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Sign changes in restricted coefficients of Hilbert modular forms

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Abstract
Let $\mathbf{f}$ be an adelic Hilbert cusp form of weight $\boldsymbol{k}$ and level $\mathfrak{n}$ over a totally real number field F. In this paper, we study the sign changes in the Fourier coefficients of $\mathbf{f}$ when restricted to square-free integral ideals and integral ideals in "arithmetic progression". In both cases we obtain qualitative results and in the former case we obtain a quantitative result as well. Our results are general in the sense that we do not impose any restriction to the totally real number field F, the weight $\boldsymbol{k}$ or the level $\mathfrak{n}$.

Keywords Sign changes · Hilbert modular forms · Arithmetic progressions · Square-free coefficients

Mathematics Subject Classification Primary 11F41, 11F30 · Secondary 11F66

1 Introduction

The modern theory of modular forms was probably seriously initiated first by Ramanujan when he studied the now famous Ramanujan Δ function. The attempts at proving the conjectures he proposed in his seminal 1916 paper have led to various interesting theories culminating in the work of Deligne. By that time, the study of modular forms had become a centerpiece of modern analytic number theory. Modular forms

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owe their importance in number theory to the fact that their “Fourier coefficients" are important arithmetic objects. Therefore the systematic study of modular forms is directly connected to the arithmetic of these coefficients. Nowadays, in the framework of the Langlands program, modular forms and its generalizations play a crucial role in completing the rather remarkable (although conjectural) picture of connections with various other branches of mathematics including Galois representations, motives etc.

Hilbert modular forms are analogues of classical modular forms over totally real number fields. These are natural generalizations of classical (elliptical) modular forms. Starting with a totally real number field F of degree n over $\mathbb{Q}$, one defines a Hilbert cusp form on an n-fold copy of the Poincaré upper half plane $\mathcal{H}$. Of particular interest to the study of modular forms is the L-function attached to a modular form. The analytic properties of the L-function is closely tied up with the arithmetic of the Fourier coefficients of the modular form f. Of particular interest is the question of sign changes in the Fourier coefficients. The Ramanujan conjectures that we mentioned earlier provide bounds on the absolute values of the Fourier coefficients of a cusp form but does not provide any information regarding the signs of these coefficients. Apart from containing arithmetic information, the Fourier coefficients of certain cusp forms, specifically Hecke eigenforms are the eigenvalues of the form with respect to certain linear operators known as Hecke operators. The “multiplicity one theorem" states that these coefficients uniquely determine the form. A rather remarkable result of Kowalski and others [9] states that, in the case of classical forms, the *signs* of these Fourier coefficients uniquely determine the form. Furthermore, as we shall see, the existence and frequency of sign changes is intricately related to the analytic properties of the associated Dirichlet series.

Consequently over the years, sign changes in the Fourier coefficients of modular forms have become quite an active area of research. In fact sign changes for Fourier coefficients of classical cusp forms were shown to exist in restricted sets of coefficients, including prime numbers, prime powers, arithmetic progressions and square-free integers. In 1983, Ram Murty studied the sign change of cusps forms at prime numbers [15]. In [11], Kohnen and Martin proved the existence of infinitely many sign change in sequence $\{a(p^{ln})\}_{n\geq 0}$ for all l in $\mathbb{N}$ and for all primes p. Similar results may be found in [3, 5, 12, 14].

In the case of Hilbert modular forms, the qualitative question of sign changes in the coefficients of an adelic Hilbert cusp form was first answered by Meher and Tanabe [13] who showed the existence of infinitely many sign changes for the Fourier coefficients provided all Fourier coefficients are real numbers. Later Kumar et al. [10] studied simultaneous sign changes in Fourier coefficients of two Hilbert cusp forms. Recent works include [17] among others.

It seemed natural for us to consider the question of the existence and the frequency of sign changes in the Fourier coefficients of a Hilbert cusp form inside restricted classes of coefficients. To the best of the authors’ knowledge, questions of this type have not been studied. We take up this task in the present paper. In particular we answer this question in the affirmative for the cases of square-free Fourier coefficients and Fourier coefficients in “arithmetic progression" (these notions will be made precise later on). Before we proceed forward, we let F be a totally real number field and let

$\mathbf{f}$ be an adelic Hilbert cusp form on F of weight $\boldsymbol{k}$ and level $\mathfrak{n}$ (see Sect. 2 below for details) for the remainder of the paper.

