The Ramanujan Journal (2023) 60:287–294
https://doi.org/10.1007/s11139-022-00643-8

Proofs of conjectures of Chan for $d(n)$

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Received: 5 September 2019 / Accepted: 6 August 2022 / Published online: 11 October 2022

Abstract
In this paper, we prove some congruences for the coefficients of a function related to Ramanujan's sixth-order mock theta function $\phi(q)$, which was conjectured by Song Heng Chan.

Keywords Partial theta function · Mock theta function · Congruence

Mathematics subject classification 11P83 · 33D15

1 Introduction

Here and in what follows, we have made use of the standard q-series notation [9]

$$\begin{aligned}(a;q)_0 &:= 1,\\ (a;q)_n &:= \prod_{k=0}^{n-1}(1-aq^k), \quad n\geq 1,\\ (a;q)_\infty &:= \prod_{k=0}^{\infty}(1-aq^k), \quad |q|<1\end{aligned}$$

and

$$(a_1,a_2,\cdots,a_k;q)_n := (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n.$$

We thank the referee for valuable comments and Dazhao Tang's suggestions. This work was supported by the National Natural Science Foundation of China (Grant No. 12001309) and the Natural Science Foundation Youth Fund of Qinghai (Grant No. 2022-ZJ-972Q).

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We define

$$j(z;q) := (z;q)_\infty (q/z;q)_\infty (q;q)_\infty.$$

For convenience, define f_k as

$$f_k := (q^k;q^k)_\infty.$$

Definition 1.1 Let $x, z \in C^*$ with neither z nor xz an integer power of q. Then

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} z}{1 - q^{r-1}xz}.$$

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Ramanujan found the following famous congruences:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5},\\ p(7n+5) &\equiv 0 \pmod{7},\\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Recently, arithmetic properties of congruences for generating functions have received a great deal of attention. For example, Andrews [1] found the smallest part partition function and established that for $n \geq 0$,

$$\begin{aligned} spt(5n+4) &\equiv 0 \pmod{5},\\ spt(7n+5) &\equiv 0 \pmod{7},\\ spt(13n+6) &\equiv 0 \pmod{13}, \end{aligned}$$

where $spt(n)$ denotes the number of the smallest part partitions of n. Ramanujan [10] presented the following function which is related to one of the Ramanujan's sixth-order mock theta functions:

$$\phi(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{n+1}}{(q;q^2)_{n+1}^2}.$$

In addition, Ramanujan [10] provided a rank type generating function which is stated as follows.

$$\sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{n+1}}{(xq, q/x; q^2)_{n+1}}.$$

Notice that $\phi(q)$ can be derived by setting $x = 1$ in the above function. Chan [4] obtained that

$$\phi(q) = \frac{1}{2} \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{1-q^{2n+1}}.$$

Motivated by a search for similar functions which satisfy simple congruences, Chan [4] presented some congruences for mock theta functions. Let

$$\sum_{n=1}^{\infty} a(n) q^n := \phi(q).$$

Chan found that for any integer $p \geq 2$ and $1 \leq j \leq p-1$ with p and j coprime, the coefficients of the Appell–Lerch sum

$$\sum_{n=0}^{\infty} a_{j,p}(n) q^n = \frac{1}{(q^j;q^p)_\infty (q^{p-j};q^p)_\infty (q^p;q^p)_\infty} \sum_{-\infty}^{\infty} \frac{(-1)^n q^{pn(n+1)/2+jn+j}}{1-q^{pn+j}}$$

satisfy the congruences

$$a_{j,p}\left(pn + pj - j^2\right) \equiv 0 \pmod{p}.$$

In addition, Chan [4, Conjecture 7.1] conjectured that

$$a(50n+19) \equiv a(50n+39) \equiv a(50n+49) \equiv 0 \pmod{25}, \tag{1.1}$$
$$a_{1,3}(5n+3) \equiv a_{1,3}(5n+4) \equiv 0 \pmod{5}, \tag{1.2}$$
$$a_{1,6}(2n) \equiv 0 \pmod{2}, \tag{1.3}$$
$$a_{1,10}(2n) \equiv a_{3,10}(2n) \equiv 0 \pmod{2}, \tag{1.4}$$
$$a_{1,6}(6n+3) \equiv 0 \pmod{3}, \tag{1.5}$$
$$a_{1,10}(10n+5) \equiv a_{3,10}(10n+5) \equiv 0 \pmod{5}. \tag{1.6}$$

