The Ramanujan Journal (2023) 60:157–173
https://doi.org/10.1007/s11139-022-00625-w

Some supercongruences related to truncated hypergeometric series

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Received: 12 October 2021 / Accepted: 1 July 2022 / Published online: 3 September 2022

Abstract
In this manuscript, we obtain some supercongruences between truncated classical hypergeometric series using the theory of classic hypergeometric series and p-adic analysis. Using these supercongruences, we obtain some supercongruences for binomial coefficients. We also derive some congruences for the p-adic Gamma function.

Keywords Supercongruences · Classical hypergeometric series · Truncated hypergeometric series · p-Adic gamma functions

Mathematics Subject Classification 11A07 · 33D15

1 Introduction and statement of results

For any complex number a, the Pochhammer symbol is define as $(a)_0 = 1$, $(a)_k = a(a+1)\dots(a+k-1)$, for $k \geq 1$. For $a_i, b_i \in \mathbb{C}$ such that $b_i \notin \mathbb{Z}_{\leq 0}$, and for any non negative integer n, the generalized hypergeometric series ${}_{n+1}F_n$ is defined by

$$ {}_{n+1}F_n \left(\begin{matrix} a_1, \ a_2, \ \dots, \ a_{n+1} \\ \\ b_1, \ \dots, \ b_n \end{matrix} \ | z \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{n+1})_k}{(b_1)_k \cdots (b_n)_k} \frac{z^k}{k!}. \tag{1.1} $$

This series converges absolutely for $|z| < 1$, and also converges absolutely for $|z| = 1$ if $\mathrm{Re}(\sum b_i - \sum a_i) > 0$. For more details, see [2]. If one of the a_i is negative integer

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then the hypergeometric series (1.1) terminates after finitely many terms. If m is a positive integer, the truncated hypergeometric series is defined by

$$
{}_{n+1}F_n\left(\begin{matrix} a_1, a_2, \ldots, a_{n+1} \\ b_1, \ldots, \quad b_n \end{matrix} \;|z\right)_m := \sum_{k=0}^{m} \frac{(a_1)_k \cdots (a_{n+1})_k}{(b_1)_k \cdots (b_n)_k} \frac{z^k}{k!}. \tag{1.2}
$$

More details on hypergeometric series can be found in [1, 3, 11]. Hypergeometric series were first studied systematically by Heine. After that many other mathematicians such as Euler, Gauss and Jacobi studied these hypergeometric series and related them to other mathematical objects. The theory of partitions, founded by Euler, has led in a natural way to the idea of basic hypergeometric series.

In [14], Greene introduced the hypergeometric function over a finite field $\mathbb{F}_q$, q is a prime power, analogous to classical hypergeometric series as finite character sums. Many authors studied the hypergeometric function over a finite field in a manner that is parallel to that of the classical hypergeometric series. Recently, lots of mathematicians evaluated the number of $\mathbb{F}_q$-points of certain algebraic varieties with the help of the hypergeometric function over a finite field (for more details, see [4–6, 12, 19]).

Fundamental importance of classical hypergeometric series and Gaussian hypergeometric series lies in many areas such as Partition theory, Representation theory of $SL(2, \mathbb{R})$, Real periods of algebraic curves, Modular forms, Combinatorics etc. In [27], Rouse provided uniform formulas for the real period and the trace of Frobenious, associated to a family of elliptic curves $E_\lambda : y^2 = x(x-1)(x-\lambda), \lambda \neq 1, 0$ in terms of ${}_2F_1$-hypergeometric functions. In [5], Barman et al. defined a period analogue for the algebraic curves $y^l = x(x-1)(x-\lambda)$, $l \geq 2$ in terms of ${}_2F_1$-hypergeometric series. In [22], McCarthy discussed the real period of elliptic curves $y^2 = (x-1)(x^2+\lambda)$ in terms of ${}_3F_2$-hypergeometric function. In general periods are complicated transcendental numbers. In the case of CM elliptic curves any period is an algebraic multiple of a quotient of gamma values.

Supercongruences are congruences which happen to hold modulo some higher power of a prime p. In 2009, Zudilin [32] proved several Ramanujan type supercongruences using the Wilf-Zeilberger method. In 2011, Long [20] proved Van Hamme conjecture:

$$
\sum_{k=0}^{\frac{p-1}{2}} (4k+1)\left(\frac{\left(\frac{1}{2}\right)_k}{k!}\right)^3 (-1)^k \equiv (-1)^{\frac{p-1}{2}} p \pmod{p^3}, \tag{1.3}
$$

where $\binom{a}{k}$ is the binomial coefficient defined in Eq. (2.5). The first proof of (1.3) was given by Mortenson [25]. It is said to be of Ramanujan-type because it is a p-adic version of Ramanujan's formula

$$
\sum_{k=0}^{\infty} (4k+1)\left(\frac{\left(\frac{1}{2}\right)_k}{k!}\right)^3 (-1)^k = \frac{2}{\pi}.
$$

In 2011, Long gives a new proof of (1.3) and she proved several similar types of supercongruences. For example, Long proved the following supercongruence conjectured by Van Hamme [31], for any prime $p \geq 3$

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k \frac{6k+1}{4^k} \binom{-\frac{1}{2}}{k}^3 \equiv -\frac{p}{\Gamma_p(\frac{1}{2})^2} \pmod{p^3}. \tag{1.4}$$