We define the notion of a square-free integral ideal similar to that of a square-free integer. For an integral ideal $\mathfrak{m}$ of F, we denote by $C(\mathfrak{m}, \mathbf{f})$ the $\mathfrak{m}$-th Fourier coefficient of $\mathbf{f}$. Suppose $\mathfrak{m} = \prod_{i=1}^{l} \mathfrak{p}_i^{e_i}$ is the unique factorization of $\mathfrak{m}$ into prime ideals $\mathfrak{p}_i$'s. The integral ideal $\mathfrak{m}$ is said to be *square-free* if and only if $e_1 = e_2 = \cdots = e_l = 1$. Now we state the first main result of the paper.

Theorem 1.1 *Let* $\mathbf{f}$ *be a primitive adelic Hilbert cusp form of weight* $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$ *and level* $\mathfrak{n}$ *with trivial Hecke character* Ψ*. Then the sequence* $\{C(\mathfrak{m}, \mathbf{f})\}_{\mathfrak{m}}$ *has infinitely many sign changes where* $\mathfrak{m}$ *runs through the square-free integral ideals of* F*. Furthermore the number of sign changes in* $C(\mathfrak{m}, \mathbf{f})$ *with* $N(\mathfrak{m}) \leq X$ *is* $\gg X^{1/2}$ *for large enough* X*.*

In Sect. 3 below we prove the above theorem by showing that the Dirichlet series with coefficients $C(\mathfrak{m}, \mathbf{f})$, where $\mathfrak{m}$ runs through the square-free integral ideals of F, is absolutely convergent for $\mathrm{Re}(s) > 1$ and has an analytic continuation to the half plane $\mathrm{Re}(s) > \frac{1}{2}$. Now the result follows from a well-known theorem of Meher and Murty and a previous result [1].

Remark 1.1 We extend the proof of Theorem 1.1 to slighter generality, where we remove the assumption of primitiveness on $\mathbf{f}$ at the expense of the assumption that $\mathfrak{n}$ is square-free.

In order to state the second main result of the paper, we make precise the notion of “arithmetic progression". Let $\mathfrak{m}_0$ be an integral ideal coprime to $\mathfrak{n}$ and $\mathfrak{m} = \mathfrak{m}_0\mathfrak{P}_\infty$ be an F-modulus where $\mathfrak{P}_\infty$ denotes the formal product of all the real embeddings of F. Let $\mathfrak{R}_{\mathfrak{m}}^{+}$ denote the strict ray class group of F for $\mathfrak{m}$ and $h_{\mathfrak{m}}$ its cardinality. We say two ideals[1] $\mathfrak{a}$, $\mathfrak{b}$ are in arithmetic progression if they belong to the same class in $\mathfrak{R}_{\mathfrak{m}}^{+}$. This definition is natural in the following sense. We say two integers are in an arithmetic progression if they belong to the same equivalence class in $(\mathbb{Z}/N\mathbb{Z})^\times$ for some natural number N. For the number field F, a natural analogue of $(\mathbb{Z}/N\mathbb{Z})^\times$, in the sense of class field theory, is the strict ray class group. The role of Dirichlet character is played by Hecke characters. Now we state the second main theorem of the paper.

Theorem 1.2 *Let* $\mathbf{f}$ *be a primitive adelic Hilbert cusp form of weight* $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$ *and level* $\mathfrak{n}$ *with trivial Hecke character* Ψ*. Then for any given* $\mathfrak{m}$ *(as above) coprime to the level* $\mathfrak{n}$ *and for any ideal class* $[\mathfrak{a}]$ *in* $\mathfrak{R}_{\mathfrak{m}}^{+}$*, the sequence* $\{C(\mathfrak{b}, \mathbf{f})\}_{\mathfrak{b}}$ *has infinitely many sign changes where* $\mathfrak{b}$ *runs through the integral ideals lying in* $[\mathfrak{a}]$*.*

Remark 1.2 When we say a modulus $\mathfrak{m}$ is coprime to an integral ideal $\mathfrak{n}$, we mean that the finite part of $\mathfrak{m}$, namely $\mathfrak{m}_0$ is coprime to $\mathfrak{n}$.

