_1+j_1}=i_1\\ b_1+\cdots+b_{i_2+j_2}=i_2}} \int_{\epsilon<t<\mu_1<\cdots<\mu_{i_1+j_1}<u<\nu_1<\cdots<\nu_{i_2+j_2}<X} \frac{d\beta_n(t)}{\beta_n(t)-1}$$
$$\times \prod_{k=1}^{i_1+j_1}\frac{d\beta_n(\mu_k)}{\beta_n(\mu_k)-a_k}\frac{d\beta_n(u)}{\beta_n(u)-1}\prod_{l=1}^{i_2+j_2}\frac{d\beta_n(\nu_l)}{\beta_n(\nu_l)-b_l}.$$

So we obtain

$$\int_{\epsilon<X<1-\epsilon} I_{r,s}^{(1)}(X)\frac{d\beta_n(X)}{\beta_n(X)-1}$$
$$= \sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} \lambda_1^{i_1}(\lambda_1+\lambda_2)^{i_2}\xi_1^{j_1}(\xi_1+\xi_2)^{j_2}$$
$$\times \sum_{\substack{a_1+\cdots+a_{i_1+j_1}=i_1\\ b_1+\cdots+b_{i_2+j_2}=i_2}} \int_{\epsilon<t<\mu_1<\cdots<\mu_{i_1+j_1}<u<\nu_1<\cdots<\nu_{i_2+j_2}<X<1-\epsilon} \frac{d\beta_n(t)}{\beta_n(t)-1}$$
$$\times \prod_{k=1}^{i_1+j_1}\frac{d\beta_n(\mu_k)}{\beta_n(\mu_k)-a_k}\frac{d\beta_n(u)}{\beta_n(u)-1}\prod_{l=1}^{i_2+j_2}\frac{d\beta_n(\nu_l)}{\beta_n(\nu_l)-b_l}\frac{d\beta_n(X)}{\beta_n(X)-1}.$$

Similarly, the calculation of $I_{r,s}^{(2)}(X)$ yields

$$\int_{\epsilon<X<1-\epsilon} I_{r,s}^{(2)}(X)\frac{d\beta_n(X)}{\beta_n(X)-1}$$

$$= \sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} \lambda_2^{i_1}(\lambda_1+\lambda_2)^{i_2}\xi_2^{j_1}(\xi_1+\xi_2)^{j_2}$$
$$\times \sum_{\substack{a_1+\cdots+a_{i_1+j_1}=i_1\\ b_1+\cdots+b_{i_2+j_2}=i_2}} \int_{\epsilon<t<\mu_1<\cdots<\mu_{i_1+j_1}<u<\nu_1<\cdots<\nu_{i_2+j_2}<X<1-\epsilon} \frac{d\beta_n(t)}{\beta_n(t)-1}$$
$$\times \prod_{k=1}^{i_1+j_1} \frac{d\beta_n(\mu_k)}{\beta_n(\mu_k)-a_k}\frac{d\beta_n(u)}{\beta_n(u)-1}\prod_{l=1}^{i_2+j_2}\frac{d\beta_n(\nu_l)}{\beta_n(\nu_l)-b_l}\frac{d\beta_n(X)}{\beta_n(X)-1}.$$

Thus

$$\frac{(-1)^{r+2}}{2\pi in}\int_{\epsilon<X<1-\epsilon} I_{r,s}(X)\frac{d\beta_n(X)}{\beta_n(X)-1}$$
$$= \frac{(-1)^{r+2}}{2\pi in}\int_{\epsilon<X<1-\epsilon}\left(I_{r,s}^{(1)}(X)+I_{r,s}^{(2)}(X)\right)\frac{d\beta_n(X)}{\beta_n(X)-1}$$
$$= \sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} (\lambda_1^{i_1}\xi_1^{j_1}+\lambda_2^{i_1}\xi_2^{j_1})(\lambda_1+\lambda_2)^{i_2}(\xi_1+\xi_2)^{j_2} \sum_{\substack{a_1+\cdots+a_{i_1+j_1}=i_1\\ b_1+\cdots+b_{i_2+j_2}=i_2}} \frac{(-1)^{k+2}}{2\pi in}$$
$$\times \int_{\epsilon<t<\mu_1<\cdots<\mu_{i_1+j_1}<u<\nu_1<\cdots<\nu_{i_2+j_2}<X<1-\epsilon} \frac{d\beta_n(t)}{\beta_n(t)-1}$$
$$\times \prod_{k=1}^{i_1+j_1} \frac{d\beta_n(\mu_k)}{\beta_n(\mu_k)-a_k}\frac{d\beta_n(u)}{\beta_n(u)-1}\prod_{l=1}^{i_2+j_2}\frac{d\beta_n(\nu_l)}{\beta_n(\nu_l)-b_l}\frac{d\beta_n(X)}{\beta_n(X)-1}. \tag{3.3}$$