Here Γ_p is the p-adic Gamma function defined in Sect. 2 (for more details, see [20]). In 2016, using the p-adic Gamma function and formulas on hypergeometric series, Long and Ramakrishna [21] established many supercongruences. In particular, for any prime $p \geq 5$, they established

$$\sum_{k=0}^{p-1} (6k+1) \frac{(\frac{1}{3})_k^6}{k!^6} \equiv \begin{cases} -p\Gamma_p(\frac{1}{3})^9 & \text{if } p \equiv 1 \pmod 6 \\ -\frac{10}{27} p^4 \Gamma_p(\frac{1}{3})^9 & \text{if } p \equiv 5 \pmod 6 \end{cases} \pmod{p^6}. \tag{1.5}$$

Deines et al. [9], propose several supercongruences for truncated hypergeometric series and p-adic Γ-function based on numeric observations. Barman et al. [7], proved Deines observation [9, Eqn. (7.4)] is correct for prime $p \equiv 1 \pmod 5$, and gave a generalization. Various Supercongruences have been conjectured by many mathematician including Van Hamme [31], Rodriguez-Villegas [26], Zudilin [32], Sun [28], Sun [29, 30] and Barman [7]. Recently, He [16] proved several supercongruences using a technique which relies on the relation between the classical and the p-adic Γ-functions. For prime $p \geq 3$, He [15] established the supercongruence

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{(3k+1)\binom{2k}{k}^3}{16^k} \equiv p \pmod{p^2}, \tag{1.6}$$

and in [17], he proved the following supercongruence

$$\sum_{k=0}^{\frac{p^2-1}{2}} \frac{(3k+1)\binom{2k}{k}^3}{16^k} \equiv 0 \pmod{p^2}. \tag{1.7}$$

In this paper, we derive supercongruences which give extensions of (1.6) and (1.7). First, we derive a supercongruence modulo p^2 between truncated ${}_4F_3$ hypergeometric series, with the help of this supercongruence we give a generalization of supercongruences (1.6) and (1.7).

Theorem 1.1 *If p is an odd prime and $r \in \mathbb{N}$, then*

$${}_4F_3\left(\begin{matrix} \frac{1-p^r}{2}, & \frac{1+p^r}{2}, & \frac{4}{3}, & \frac{1}{2} \\ & 1-p^r, & 1+p^r, & \frac{1}{3} \end{matrix} \,|4\right)_{\frac{p^r-1}{2}} \equiv {}_4F_3\left(\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{4}{3}, & \frac{1}{2} \\ & 1, & 1, & \frac{1}{3} \end{matrix} \,|4\right)_{\frac{p^r-1}{2}} \pmod{p^2}.$$

and

$$\sum_{k=0}^{\frac{p^r-1}{2}} \frac{(3k+1)\binom{2k}{k}^3}{16^k} \equiv p^r \pmod{p^2}.$$

In the following theorem we establish supercongruences between truncated hypergeometric series and using this, we obtain a binomial coefficient sum p^2.

Theorem 1.2 *Let p be a prime such that $p \equiv 1 \pmod 4$. Then*

$${}_4F_3\left(\begin{matrix} \frac{1-p}{2}, & 2, & \frac{p+1}{2}, & \frac{5}{3} \\ & \frac{7-p}{4}, & \frac{2}{3}, & \frac{7+p}{4} \end{matrix} \Big| \frac{1}{4}\right)_{p-1} \equiv {}_4F_3\left(\begin{matrix} \frac{1}{2}, & 2, & \frac{1}{2}, & \frac{5}{3} \\ & \frac{7}{4}, & \frac{2}{3}, & \frac{7}{4} \end{matrix} \Big| \frac{1}{4}\right)_{p-1} \pmod{p^2}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{k-\frac{1}{2}}{k}^2 (k+1)(3k+2)}{\binom{k+\frac{3}{4}}{k}^2 2^{2k+1}} \equiv \frac{9}{8} \pmod{p^2}.$$

In the following theorem, we obtain a supercongruence for the ratio of two truncated ${}_3F_2$ hypergeometric series using a technique which relies on the relation between p-adic and classic Γ-functions.

Theorem 1.3 *For a prime $p \geq 5$ and any integer $r > 1$,*

$$\frac{{}_3F_2\left(\begin{matrix} 1, & \frac{3}{2}, & \frac{1-p^r}{2} \\ & 2, & \frac{4-p^r}{2} \end{matrix} \Big| 1\right)_{p^r-1}}{{}_3F_2\left(\begin{matrix} 1, & \frac{3}{2}, & \frac{1-p^{r-1}}{2} \\ & 2, & \frac{4-p^{r-1}}{2} \end{matrix} \Big| 1\right)_{p^{r-1}-1}} \equiv 0 \pmod{p^r}.$$

In the last two theorems of this section, we obtain congruences modulo p for the p-adic Γ-function using p-adic analysis and combinatorial identities which were given by Mortenson in [24].

Theorem 1.4 *If p is an odd prime, then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\Gamma_p(\frac{1}{2}+k)^2}{\Gamma_p(1+k)^2} \equiv -1 \pmod{p}.$$

We recall the following identity which is equivalent to Ramanujan-π like series.

$$\frac{1}{\pi} = \frac{1}{4} \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2}{(k+1)2^{4k}}. \tag{1.8}$$

The above identity is equivalent to

$$\sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k)^2}{(k+1)\Gamma(k+1)^2} = 4. \tag{1.9}$$

In the following theorem, we derive a p-adic version of the above identity.