2 Preliminaries

In this section, we give a brief overview of the basic theory of Hilbert modular forms and fix the notation along the way. Our account is in no way complete and our primary

[1] It is safe to assume that $\mathfrak{a}$, $\mathfrak{b}$ are integral ideals where the definition is much more natural.

focus will be on reviewing the theory which shall be used in the sequel. We denote by F a totally real number field of degree n over $\mathbb{Q}$, and $\mathcal{O}_F$ the ring of integers of F. Let h denotes the strict class number of F. We denote by gothic letters $\mathfrak{m}$, $\mathfrak{n}$, (integral) ideals of $\mathcal{O}_F$. Given an F modulus $\mathfrak{m} = \mathfrak{m}_0\mathfrak{P}_\infty$ as above, we denote the strict ray class group associated to $\mathfrak{m}$ by $\mathfrak{R}^+_{\mathfrak{m}}$. Hecke characters are characters on the groups $\mathfrak{R}^+_{\mathfrak{m}}$ for various moduli $\mathfrak{m}$. As a special case the usual strict class group of F arises when we choose our modulus to be $\mathcal{O}_F$ (with trivial infinite part). We hasten to warn the reader that the above definitions of Hecke characters and moduli are but special cases of a much general definition. We content ourselves with these restrictive versions because they are sufficient for our purposes. For a full and brief account of Hecke characters in the classical and adelic setting, we refer the beautiful article of Shurman [20]. We denote Hecke characters by upper case greek letters Φ, Ψ etc.

Suppose two ideals $\mathfrak{a}$, $\mathfrak{b}$ belong to the same class in $\mathfrak{R}^+_{\mathfrak{m}}$. This means that the ideal $\mathfrak{a}\mathfrak{b}^{-1}$ is a principal ideal. Furthermore we can find a totally positive element ξ such that $\mathfrak{a}\mathfrak{b}^{-1} = \xi\mathcal{O}_F$ (ξ is totally positive because the “infinite part" of the modulus $\mathfrak{m}$ has all the real embeddings of F).

2.1 Classical Hilbert modular forms

As before, let $\mathcal{H}$ denote the Poincaré upper half plane. Once and for all we fix an embedding of $F \to \mathbb{R}^n$, where the map is given by $\xi \mapsto (\sigma_1(\xi), \sigma_2(\xi), \ldots, \sigma_n(\xi))$ and $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ are the real embeddings of F. There is an action of $GL_2^+(\mathbb{R})^n$ on $\mathcal{H}^n$ by componentwise Möbius transformation. A subgroup $\Gamma \subset GL_2^+(\mathbb{R})^n$ is called a *congruence subgroup* if it contains a subgroup of the form

$$\Gamma_N = \left\{ \gamma \in SL_2(\mathcal{O}_F) \,\middle|\, \gamma - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in N \cdot M_2(\mathcal{O}_F) \right\} \tag{2.1}$$

for some positive integer N and if $\Gamma/(\Gamma \cap F)$ is commensurable with $SL_2(\mathcal{O}_F)/\{\pm I\}$. Classical Hilbert modular forms are (complex) analytic functions on $\mathcal{H}^n$ which satisfy certain symmetries with respect to the action of congruence subgroups on $\mathcal{H}^n$. We shall only define classical forms for special types of congruence subgroups[2]

Given an integral ideal $\mathfrak{c}$ and a fractional ideal $\mathfrak{f}$ in F, define

$$\Gamma(\mathfrak{f}, \mathfrak{c}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F) \,\middle|\, a \in \mathcal{O}_F, b \in \mathfrak{f}^{-1}, c \in \mathfrak{c}\mathfrak{f}, d \in \mathcal{O}_F, ad - bc \in \mathcal{O}_F^\times \right\}. \tag{2.2}$$

Let $(\mathcal{O}_F{}^\times)^+$ denotes the group of totally positive units of $\mathcal{O}_F$. Define a character χ on $\Gamma(\mathfrak{f}, \mathfrak{c})$ as