Hence the regularization of (3.3) with the replacement $2\pi in$ by T is equal to

$$\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} (\lambda_1^{i_1}\xi_1^{j_1}+\lambda_2^{i_1}\xi_2^{j_1})(\lambda_1+\lambda_2)^{i_2}(\xi_1+\xi_2)^{j_2} \sum_{\substack{\mathbf{k}\in I(i_1+j_1+1,i_1+1)\\ \mathbf{l}\in I(i_2+j_2+1,i_2+1)}} \zeta_{RS}(\mathbf{k},\mathbf{l};T). \tag{3.4}$$

By (3.2) and (3.4), we have

$$\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} \left((-1)^{i_2+j_2}\lambda_1^{i_1}\xi_1^{j_1}\lambda_2^{i_2}\xi_2^{j_2}+(\lambda_1^{i_1}\xi_1^{j_1}+\lambda_2^{i_1}\xi_2^{j_1})(\lambda_1+\lambda_2)^{i_2}(\xi_1+\xi_2)^{j_2}\right)$$
$$\times \sum_{\substack{\mathbf{k}\in I(i_1+j_1+1,i_1+1)\\ \mathbf{l}\in I(i_2+j_2+1,i_2+1)}} \zeta_{RS}(\mathbf{k},\mathbf{l};T) = P_{r,s}^{\mathrm{reg}}(T), \tag{3.5}$$

where $P^{\mathrm{reg}}_{r,s}(T)$ is equal to

$$\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} (-1)^{i_2+j_2+1}\lambda_1^{i_1}\xi_1^{j_1}\lambda_2^{i_2}\xi_2^{j_2}$$
$$\times \sum_{\substack{\mathbf{k}\in I(i_1+j_1+1,i_1+1)\\ \mathbf{l}\in I(i_2+j_2+1,i_2+1)}} \sum_{i=0}^{i_2}\sum_{m=0}^{l_i-1} \frac{(-1)^{k+i}}{T}$$
$$\times L(1,\{0\}^{k_0-1},\dots,1,\{0\}^{k_{i_1}-1},1,\{0\}^{l_{i_2}-1},1,\dots,\{0\}^{l_{i+1}-1},1,\{0\}^{m+1};T)$$
$$\times L(1,\{0\}^{l_0-1},\dots,1,\{0\}^{l_i-m};T).$$

We see that $P^{\mathrm{reg}}_{r,s}(T)$ is a polynomial in T without constant term because $L(a_1,\dots,a_m;T)\in T\mathcal{Z}[T]$. Therefore, by putting $T=0$ in (3.5), we have (1.2). This completes the proof. □

Acknowledgements The authors would like to thank M. Hirose for helpful comments and detailed information about refined symmetric multiple zeta values. We are grateful to anonymous referees for valuable advice including the shuffle product and iterated integrals.

References

1. Burgos Gil, J.I., J. Fresán, J.: Multiple zeta values: from numbers to motives (To Appear)
2. Kadota, S.: Certain weighted sun formulas for multiple zeta values with some parameters. Comment. Math. Univ. St. Pauli **66**, 1–13 (2017)
3. Kamano, K.: Weighted sum formulas for finite multiple zeta values. J. Number Theory **192**, 168–180 (2018)
4. Ihara, K., Kaneko, M., Zagier, D.: Derivation and double shuffle relations for multiple zeta values. Compos. Math. **142**, 307–338 (2006)
5. Kaneko, M., Zagier, D.: Finite Multiple Zeta Values (In Preparation)
6. Hirose, M.: Double shuffle relations for refined symmetric multiple zeta values. Doc. Math. **25**, 365–380 (2020)
7. Murahara, H.: A note on finite real multiple zeta values. Kyushu J. Math. **70**, 197–204 (2016)