Theorem 1.5 *If p is an odd prime and $r \geq 1$ is any integer, then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\Gamma_p(\frac{1}{2}+k)^2}{\Gamma_p(1+k)^2} \frac{1}{k+r} \equiv \begin{cases} 0, & \textit{if } 1 \leq r \leq \frac{p-1}{2} \\ \frac{\Gamma_p(\frac{1}{2})^2}{r}, & \textit{if } r = mp, m \in \mathbb{N} \end{cases} \pmod{p}.$$

We note that in this paper we have not considered finite field hypergeometric series.

2 Preliminaries

We note that we use p as an odd prime in this paper. In this section, we recall some preliminaries on p-adic numbers, the p-adic Γ-function, classical hypergeometric series and the Pochhammer symbol. First, we recall the p-adic valuation on the field of rational numbers. Let x be any non zero rational number, then it can be represented by $x = \frac{p^a r}{s}$, where p is a prime, r and s are integers relatively prime to p, then the p-adic valuation of x is defined by

$$\upsilon_p(x) = \begin{cases} a, & \text{if } x \neq 0; \\ \infty, & \text{if } x = 0. \end{cases} \tag{2.1}$$

and the p-adic norm is defined by

$$|x|_p = \begin{cases} p^{-\upsilon_p(x)}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases} \tag{2.2}$$

The set of p-adic numbers is the completion of the rational numbers $\mathbb{Q}$ with respect to the p-adic norm. The set of p-adic numbers forms a field. It is denoted by $\mathbb{Q}_p$. Any p-adic number can be uniquely written as $\sum_{k=m}^{\infty} a_k p^k$, where m is some integer such that $a_m \neq 0$ and $a_k \in \{0, 1, \ldots, p-1\}$.

A p-adic integer is a p-adic number of the form $\sum_{k=m}^{\infty} a_k p^k$, where $m \geq 0$, and $a_k \in \{0, 1, \ldots, p-1\}$. The set of p-adic integers forms a ring. It is denoted by $\mathbb{Z}_p$. Note that $\mathbb{Z}_p$ is the unit ball with center 0 in $\mathbb{Q}_p$.

The gamma function $\Gamma(n)$ is defined to be an extension of the factorial to complex and real number arguments. It is related to the factorial by $\Gamma(n) = (n-1)!$, if n is a positive integer. It is analytic everywhere except at $n = 0, -1, -2, \ldots$. The p-adic Γ-function is a function of a p-adic variable analogous to the Γ-function. It was first explicitly defined by Morita [23] in 1975. In 1980, Boyarsky [8] pointed out that Dwork [10] implicitly used the same function in 1964.

Definition 2.1 [18] We define the p-adic Γ-function by setting $\Gamma_p(0) = 1$, and for $n \in \mathbb{Z}^+$ by

$$\Gamma_p(n) := (-1)^n \prod_{0<j<n,\, p\nmid j} j, \ \text{for } n \in \mathbb{N}.$$

The function has a unique extension to a continuous function on the ring of p-adic integer $\mathbb{Z}_p$. If $x(\neq 0) \in \mathbb{Z}_p$, then $\Gamma_p(x)$ is defined by

$$\Gamma_p(x) := \lim_{x_n \to x} \Gamma_p(x_n), \tag{2.3}$$

where in the limit, we take any sequence of positive integers p-adically approaching to x.

Proposition 2.2 [18] *If p is a prime and $x, y \in \mathbb{Z}_p$, then the following are true:*

(1) $\Gamma_p(0) = 1$ *and* $\Gamma_p(1) = -1$.
(2) $\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x), & \text{if } x \in \mathbb{Z}_p^*; \\ -\Gamma_p(x), & \text{if } x \in p\mathbb{Z}_p. \end{cases}$
(3) If $n \geq 1$ *and* $x \equiv y \pmod{p^n}$, *then* $\Gamma_p(x) \equiv \Gamma_p(y) \pmod{p^n}$.
(4) $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)}$, *where* $a_0(x) \in \{1, \ldots, p\}$ *satisfies* $x - a_0(x) \equiv 0 \pmod{p}$.

For a complex number a and a non negative integer k, we define the Pochhammer symbol or the rising factorial as

$$(a)_k = \prod_{j=0}^{k-1}(a+j), \quad k > 0 \tag{2.4}$$

and $(a)_0 = 1$. If $a \in \mathbb{R}$ and $k \in \mathbb{N}$, then the binomial coefficient is defined by

$$\binom{a}{k} = \frac{a(a-1)\ldots(a-k+1)}{k!}. \tag{2.5}$$

The rising factorial can be used to express the binomial coefficient as

$$(a)_k = \binom{a+k-1}{k} k!, \tag{2.6}$$

Definition 2.3 [21] Let a be a rational number with $v_p(a) = 0$. Let $i \in \{1, 2, \ldots, p-1\}$ be a unique integer such that $v_p(a+i) > 0$. We define $a' \in \mathbb{Q}$ by $a + i = pa'$.

Lemma 2.4 [21] *For* $a \in \left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right\}$ *and* $p \equiv 1 \pmod{n}$, $a' = a$.

The following Lemma allows us to replace Γ-quotients with Γ_p-quotients.

Lemma 2.5 [21] *Let a be a rational in* $(0, 1]$

(1) If $v_p(a) = 0$ and $m, r \in \mathbb{N}$, then

$$\frac{\Gamma(a + mp^r)}{\Gamma(a + mp^{r-1})} = (-1)^m p^{mp^{r-1}} \frac{\Gamma_p(a + mp^r)}{\Gamma_p(a)} \frac{(a^{'})_{mp^{r-1}}}{(a)_{mp^{r-1}}}.$$

(2) Suppose $a + mp^r \in \mathbb{N}$. (Here, $a, m \in \mathbb{Q}$ but need not be in $\mathbb{Z}$.) Then

$$\frac{\Gamma(a + mp^r)}{\Gamma(a + mp^{r-1})} = (-1)^{a+mp^r} p^{a+mp^{r-1}-1} \Gamma_p(a + mp^r).$$

Next, we recall the following combinatorial identities from [24], which are needed in the proof of our main theorems.