Let

$$\sum_{n=0}^{\infty} a_{k,j,2p}(n) q^n := -m(q^{kp}, q^p, q^j).$$

Chan and Mao [5] derived that for any integer k and any two coprime integers $2p$ and j such that $p \geq 1$ and $1 \leq j \leq 2p-1$,

$$a_{k,j,2p}(2pn+p) \equiv 0 \pmod{p},$$

which generalizes (1.4) and (1.6). Qu et al. [11] found (1.3) and (1.4) hold. In 2018, Baruah and Begum [3] not only derived (1.1), but also proved (1.6). Recently, in view of elementary methods, Ding and Xia [7] derived

$$a_{1,3}(15n+8) \equiv a_{1,3}(15n+14) \equiv 0 \pmod{5},$$

which are special cases of (1.2). Then by means of Appell–Lerch sums and the theory of modular forms, Fan et.al [6] proved (1.2). Chan [4] also provided the following conjecture.

Conjecture 1.2 [4, Conjecture 7.4] *Let*

$$\sum_{n=1}^{\infty} d(n)q^n = \sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{2n+1}}{(q;q^2)_{n+1}^2}.$$

We conjecture that for any nonnegative integer n,

$$\begin{aligned} d\,(18n+5) &\equiv d\,(18n+8) \equiv d\,(18n+11) \equiv d\,(18n+14) \\ &\equiv d\,(18n+17) \equiv 0 \pmod{3}. \end{aligned}$$

In this paper, we derive the following theorems related to $d(n)$.

Theorem 1.3 *We have*

$$\sum_{n=1}^{\infty} d(n)q^n = \sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{2n+1}}{(q;q^2)_{n+1}^2} = \frac{1}{2}\frac{f_2^3}{f_1^3}\sum_{n=0}^{\infty}(-1)^n q^{\frac{n^2+n}{2}} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{n^2+n}$$

Theorem 1.4 *For $n \geq 0$, we have*

$$d(6n+5) \equiv 0 \pmod{3}, \tag{1.7}$$
$$d(9n+5) \equiv 0 \pmod{3}, \tag{1.8}$$
$$d(9n+8) \equiv 0 \pmod{3}. \tag{1.9}$$

Theorem 1.5 *For any odd prime p and $n \geq 0$,*

$$d\left(9p\,(pn+i) + \frac{9p^2-1}{4}\right) \equiv 0 \pmod{3},$$

for $i = 1, 2, \cdots, p-1$.

The paper is organized as follows. We present some lemmas in Sect. 2. Theorems 1.3–1.5 are proved in Sect. 3.

2 Some lemmas

In order to prove the main results, the following lemmas are needed.

Lemma 2.1 [2, Agrarwal] *For any parameters* A, B, $a \neq 0$, *and* b,

$$\sum_{n=0}^{\infty} \frac{(B;q)_n(-Abq;q)_n q^n}{(-aq;q)_n(-bq;q)_n} = -\frac{(B;q)_\infty(-Abq;q)_\infty}{a(-aq;q)_\infty(-bq;q)_\infty} \sum_{m=0}^{\infty} \frac{(1/A;q)_m}{(-B/a;q)_{m+1}} \left(\frac{Abq}{a}\right)^m + (1+b)\sum_{m=0}^{\infty} \frac{(-1/a;q)_{m+1}(-ABq/a;q)_m}{(-B/a;q)_{m+1}(Abq/a;q)_{m+1}}(-b)^m.$$

Lemma 2.2 [8, p. 15]*The Rogers-Fine identity*

$$\sum_{n=0}^{\infty} \frac{(\alpha;q)_n \tau^n}{(\beta;q)_n} = \sum_{n=0}^{\infty} \frac{(\alpha;q)_n(\alpha\tau q/\beta;q)_n \beta^n \tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta;q)_n(\tau;q)_{n+1}}.$$

Lemma 2.3 [9, Eq. (III.17), p. 242] *Watson-Whipple transformation*

$$\sum_{n=0}^{\infty} \frac{1-aq^{2n}}{1-a} \cdot \frac{(a,b,c,d,e;q)_n(-1)^n q^{\frac{n^2-n}{2}}(aq)^{2n}}{(q,\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{e};q)_n(bcde)^n} = \frac{(aq,\frac{aq}{de};q)_\infty}{(\frac{aq}{d},\frac{aq}{e};q)} \sum_{n=0}^{\infty} \frac{(\frac{aq}{bc},d,e;q)_n}{(q,\frac{aq}{b},\frac{aq}{c};q)_n} \cdot (\frac{aq}{de})^n.$$