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \chi_1(ad - bc)\chi_0(a \mod \mathfrak{c}, d \mod \mathfrak{c}),$$

[2] We omit a certain technical condition in the definition which can safely be ignored for the purposes of this paper.

where χ_0 and χ_1 are finite order characters of $(\mathcal{O}_F/\mathfrak{c})^\times \times (\mathcal{O}_F/\mathfrak{c})^\times$ and $(\mathcal{O}_F{}^\times)^+$ respectively. Suppose $\boldsymbol{k} = (k_1, k_2, \ldots, k_n) \in (\mathbb{Z}_+)^n$, $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Gamma$ and a is a scalar. Then we adopt the following notation:

(1) $\mathbf{z}^{\boldsymbol{k}} := \prod_{j=1}^n z_j^{k_j}$,
(2) $k_0 := \max\{k_1, k_2, \ldots, k_n\}$,
(3) $\det(\gamma) := (\det(\gamma_1), \det(\gamma_2), \ldots, \det(\gamma_n))$,
(4) $a^{\boldsymbol{k}} := a^{\sum_{j=1}^n k_j}$, and
(5) $c\mathbf{z} + d := (c_1 z_1 + d_1, c_2 z_2 + d_2, \ldots, c_n z_n + d_n)$,

where we have set

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} = (\gamma_1, \gamma_2, \ldots, \gamma_n) = \left(\begin{pmatrix} * & * \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} * & * \\ c_2 & d_2 \end{pmatrix}, \ldots, \begin{pmatrix} * & * \\ c_n & d_n \end{pmatrix} \right).$$

Consider a congruence subgroup Γ as in (2.2) and a character χ on Γ. A *classical Hilbert modular form* of weight $\boldsymbol{k}$ is a complex analytic function $f : \mathcal{H}^n \to \mathbb{C}$ such that for every $\gamma \in \Gamma$ and for every $\mathbf{z} \in \mathcal{H}^n$, we have

$$f(\mathbf{z}) = \chi^{-1}(\gamma) \det(\gamma)^{\frac{\boldsymbol{k}}{2}} (c\mathbf{z} + d)^{-\boldsymbol{k}} f(\gamma \mathbf{z}),$$

which are also holomorphic at the cusps of Γ. Such a function f has a Fourier expansion of the form

$$f(\mathbf{z}) = \sum_{\xi} c(\xi) \exp(Tr(\xi \mathbf{z})), \tag{2.3}$$

where we have set $\exp(x) = e^{2\pi i x}$ and $Tr(\xi \mathbf{z}) = \sum_{j=1}^n \sigma_j(\xi) z_j$. The summation in (2.3) runs over zero and the totally positive elements of a lattice (determined by Γ). If in the Fourier expansion at each cusp, we have $c(0) = 0$, then we say that f is a *cusp form.*

2.2 Adelization of Hilbert modular Forms

Suppose we write $\mathfrak{R}^+$ for the strict ray class group associated to the modulus $\mathcal{O}_F \mathfrak{P}_\infty$ and we let $h = |\mathfrak{R}^+|$. Let $\{t_j\}_{j=1}^h$ be a collection of ideles with $(t_j)_\infty = 1$ for all $1 \le j \le h$ such that $\{\mathfrak{t}_j\}_{j=1}^h$ is a complete set of representatives for $\mathfrak{R}^+$. Here we have let $\mathfrak{t}_j$ denote the fractional ideal of $\mathcal{O}_F$ corresponding to the idele t_j. For each j and for a fixed integral ideal $\mathfrak{n}$ let $\Gamma_j(\mathfrak{n})$ denote the congruence subgroup $\Gamma(\mathfrak{t}_j \mathfrak{d}, \mathfrak{n})$ where $\mathfrak{d}$ is the different ideal of F. For every $1 \le j \le h$ let f_i be a classical Hilbert modular form of fixed weight $\boldsymbol{k}$ for the congruent subgroup $\Gamma_j(\mathfrak{n})$.