Lemma 2.6 [24] *If $n \geq 0$ is an integer, then*

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{n}{k} = (-1)^n.$$

Lemma 2.7 [24] *If $n, r \geq 1$ are integers, then*

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{n}{k} \frac{1}{k+r} = \frac{(-1)^n}{r} \prod_{j=1}^{n} \left(\frac{r-j}{r+j} \right).$$

Lemma 2.8 [24] *If $n, r \geq 1$ are integers, then*

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{n}{k} \frac{k}{k+r} = (-1)^n - (-1)^n \prod_{j=1}^{n} \left(\frac{r-j}{r+j} \right).$$

Remark 2.9 The identity in the right hand side of Lemma 2.8 becomes $(-1)^n$ when $1 \leq r \leq n$.

3 Proof of the results

In this section, we prove our main results using the Gaussian hypergeometric series and p-adic analysis.

Proof of Theorem 1.1 We recall the following result from [13], for any positive integer n

$$ {}_5F_4 \left(\begin{matrix} -n, \ 1-2b, \ 1-2b+n, & \frac{4-4b}{3}, & \frac{1-4b}{2} \\ -2n, \quad 1-b, \quad 2-4b+2n, & \frac{1-4b}{3} \end{matrix} |4 \right)$$

$$= \frac{(\frac{3}{2} - 2b)_n}{(\frac{1}{2})_n},$$

Substituting $n = \frac{p^r - 1}{2}$ and $b = 0$ in above result, in view of Eqs. (1.1) and (1.2), we obtain

$$ {}_5F_4\left(\begin{matrix} \frac{1-p^r}{2}, & 1, & \frac{1+p^r}{2}, & \frac{4}{3}, & \frac{1}{2} \\ & 1-p^r, & 1, & 1+p^r, & \frac{1}{3} \end{matrix} |4\right)_{\frac{p^r-1}{2}} = \frac{(\frac{3}{2})_{\frac{p^r-1}{2}}}{(\frac{1}{2})_{\frac{p^r-1}{2}}}.$$

Canceling equal entries from the top and bottom rows of the hypergeometric series, we have

$$ {}_4F_3\left(\begin{matrix} \frac{1-p^r}{2}, & \frac{1+p^r}{2}, & \frac{4}{3}, & \frac{1}{2} \\ & 1-p^r, & 1+p^r, & \frac{1}{3} \end{matrix} |4\right)_{\frac{p^r-1}{2}} = \frac{(\frac{3}{2})_{\frac{p^r-1}{2}}}{(\frac{1}{2})_{\frac{p^r-1}{2}}}. \tag{3.1}$$

Now we see that $(\frac{3}{2})_{\frac{p^r-1}{2}} = p^r(\frac{1}{2})_{\frac{p^r-1}{2}}$. In view of Eq. (3.1), we arrive at

$$ {}_4F_3\left(\begin{matrix} \frac{1-p^r}{2}, & \frac{1+p^r}{2}, & \frac{4}{3}, & \frac{1}{2} \\ & 1-p^r, & 1+p^r, & \frac{1}{3} \end{matrix} |4\right)_{\frac{p^r-1}{2}} = p^r. \tag{3.2}$$

From Eq. (1.2), we obtain

$$ {}_4F_3\left(\begin{matrix} \frac{1-p^r}{2}, & \frac{1+p^r}{2}, & \frac{4}{3}, & \frac{1}{2} \\ & 1-p^r, & 1+p^r, & \frac{1}{3} \end{matrix} |4\right)_{\frac{p^r-1}{2}} = \sum_{k=0}^{\frac{p^r-1}{2}} \frac{\left(\frac{1-p^r}{2}\right)_k \left(\frac{1+p^r}{2}\right)_k \left(\frac{4}{3}\right)_k \left(\frac{1}{2}\right)_k}{(1-p^r)_k(1+p^r)_k\left(\frac{1}{3}\right)_k} \frac{4^k}{k!}. \tag{3.3}$$

By Eq. (2.4), we have

$$\frac{\left(\frac{1-p^r}{2}\right)_k}{(1-p^r)_k} = \prod_{j=0}^{k-1} \frac{\left(\frac{1}{2} - \frac{p^r}{2} + j\right)}{(1-p^r+j)} = \frac{\left(\frac{1}{2}\right)_k}{(1)_k} \prod_{j=0}^{k-1} \frac{\left(1 - \frac{p^r}{1+2j}\right)}{\left(1 - \frac{p^r}{1+j}\right)}.$$

Note that any p appearing in $(1)_k$ is canceled by p appearing in $\left(\frac{1}{2}\right)_k$ (p appears first in $\left(\frac{1}{2}\right)_k$ then in $(1)_k$ and then appears at gap of p in both $\left(\frac{1}{2}\right)_k$ and $(1)_k$). Thus, $\frac{\left(\frac{1}{2}\right)_k}{(1)_k} \in \mathbb{Z}_p$. For $k \le \frac{p^r-1}{2}$, we observe that $1+2j$ and $1+j$, $0 \le j \le k-1$ are not a multiple of p^r and $\upsilon_p(1+2j)$, $\upsilon_p(1+j) \leq r-1$ and $\left|\frac{p^r}{1+j}\right|_p < 1$. Thus, for $k \leq \frac{p^r-1}{2}$, there exist constants $A_{k,r}$ and $B_{k,r}$ such that $A_{k,r}p^r, B_{k,r}p^r \in \mathbb{Z}_p$ and

$$\frac{\left(\frac{1-p^r}{2}\right)_k}{(1-p^r)_k} \equiv \frac{\left(\frac{1}{2}\right)_k}{(1)_k}(1-A_{k,r}p^r)(1+B_{k,r}p^r) \pmod{p^2}. \tag{3.4}$$