3 Proof of Theorem 1.3–1.5.

Proof of Theorem 1.3 In Lemma 2.1, replace q by q^2, then set $a = b = -q$, and finally replace A and B by $-q^{-2}$ and $-q^2$, respectively, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{2n+1}}{(q;q^2)_{n+1}^2} &= \frac{(-q;q^2)_\infty(-q^2;q^2)_\infty}{(q;q^2)_\infty^2} \cdot \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n(-1)^n}{(-q;q^2)_{n+1}} \\ &\quad + \frac{q}{1-q}\sum_{n=0}^{\infty} \frac{(q^{-1};q^2)_{n+1}(q;q^2)_n q^n}{(-q;q^2)_{n+1}(-1;q^2)_{n+1}} \\ &= \frac{(-q;q)_\infty}{(1+q)(q;q^2)_\infty^2} \cdot \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n(-1)^n}{(-q^3;q^2)_n} \\ &\quad - \frac{q}{2(1+q)}\sum_{n=0}^{\infty} \frac{(q;q^2)_n(q;q^2)_n q^n}{(-q^3;q^2)_n(-q^2;q^2)_n} \end{aligned} \tag{3.1}$$

For the first summation in (3.1), we apply Lemma 2.2 with $\alpha = -q^2$, q replaced by q^2, $\beta = -q^3$ and $\tau = -1$ to find that

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n(-1)^n}{(-q^3;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n(-q^2;q^2)_n q^{2n^2+n}(1-q^{4n+2})}{(-q^3;q^2)_n(-1;q^2)_{n+1}}$$

$$= \frac{(1+q)}{2} \sum_{n=0}^{\infty} \left(q^{2n^2+n} - q^{2n^2+3n+1} \right)$$

$$= \frac{(1+q)}{2} \sum_{n=0}^{\infty} (-1)^n q^{\frac{n^2+n}{2}}. \tag{3.2}$$

For the second summand in (3.1), we set $a = q^2$ and $b = -q^2$, and replace c, d, e and q by $-q, q^2, q$ and q^2, respectively. Then we find that

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n (q;q^2)_n q^n}{(-q^3;q^2)_n (-q^2;q^2)_n} = \frac{(q^2;q^2)_\infty (q^3;q^2)_\infty}{(q^4;q^2)_\infty (q;q^2)_\infty}$$

$$\cdot \sum_{n=0}^{\infty} \frac{1-q^{4n+2}}{1-q^2} \frac{(q^2, -q^2, -q, q^2, q; q^2)_n \cdot (-1)^n q^{n^2+n}}{(q^2, -q^2, -q^3, q^2, q^3; q^2)_n}$$

$$= (1+q) \sum_{n=0}^{\infty} (-1)^n q^{n^2+n}. \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1) yields that

$$\sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{2n+1}}{(q;q^2)_{n+1}^2} = \frac{1}{2} \frac{(-q;q)_\infty}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} (-1)^n q^{\frac{n^2+n}{2}} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{n^2+n}$$

$$= \frac{1}{2} \frac{f_2^3}{f_1^3} \sum_{n=0}^{\infty} (-1)^n q^{\frac{n^2+n}{2}} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{n^2+n}.$$

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4 First, using Theorem 1.3, we find that

$$\sum_{n=0}^{\infty} d(n) q^n \equiv \frac{1}{2} \frac{f_6}{f_3} \sum_{n=0}^{\infty} (-1)^n q^{\frac{n^2+n}{2}} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{n^2+n} \pmod 3.$$

$$= \frac{1}{2} \frac{f_6}{f_3} \sum_{k=0}^{2} \sum_{n=0}^{\infty} (-1)^{n+k} q^{\frac{(pn+k)(pn+k+1)}{2}} - \frac{1}{2} \sum_{k=0}^{2} \sum_{n=0}^{\infty} (-1)^{n+k} q^{(pn+k)(pn+k+1)}$$

$$= \frac{1}{2} \frac{f_6}{f_3} \left(\sum_{n=0}^{\infty} (-1)^n q^{\frac{9n^2+3n}{2}} - \sum_{n=0}^{\infty} (-1)^n q^{\frac{9n^2+9n}{2}+1} + \sum_{n=0}^{\infty} (-1)^n q^{\frac{9n^2+15n}{2}+3} \right)$$

$$- \frac{1}{2} \left(\sum_{n=0}^{\infty} (-1)^n q^{9n^2+3n} - \sum_{n=0}^{\infty} (-1)^n q^{9n^2+9n+2} + \sum_{n=0}^{\infty} (-1)^n q^{9n^2+15n+6} \right).$$