Following Shimura [18], one can associate to the h-tuple $(f_1, f_2, \ldots, f_h)$ of classical Hilbert modular forms, an *adelic Hilbert modular form* $\mathbf{f}$. An adelic Hilbert modular form is a function on $GL_2(\mathbb{A}_F)$, where $\mathbb{A}_F$ is the adele ring of F. The Fourier expansion of f_j takes on the following form:

$$f_j(\mathbf{z}) = \sum_{\substack{0 \ll \xi \in \mathfrak{t}_j \\ \xi = 0}} a_j(\xi) \exp(Tr(\xi \mathbf{z})).$$

Here we have used the shorthand $\xi \gg 0$ to denote that ξ is totally positive. The Fourier coefficients of the adelic form $\mathbf{f}$ are parameterized by integral ideals $\mathfrak{m}$ of F. As we have seen earlier for any given integral ideal $\mathfrak{m}$ it is possible to choose a *unique* $1 \le \lambda \le h$ and a totally positive element $\xi \in \mathfrak{t}_\lambda$ such that $\mathfrak{m} = \xi \mathfrak{t}_\lambda^{-1}$. With this choice, we have the following relation between the Fourier coefficients of $\mathbf{f}$ and the Fourier coefficients of f_λ,

$$C(\mathfrak{m}, \mathbf{f}) = a_\lambda(\xi) \xi^{-\frac{\boldsymbol{k}}{2}} N(\mathfrak{m})^{\frac{k_0}{2}}.$$

Under some suitable conditions, there exists a finite order Hecke character (viewed as an idelic character) Ψ such that $\mathbf{f}(\tau g) = \Psi(\tau)\mathbf{f}(g)$ for all $\tau \in \mathbb{A}_F^\times$ and $g \in GL_2(\mathbb{A}_F)$. The space of such adelic Hilbert cusp forms of weight $\boldsymbol{k}$ and level $\mathfrak{n}$ and character Ψ forms a finite dimensional vector space and will be denoted by $S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$. We also mention in passing that like the classical modular forms, there is a well-understood newform theory for Hilbert modular forms (see [18, 19]).

There is an algebra of operators (once again indexed by the integral ideals of F) on the space $S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ called the Hecke operators. A cusp form $\mathbf{f}$ is said to be *primitive* if it is a common Hecke eigenfunction in the space of newforms and if $C(\mathcal{O}_F, \mathbf{f}) = 1$. In this case the eigenvalue of a primitive new form $\mathbf{f}$ with respect to the Hecke operator $T(\mathfrak{m})$ is precisely $C(\mathfrak{m}, \mathbf{f})$. Note that all the Fourier coefficients are real numbers if $\mathbf{f}$ is a primitive form. For more details on the Hecke theory of adelic Hilbert newforms we refer the reader to [6, 18].

Let $\mathbf{f} \in S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ be a primitive adelic Hilbert cusp form and $\{C(\mathfrak{m}, \mathbf{f})\}_{\mathfrak{m}}$ denote its Fourier coefficients. It is possible to associate an L-series to $\mathbf{f}$. The L-series associated to $\mathbf{f}$ is defined as

$$L(s, \mathbf{f}) = \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \mathfrak{m} \neq \{0\}}} \frac{C(\mathfrak{m}, \mathbf{f})}{N(\mathfrak{m})^s}. \tag{2.4}$$

The Ramanujan conjecture is true for Hilbert modular forms [4]. More precisely[3] for every $\epsilon > 0$,

$$C(\mathfrak{m}, \mathbf{f}) \ll_\epsilon N(\mathfrak{m})^\epsilon,$$

and therefore the above L-series is absolutely convergent for $\mathrm{Re}(s) > 1$. The strip $0 < \mathrm{Re}(s) < 1$ is called the critical strip. The coefficients $C(\mathfrak{m}, \mathbf{f})$ are multiplicative in $\mathfrak{m}$ and satisfy a three term recursive relation connecting the values of $C(\mathfrak{p}^i, \mathbf{f}), C(\mathfrak{p}^{i+1}, \mathbf{f}), C(\mathfrak{p}^{i+2}, \mathbf{f})$ for prime ideals $\mathfrak{p}$ and $i \in \mathbb{N}$. As a consequence the L-series defined in (2.4) satisfies an Euler product formula, valid when $\mathrm{Re}(s) > 1$;

[3] After a suitable re-normalization, which we assume from now on.