Similarly,

$$\frac{\left(\frac{1+p^r}{2}\right)_k}{(1+p^r)_k} \equiv \frac{\left(\frac{1}{2}\right)_k}{(1)_k}(1+A_{k,r}p^r)(1-B_{k,r}p^r) \pmod{p^2}. \tag{3.5}$$

From Eqs. (3.3)–(3.5) and (1.2), we conclude that

$${}_4F_3\left(\begin{matrix}\frac{1-p^r}{2}, & \frac{1+p^r}{2}, & \frac{4}{3}, & \frac{1}{2} \\ 1-p^r, & 1+p^r, & \frac{1}{3}\end{matrix} \,|4\right)_{\frac{p^r-1}{2}} \equiv {}_4F_3\left(\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{4}{3}, & \frac{1}{2} \\ 1, & 1, & \frac{1}{3}\end{matrix} \,|4\right)_{\frac{p^r-1}{2}} \pmod{p^2}. \tag{3.6}$$

In view of Eqs. (3.2) and (3.6), we write

$${}_4F_3\left(\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{4}{3}, & \frac{1}{2} \\ 1, & 1, & \frac{1}{3}\end{matrix} \,|4\right)_{\frac{p^r-1}{2}} \equiv p^r \pmod{p^2}. \tag{3.7}$$

From Eqs. (1.2) and (3.7), we have

$$\sum_{k=0}^{\frac{p^r-1}{2}} \frac{\left(\frac{4}{3}\right)_k \left(\frac{1}{2}\right)_k^3}{\left(\frac{1}{3}\right)_k (1)_k^2} \frac{4^k}{k!} \equiv p^r \pmod{p^2}.$$

By the definition of the Pochhammer symbol, we obtain

$$\sum_{k=0}^{\frac{p^r-1}{2}} \frac{(3k+1)\binom{2k}{k}^3}{16^k} \equiv p^r \pmod{p^2}$$

This completes the proof of the theorem. □

Proof of Theorem 1.2 From [13], we recall the following result:

$${}_5F_4\left(\begin{matrix}-2n, & 1+2a+2b, & b+1, & 2n-2a+2, & \frac{2b+5}{3} \\ & 1+a+b-n, & \frac{2b+2}{3}, & 2+b+n, & \frac{3-2a}{2}\end{matrix} \,|\frac{1}{4}\right) = \frac{(\frac{1}{2})_n(2+b)_n}{(-a-b)_n(\frac{3}{2}-a)_n},$$

where n is a positive integer. Set $a=\frac{1}{2},\ b=0$ and $n=\frac{p-1}{4}$ in above result, we have

$$ {}_5F_4\left(\begin{matrix}\frac{1-p}{2}, & 2, & 1, & \frac{p+1}{2}, & \frac{5}{3} \\ & \frac{7-p}{4}, & \frac{2}{3}, & \frac{7+p}{4}, & 1\end{matrix}\ \Big|\ \frac{1}{4}\right)_{\frac{p-1}{2}} = \frac{\left(\frac{1}{2}\right)_{\frac{p-1}{4}}(2)_{\frac{p-1}{4}}}{\left(\frac{-1}{2}\right)_{\frac{p-1}{4}}(1)_{\frac{p-1}{4}}}. \tag{3.8} $$

Canceling equal entries from the top and bottom rows of the hypergeometric series, we have

$$ {}_4F_3\left(\begin{matrix}\frac{1-p}{2}, & 2, & \frac{p+1}{2}, & \frac{5}{3} \\ & \frac{7-p}{4}, & \frac{2}{3}, & \frac{7+p}{4}\end{matrix}\ \Big|\ \frac{1}{4}\right)_{\frac{p-1}{2}} = \frac{\left(\frac{1}{2}\right)_{\frac{p-1}{4}}(2)_{\frac{p-1}{4}}}{\left(\frac{-1}{2}\right)_{\frac{p-1}{4}}(1)_{\frac{p-1}{4}}}. $$

Now we see that $\left(-\frac{1}{2}\right)_{\frac{p-1}{4}} = -\frac{2}{p-3}\left(\frac{1}{2}\right)_{\frac{p-1}{4}}$ and $(2)_{\frac{p-1}{4}} = \frac{p+3}{4}(1)_{\frac{p-1}{4}}$. In view of Eq. (3.8), we obtain

$$ {}_4F_3\left(\begin{matrix}\frac{1-p}{2}, & 2, & \frac{p+1}{2}, & \frac{5}{3} \\ & \frac{7-p}{4}, & \frac{2}{3}, & \frac{7+p}{4}\end{matrix}\ \Big|\ \frac{1}{4}\right)_{\frac{p-1}{2}} = -\frac{p^2-9}{8}. \tag{3.9} $$