From the above congruence, we arrive at

$$\sum_{n=0}^{\infty} d(3n+2) q^n = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{3n^2+3n}. \tag{3.4}$$

In view of (3.4), we derive that

$$d\left(3\left(2n+1\right)+2\right)=d(6n+5)\equiv 0 \pmod 3,$$
$$d\left(3\left(3n+1\right)+2\right)=d(9n+5)\equiv 0 \pmod 3,$$
$$d\left(3\left(3n+2\right)+2\right)=d(9n+8)\equiv 0 \pmod 3.$$

Proof of Theorem 1.5 By means of (3.4), we derive that for any odd prime p,

$$\begin{aligned}\sum_{n=0}^{\infty} d\left(3\left(3n\right)+2\right)q^n &= \sum_{n=0}^{\infty} d\left(9n+2\right)q^n \\ &\equiv \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)} \pmod 3 \\ &= \frac{1}{2}\sum_{k=0}^{p-1}\sum_{n=0}^{\infty}(-1)^{n+k} q^{(pn+k)(pn+k+1)} \\ &= \frac{1}{2}\sum_{k=0}^{\frac{p-3}{2}} q^{k^2+k}\sum_{n=0}^{\infty}(-1)^{n+k} q^{p^2n^2+(2k+1)pn} \\ &\quad + \frac{1}{2}(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{4}}\sum_{n=0}^{\infty}(-1)^n q^{p^2n(n+1)} \\ &\quad + \frac{1}{2}\sum_{k=\frac{p+1}{2}}^{p-1} q^{k^2+k}\sum_{n=0}^{\infty}(-1)^{n+k} q^{p^2n^2+(2k+1)pn}.\end{aligned}$$

Notice that for any integer $0\le k\le p-1$,

$$k^2+k\equiv \frac{p^2-1}{4} \pmod p$$

if and only if $k=(p-1)/2$. Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} d\left(9\left(pn+\frac{p^2-1}{4}\right)+2\right)q^n &= \sum_{n=0}^{\infty} d\left(9pn+\frac{9p^2-1}{4}\right)q^n \\ &\equiv \frac{1}{2}(-1)^{\frac{p-1}{2}}\sum_{n=0}^{\infty}(-1)^n q^{\frac{pn(n+1)}{2}} \pmod 3.\end{aligned} \tag{3.5}$$

Since there are no terms on the right of (3.5) in which the powers of q are congruent to $1, 2, \cdots, p-1$ modulo p,

$$d\left(9p\,(pn+i)+\frac{9p^2-1}{4}\right)\equiv 0 \pmod 3,$$

for $i = 1, 2, \cdots, p-1$. We complete the proof. □

Remark. After the completion of the first version of this article, we learned from Prof. Dazhao Tang that Theorem 1.3 is one special case of Entry 6.3.7 in book "Ramanujan's Lost Notebook Part II".

References

1. Andrews, G.E.: The number of smallest parts in the partition of n. J. Reine Angew. Math. **624**, 133–142 (2008)
2. Agrarwal, R.P.: On the paper, "A 'lost' notebook of Ramanujan". Adv. Math. **53**, 291–300 (1984)
3. Baruah, N.D., Begum, N.M.: Proofs of some conjectures of Chan on Appell–Lerch sums. Ramanujan J. **51**, 99–115 (2020)
4. Chan, S.H.: Congruences for Ramanujan's ϕ function. Acta Arith. **153**, 161–189 (2012)
5. Chan, S.H., Mao, R.R.: Two congruences for Appell–Lerch sums. Int. J. Number Theory **8**, 111–123 (2012)
6. Fan, Y., Wang, L., Xia, E.X.W.: Proof of a conjectural congruence of Chan for Appell–Lerch sums. Int. J. Number Theory **17**, 1003–1011 (2021)
7. Ding, W.H., Xia, E.X.W.: Arithmetic properties for Appell–Lerch sums. Ramaujan J. **56**, 763–783 (2021)
8. Fine, N.J.: Basic Hypergeometric Series and Applications. American Mathematical Society, Providence, RI (1988)
9. Gasper, G., Rahman, M.: Basic Hypergeometric Series, 2nd edn. Cambridge University Press, Cambridge (2004)
10. Ramanujan, S.: The Lost Notebook and Other Unpublished Paper. Narosa, New Delhi (1988)
11. Qu, Y.K., Wang, Y.J., Yao, O.X.M.: Generalizations of some conjectures of Chan congruences for Appell–Lerch sums. J. Math. Anal. Appl. **460**, 232–238 (2018)