$$
\begin{aligned}
L(s, \mathbf{f}) &= \prod_{\mathfrak{p}} \left(1 + \frac{C(\mathfrak{p}, \mathbf{f})}{N(\mathfrak{p})^s} + \frac{C(\mathfrak{p}^2, \mathbf{f})}{N(\mathfrak{p}^2)^s} + \cdots \right) \\
&= \prod_{\mathfrak{p}} \left(1 - C(\mathfrak{p}, \mathbf{f}) N(\mathfrak{p})^{-s} + \Psi^*(\mathfrak{p}) N(\mathfrak{p})^{-2s}\right)^{-1},
\end{aligned}
$$

where $N(\mathfrak{m})$ denotes the norm of the ideal $\mathfrak{m}$ and Ψ^* denotes the ideal version of the idelic Hecke character Ψ. We also mention that the L-series can be analytically continued to the whole complex plane. We omit the details for the sake of brevity.

Finally we introduce the shift operator. More details can be found in [19]. Let $\mathfrak{q}$ be an integral ideal and $q \in \mathbb{A}_F^\times$ such that $q_\infty = 1$ and $q\mathcal{O}_F = \mathfrak{q}$. The shift operator $B_\mathfrak{q}$ is defined as;

$$
\mathbf{f} \,|B_\mathfrak{q} = N(\mathfrak{q})^{-k_0/2} \mathbf{f} \left| \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix} \right. .
$$

It is clear that $B_\mathfrak{q}$ maps $S_{\boldsymbol{k}}(\mathfrak{n})$ to $S_{\boldsymbol{k}}(\mathfrak{n}\mathfrak{q})$ and $C(\mathfrak{m}, \mathbf{f}\,|B_\mathfrak{q}) = C(\mathfrak{m}\mathfrak{q}^{-1}, \mathbf{f})$ (see Notation).

Notation

(1) The symbol ϵ is used to denote a small positive quantity, which might be different in each instance.
(2) If $\mathfrak{a}$ is a fractional ideal which is not an integral ideal, then the Fourier coefficient $C(\mathfrak{a}, \mathbf{f})$ is understood to be zero.

3 Sign changes in square-free coefficients

In this section we study the sign change in the coefficients $C(\mathfrak{m}, \mathbf{f})$ when $\mathfrak{m}$ is a square-free integral ideal of F. From the definition of square-free above, we can generalize the Möbius function (which we again denote by μ) to a function of all the integral ideals of $\mathcal{O}_F$ as follows:

$$
\mu(\mathfrak{m}) = \begin{cases} (-1)^l & \text{if } \mathfrak{m} \text{ is square-free and } \mathfrak{m} = \prod_{i=1}^{l} \mathfrak{p}_i, \\ 0 & \text{otherwise.} \end{cases}
$$

Note that μ satisfies properties analogous to the classical Möbius function defined on the integers, and in particular we have

$$
\sum_{\mathfrak{r}^2 | \mathfrak{m}} \mu(\mathfrak{r}) = \begin{cases} 1 & \mathfrak{m} \text{ is square-free,} \\ 0 & \text{otherwise.} \end{cases}
$$

The proof of the Theorem 1.1 follows from comparing the growth of the sums,

$$\sum_{N(\mathfrak{m})<X}{}^{\#} C(\mathfrak{m}, \mathbf{f}) \qquad \text{and} \qquad \sum_{N(\mathfrak{m})<X}{}^{\#} C^2(\mathfrak{m}, \mathbf{f}) \tag{3.1}$$

and then appealing to a theorem of Murty and Meher [12] which guarantees the existence of sign changes in some short intervals. Here # in the summation indicates that we are taking sum over square-free integral ideals[4]. The second sum in (3.1) has already been investigated in [1] and hence we have to estimate the first sum. Our approach is inspired by the work of Hulse et al. [8]. We start by considering the following Dirichlet series:

$$L^{\#}(s, \mathbf{f}) = \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_F \\ \mathfrak{m}\neq\{0\}}}{}^{\#} \frac{C(\mathfrak{m}, \mathbf{f})}{N(\mathfrak{m})^s} = \prod_{\mathfrak{p} \text{ prime}} \left(1 + \frac{C(\mathfrak{p}, \mathbf{f})}{N(\mathfrak{p})^s}\right),$$

which is absolutely convergent for $\mathrm{Re}(s) > 1$ (this is clear by directly applying the Ramanujan bound). We present a slightly longer proof of this fact because the computations therein will be used later on.