In view of Eq. (1.2), the left side of Eq. (3.9) reduce to

$$ {}_4F_3\left(\begin{matrix}\frac{1-p}{2}, & 2, & \frac{p+1}{2}, & \frac{5}{3} \\ & \frac{7-p}{4}, & \frac{2}{3}, & \frac{7+p}{4}\end{matrix}\ \Big|\ \frac{1}{4}\right)_{\frac{p-1}{2}} = \sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1-p}{2}\right)_k (2)_k \left(\frac{1+p}{2}\right)_k \left(\frac{5}{3}\right)_k}{\left(\frac{7-p}{4}\right)_k \left(\frac{2}{3}\right)_k \left(\frac{7+p}{4}\right)_k} \frac{1}{4^k k!}. $$

From Eqs. (3.4) and (3.5), for $k \le \frac{p-1}{2}$, we have

$$ \left(\frac{1-p}{2}\right)_k \equiv \left(\frac{1}{2}\right)_k (1 - A_{k,1}p) \pmod{p^2} \tag{3.10} $$

and

$$ \left(\frac{1+p}{2}\right)_k \equiv \left(\frac{1}{2}\right)_k (1 + A_{k,1}p) \pmod{p^2}. \tag{3.11} $$

We can write

$$ \left(\frac{7-p}{4}\right)_k^{-1} = \prod_{j=0}^{k-1}\left(\frac{7}{4}-\frac{p}{4}+j\right)^{-1} = \left(\frac{7}{4}\right)_k^{-1} \prod_{j=0}^{k-1}\left(1-\frac{p}{7+4j}\right)^{-1}. $$

Since $p\equiv 1 \mod 4$, we can see that $7+4j, 0\le j\le k-1$ is not a multiple of p for $0<k\le \frac{p-1}{2}$. Thus, there exit a constant $C_k\in\mathbb{Z}_p$ such that

$$\left(\frac{7-p}{4}\right)_k^{-1}\equiv\left(\frac{7}{4}\right)_k^{-1}(1+C_kp)\pmod{p^2}. \tag{3.12}$$

Similarly,

$$\left(\frac{7+p}{4}\right)_k^{-1}\equiv\left(\frac{7}{4}\right)_k^{-1}(1-C_kp)\pmod{p^2}. \tag{3.13}$$

In view of Eqs. (3.10)–(3.13), we obtain

$${}_4F_3\left(\begin{matrix}\frac{1-p}{2}, & 2, & \frac{p+1}{2}, & \frac{5}{3} \\ & \frac{7-p}{4}, & \frac{2}{3}, & \frac{7+p}{4}\end{matrix}\,\Big|\,\frac{1}{4}\right)_{\frac{p-1}{2}}\equiv {}_4F_3\left(\begin{matrix}\frac{1}{2}, & 2, & \frac{1}{2}, & \frac{5}{3} \\ & \frac{7}{4}, & \frac{2}{3}, & \frac{7}{4}\end{matrix}\,\Big|\,\frac{1}{4}\right)_{\frac{p-1}{2}}\pmod{p^2} \tag{3.14}$$

From Eqs. (1.2), (3.9) and (3.14), we have

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{k-\frac{1}{2}}{k}^2(k+1)(3k+2)}{\binom{k+\frac{3}{4}}{k}^2 2^{2k+1}}\equiv\frac{9}{8}\pmod{p^2}.$$

This completes the proof of the theorem. □

Proof of Theorem 1.3 From [3], we recall the following result:

$${}_3F_2\left(\begin{matrix}a, & b, & -n \\ & b+\frac{1}{2}, & a-n+\frac{1}{2}\end{matrix}\,\Big|\,1\right)$$
$$=\sqrt{\pi}\frac{\Gamma(a-n+\frac{1}{2})\Gamma(b+\frac{1}{2})\Gamma(b-a+n+\frac{1}{2})}{\Gamma(a+\frac{1}{2})\Gamma(-n+\frac{1}{2})\Gamma(b-a+\frac{1}{2})\Gamma(b+a+\frac{1}{2})}.$$

Letting $a=1, b=\frac{3}{2}, n=\frac{p^r-1}{2}$ and then $n=\frac{p^{r-1}-1}{2}$ yield

$${}_3F_2\left(\begin{matrix}1, & \frac{3}{2}, & \frac{1-p^r}{2} \\ & 2, & \frac{4-p^r}{2}\end{matrix}\,\Big|\,1\right)_{\frac{p^r-1}{2}}=\sqrt{\pi}\frac{\Gamma(\frac{4-p^r}{2})\Gamma(2)\Gamma(\frac{1+p^r}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{2-p^r}{2})\Gamma(1)\Gamma(3)}$$

and

$$\left(\begin{matrix} 1, \ \frac{3}{2}, & \frac{1-p^{r-1}}{2} \\ & & |1 \\ 2, & \frac{4-p^{r-1}}{2} \end{matrix}\right)_{\frac{p^{r-1}-1}{2}} = \sqrt{\pi}\frac{\Gamma(\frac{4-p^{r-1}}{2})\Gamma(2)\Gamma(\frac{1+p^{r-1}}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{2-p^{r-1}}{2})\Gamma(1)\Gamma(3)}.$$

It follows from the above equations that

$$\frac{{}_3F_2\left(\begin{matrix} 1, \ \frac{3}{2}, & \frac{1-p^{r}}{2} \\ & & |1 \\ 2, & \frac{4-p^{r}}{2} \end{matrix}\right)_{\frac{p^{r}-1}{2}}}{{}_3F_2\left(\begin{matrix} 1, \ \frac{3}{2}, & \frac{1-p^{r-1}}{2} \\ & & |1 \\ 2, & \frac{4-p^{r-1}}{2} \end{matrix}\right)_{\frac{p^{r-1}-1}{2}}} = \frac{\Gamma(\frac{4-p^{r}}{2})\Gamma(\frac{1+p^{r}}{2})\Gamma(\frac{2-p^{r-1}}{2})}{\Gamma(\frac{2-p^{r}}{2})\Gamma(\frac{4-p^{r-1}}{2})\Gamma(\frac{1+p^{r-1}}{2})}. \tag{3.15}$$

From the relation $\Gamma(z+1) = z\Gamma(z)$, $z \in \mathbb{C}$ except on the poles $0, -1, -2, \ldots$, we obtain

$$\Gamma\left(\frac{4-p^r}{2}\right) = \frac{(2-p^r)}{2}\Gamma\left(\frac{2-p^r}{2}\right), \tag{3.16}$$

and

$$\Gamma\left(\frac{4-p^{r-1}}{2}\right) = \frac{(2-p^{r-1})}{2}\Gamma\left(\frac{2-p^{r-1}}{2}\right). \tag{3.17}$$

Using Eqs. (3.16) and (3.17) in Eq. (3.15), we have

$$\frac{{}_3F_2\left(\begin{matrix} 1, \ \frac{3}{2}, & \frac{1-p^{r}}{2} \\ & & |1 \\ 2, & \frac{4-p^{r}}{2} \end{matrix}\right)_{\frac{p^{r}-1}{2}}}{{}_3F_2\left(\begin{matrix} 1, \ \frac{3}{2}, & \frac{1-p^{r-1}}{2} \\ & & |1 \\ 2, & \frac{4-p^{r-1}}{2} \end{matrix}\right)_{\frac{p^{r-1}-1}{2}}} = \frac{(2-p^r)\Gamma\left(\frac{1+p^r}{2}\right)}{(2-p^{r-1})\Gamma\left(\frac{1+p^{r-1}}{2}\right)}. \tag{3.18}$$

Then from (2) of Lemma 2.5, we obtain

$$\frac{\Gamma\left(\frac{1+p^r}{2}\right)}{\Gamma\left(\frac{1+p^{r-1}}{2}\right)} = (-1)^{\frac{1+p^r}{2}} p^{\frac{p^{r-1}-1}{2}} \Gamma_p\left(\frac{1+p^r}{2}\right). \tag{3.19}$$

From Eqs. (3.18) and (3.19), we obtain

$$\frac{{}_3F_2\left(\begin{matrix}1,\ \frac{3}{2},\ \frac{1-p^r}{2}\\ 2,\ \frac{4-p^r}{2}\end{matrix}\Big|1\right)_{\frac{p^r-1}{2}}}{{}_3F_2\left(\begin{matrix}1,\ \frac{3}{2},\ \frac{1-p^{r-1}}{2}\\ 2,\ \frac{4-p^{r-1}}{2}\end{matrix}\Big|1\right)_{\frac{p^{r-1}-1}{2}}}=(-1)^{\frac{p^r+1}{2}}\frac{(2-p^r)(p^{\frac{p^{r-1}-1}{2}})}{(2-p^{r-1})}\Gamma_p\left(\frac{1+p^r}{2}\right).$$

For $p\geq 5$ and $r>1$, we know that $|\frac{p^{r-1}}{2}|_p<1$ and $p^{r-1}\geq 2r+1$. Thus,

$$\frac{{}_3F_2\left(\begin{matrix}1,\ \frac{3}{2},\ \frac{1-p^r}{2}\\ 2,\ \frac{4-p^r}{2}\end{matrix}\Big|1\right)_{\frac{p^r-1}{2}}}{{}_3F_2\left(\begin{matrix}1,\ \frac{3}{2},\ \frac{1-p^{r-1}}{2}\\ 2,\ \frac{4-p^{r-1}}{2}\end{matrix}\Big|1\right)_{\frac{p^{r-1}-1}{2}}}\equiv 0 \pmod{p^r}.$$

This completes the proof of the theorem. □

Proof of Theorem 1.4 Setting $n=\frac{p-1}{2}$ in Lemma 2.6, we have

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{\frac{p-1}{2}+k}{k}\binom{\frac{p-1}{2}}{k}=(-1)^{\frac{p-1}{2}}.$$

Using the properties of the binomial symbol, we deduce that

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\frac{(\frac{p-1}{2}+k)!}{k!^2(\frac{p-1}{2}-k)!}=(-1)^{\frac{p-1}{2}}.$$

In view of Definition 2.1, we obtain

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\frac{\Gamma_p(\frac{p+1}{2}+k)}{\Gamma_p(1+k)^2\Gamma_p(\frac{p+1}{2}-k)}=(-1)^{\frac{p-1}{2}}. \tag{3.20}$$

In view of Proposition 2.2, we can write

$$\Gamma_p\left(\frac{1-p}{2}+k\right)\Gamma_p\left(\frac{p+1}{2}-k\right)=(-1)^{\frac{p+1}{2}}(-1)^k. \tag{3.21}$$

First we use Eq. (3.21) in Eq. (3.20) after that using Proposition 2.2 and simplifying we obtain the desired result. □

We require the following lemma in the proof of Theorem 1.5.