Lemma 3.1 *The Dirichlet series $L^{\#}(s, \mathbf{f})$ is absolutely convergent for $Re(s) > 1$.*

Proof It can be easily seen that

$$\begin{aligned} L^{\#}(s, \mathbf{f}) &= \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_F \\ \mathfrak{m}\neq\{0\}}}{}^{\#} \frac{C(\mathfrak{m}, \mathbf{f})}{N(\mathfrak{m})^s} \\ &= \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_F \\ \mathfrak{m}\neq\{0\}}} \frac{C(\mathfrak{m}, \mathbf{f})}{N(\mathfrak{m})^s} \left(\sum_{\mathfrak{r}^2|\mathfrak{m}} \mu(\mathfrak{r})\right) \\ &= \sum_{\mathfrak{r}} \mu(\mathfrak{r}) D_{\mathfrak{r}^2}(s), \end{aligned} \tag{3.2}$$

where we have set

$$D_{\mathfrak{r}}(s) := \sum_{\mathfrak{r}|\mathfrak{m}} \frac{C(\mathfrak{m}, \mathbf{f})}{N(\mathfrak{m})^s} = \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_F \\ \mathfrak{m}\neq\{0\}}} \frac{C(\mathfrak{r}\mathfrak{m}, \mathbf{f})}{N(\mathfrak{r}\mathfrak{m})^s}$$

for an integral ideal $\mathfrak{r}$. Recalling the Ramanujan bound

$$C(\mathfrak{m}, \mathbf{f}) \ll_\epsilon N(\mathfrak{m})^\epsilon.$$

[4] We follow this convention throughout the paper.

We have

$$|D_{\mathfrak{r}}(1+\epsilon+it)| \ll \sum_{\mathfrak{m}} \left| \frac{C(\mathfrak{rm}, \mathbf{f})}{N(\mathfrak{rm})^{1+\epsilon+it}} \right| \ll \frac{1}{N(\mathfrak{r})^{1+\epsilon}} \sum_{\mathfrak{m}} \left| \frac{C(\mathfrak{rm}, \mathbf{f})}{N(\mathfrak{m})^{1+\epsilon+it}} \right|.$$

The right most summation in the previous inequality is $\ll_\epsilon N(\mathfrak{r})^\epsilon$. Therefore, for every $\epsilon > 0$, we get

$$|D_{\mathfrak{r}}(1+\epsilon+it)| \ll_\epsilon \frac{1}{N(\mathfrak{r})^{1+\epsilon}}. \tag{3.3}$$

In particular we have

$$|D_{\mathfrak{r}^2}(1+\epsilon+it)| \ll_\epsilon \frac{1}{N(\mathfrak{r})^{2+\epsilon}}. \tag{3.4}$$

Therefore from (3.2) we can see that $L^{\#}(s, \mathbf{f})$ is absolutely convergent for $\mathrm{Re}(s) > 1$. □

Now our next aim is to show that $L^{\#}(s, \mathbf{f})$ has analytic continuation up to $\mathrm{Re}(s) > \frac{1}{2}$.

Lemma 3.2 *For a given prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ and $s \in \mathbb{C}$, define*

$$S_{\mathfrak{p}}(s) := \left(-\frac{C(\mathfrak{p}, \mathbf{f})^2}{N(\mathfrak{p})^{2s}} + \Psi^*(\mathfrak{p}) N(\mathfrak{p})^{-2s} \left(\frac{C(\mathfrak{p}, \mathbf{f})}{N(\mathfrak{p})^s} + 1 \right) \right).$$

Then we have

$$L^{\#}(s, \mathbf{f}) = L(s, \mathbf{f}) \prod_{\mathfrak{p}} \left(1 + S_{\mathfrak{p}}(s)\right),$$

where product runs over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$. Consequently $L^{\#}(s, \mathbf{f})$ can be analytically continued to the half plane $Re(s) > \frac{1}{2}$.