Lemma 3.1 *If p is an odd prime and $r \geq 1$ is any integer, then*

$$\prod_{j=1}^{\frac{p-1}{2}} \left(\frac{r-j}{r+j}\right) \equiv \begin{cases} 0, & \textit{if } 1 \leq r \leq \frac{p-1}{2} \\ -\Gamma_p(\frac{1}{2})^2, & \textit{if } r = mp.\ m \in \mathbb{N} \end{cases} \pmod{p}.$$

Proof of Lemma 3.1 It is clear that $\prod_{j=1}^{\frac{p-1}{2}} \left(\frac{r-j}{r+j}\right) = 0$ if $1 \leq r \leq \frac{p-1}{2}$. If $r \geq \frac{p+1}{2}$, then we can write

$$\prod_{j=1}^{\frac{p-1}{2}} \left(\frac{r-j}{r+j}\right) = \frac{(r-1)!r!}{(r+\frac{p-1}{2})!(r-\frac{p+1}{2})!}. \tag{3.22}$$

Setting $r = mp$, $m \in \mathbb{N}$ in Eq. (3.22), we deduce that

$$\prod_{j=1}^{\frac{p-1}{2}} \left(\frac{r-j}{r+j}\right) = \frac{(mp-1)!(mp)!}{\left(\frac{(2m+1)p-1}{2}\right)!\left(\frac{(2m-1)p-1}{2}\right)!}. \tag{3.23}$$

In view of Definition 2.1 and Eq. (3.23), we can write

$$\prod_{j=1}^{\frac{p-1}{2}} \left(\frac{r-j}{r+j}\right) = -\frac{(m)!(m-1)!p^{2m-1}\Gamma_p(mp)^2}{(m)!(m-1)!p^{2m-1}\Gamma_p\left(\frac{(2m-1)p+1}{2}\right)\Gamma_p\left(\frac{(2m+1)p+1}{2}\right)}.$$

Therefore,

$$\prod_{j=1}^{\frac{p-1}{2}} \left(\frac{r-j}{r+j}\right) = -\frac{\Gamma_p(mp)^2}{\Gamma_p\left(\frac{(2m-1)p+1}{2}\right)\Gamma_p\left(\frac{(2m+1)p+1}{2}\right)}.$$

We observe that $\Gamma_p\left(\frac{(2m-1)p+1}{2}\right)\Gamma_p\left(\frac{(2m-1)p+1}{2}\right)$, for any $m \in \mathbb{N}$ is not a multiple of p. Using Proposition 2.2, we have

$$\prod_{j=1}^{\frac{p-1}{2}} \left(\frac{r-j}{r+j}\right) \equiv -\frac{1}{\Gamma_p(\frac{1}{2})^2} \pmod{p}.$$

Thus, we have the desired result. □

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5 Setting $n = \frac{p-1}{2}$ in Lemma 2.7, we have

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{\frac{p-1}{2}+k}{k}\binom{\frac{p-1}{2}}{k}\frac{1}{k+r} = \frac{(-1)^{\frac{p-1}{2}}}{r}\prod_{j=1}^{\frac{p-1}{2}}\left(\frac{r-j}{r+j}\right). \tag{3.24}$$

First we simplify the left side of Eq. (3.24) using the properties of binomial coefficients. We can write

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{\frac{p-1}{2}+k}{k}\binom{\frac{p-1}{2}}{k}\frac{1}{k+r} = \sum_{k=0}^{\frac{p-1}{2}}(-1)^k\frac{(\frac{p-1}{2}+k)!}{k!^2(\frac{p-1}{2}-k)!}\frac{1}{k+r}.$$

From Definition 2.1, we deduce that

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{\frac{p-1}{2}+k}{k}\binom{\frac{p-1}{2}}{k}\frac{1}{k+r} = \sum_{k=0}^{\frac{p-1}{2}}(-1)^k\frac{\Gamma_p\left(\frac{p+1}{2}+k\right)}{\Gamma_p(1+k)^2\Gamma_p\left(\frac{p+1}{2}-k\right)}\frac{1}{k+r}. \tag{3.25}$$

If $1 \leq j \leq \frac{p-1}{2}$, then from Proposition 2.2, we can write

$$\Gamma_p\left(\frac{1}{2}+\frac{p}{2}-k\right)\Gamma_p\left(\frac{1}{2}-\frac{p}{2}+k\right) = (-1)^{\frac{p+1}{2}-k}. \tag{3.26}$$

Combining Eqs. (3.25) and (3.26), we have

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{\frac{p-1}{2}+k}{k}\binom{\frac{p-1}{2}}{k}\frac{1}{k+r} = -\sum_{k=0}^{\frac{p-1}{2}}(-1)^{\frac{p-1}{2}}\frac{\Gamma_p\left(\frac{p+1}{2}+k\right)\Gamma_p\left(\frac{1-p}{2}+k\right)}{\Gamma_p(1+k)^2}\frac{1}{k+r}.$$

In view of Proposition 2.2, we deduce that

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{\frac{p-1}{2}+k}{k}\binom{\frac{p-1}{2}}{k}\frac{1}{k+r} \equiv -\sum_{k=0}^{\frac{p-1}{2}}(-1)^{\frac{p-1}{2}}\frac{\Gamma_p(\frac{1}{2}+k)^2}{\Gamma_p(1+k)^2}\frac{1}{k+r} \pmod{p}. \tag{3.27}$$

In view of Lemma 3.1, and combining Eqs. (3.24) and (3.27), we have

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\Gamma_p(\frac{1}{2}+k)^2}{\Gamma_p(1+k)^2}\frac{1}{k+r} \equiv \begin{cases} 0, & \text{if } 1 \le r \le \frac{p-1}{2} \\ \frac{\Gamma_p(\frac{1}{2})^2}{r}, & \text{if } r = mp, m \in \mathbb{N} \end{cases} \pmod{p}.$$

□

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