Proof Let $\mathbf{f}$ be a new form in $S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ and let $L_{\mathfrak{p}}(s)$ denote the local $\mathfrak{p}$ factor of $L(s, \mathbf{f})$. That is

$$\begin{aligned} L_{\mathfrak{p}}(s) &= \left(1 + \frac{C(\mathfrak{p}, \mathbf{f})}{N(\mathfrak{p})^s} + \frac{C(\mathfrak{p}^2, \mathbf{f})}{N(\mathfrak{p}^2)^s} + \cdots \right) \\ &= \frac{1}{\left(1 - C(\mathfrak{p}, \mathbf{f}) N(\mathfrak{p})^{-s} + \Psi^*(\mathfrak{p}) N(\mathfrak{p})^{-2s}\right)}. \end{aligned} \tag{3.5}$$

Here Ψ^* denotes the ideal version of an idelic character Ψ.

Define $L_{\mathfrak{p}}^{(r)}(s)$ to be the r tail of $L_{\mathfrak{p}}$. That is,

$$L_{\mathfrak{p}}^{(r)}(s) = \sum_{n=r}^{\infty} \frac{C(\mathfrak{p}^n, \mathbf{f})}{N(\mathfrak{p}^n)}.$$

Note that $L_{\mathfrak{p}}^{(0)}(s) = L_{\mathfrak{p}}(s)$ by definition.

Suppose $\mathfrak{r} = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}$, where $e_{\mathfrak{p}} = 0$ for almost all $\mathfrak{p}$. Then it is clear that

$$D_{\mathfrak{r}}(s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}^{(e_{\mathfrak{p}})}(s).$$

Now suppose that $\mathfrak{r}$ is a square-free ideal. Then we have

$$\begin{aligned} D_{\mathfrak{r}^2}(s) &= L(s, \mathbf{f}) \prod_{\mathfrak{p}|\mathfrak{r}} L_{\mathfrak{p}}^{(2)}(s)/L_{\mathfrak{p}}(s) \\ &= L(s, \mathbf{f}) \prod_{\mathfrak{p}|\mathfrak{r}} (L_{\mathfrak{p}}(s) - 1 - C(\mathfrak{p}, \mathbf{f}) N(\mathfrak{p})^{-s})/L_{\mathfrak{p}}(s) \\ &= L(s, \mathbf{f}) \prod_{\mathfrak{p}|\mathfrak{r}} \left(1 - \frac{1}{L_{\mathfrak{p}}(s)} - \frac{C(\mathfrak{p}, \mathbf{f})}{N(\mathfrak{p})^s L_{\mathfrak{p}}(s)}\right) \\ &= L(s, \mathbf{f}) \prod_{\mathfrak{p}|\mathfrak{r}} \Big(1 - (1 - C(\mathfrak{p}, \mathbf{f}) N(\mathfrak{p})^{-s} + \Psi^*(\mathfrak{p}) N(\mathfrak{p})^{-2s}) \\ &\qquad - \frac{C(\mathfrak{p}, \mathbf{f})}{N(\mathfrak{p})^s} \left(1 - C(\mathfrak{p}, \mathbf{f}) N(\mathfrak{p})^{-s} + \Psi^*(\mathfrak{p}) N(\mathfrak{p})^{-2s}\right)\Big) \\ &= L(s, \mathbf{f}) \prod_{\mathfrak{p}|\mathfrak{r}} \left(\frac{C(\mathfrak{p}, \mathbf{f})^2}{N(\mathfrak{p})^{2s}} - \Psi^*(\mathfrak{p}) N(\mathfrak{p})^{-2s} \left(\frac{C(\mathfrak{p}, \mathbf{f})}{N(\mathfrak{p})^s} + 1\right)\right). \end{aligned}$$

Therefore,

$$\mu(\mathfrak{r}) D_{\mathfrak{r}^2}(s) = L(s, \mathbf{f}) \prod_{\mathfrak{p}|\mathfrak{r}} \left(-\frac{C(\mathfrak{p}, \mathbf{f})^2}{N(\mathfrak{p})^{2s}} + \Psi^*(\mathfrak{p}) N(\mathfrak{p})^{-2s} \left(\frac{C(\mathfrak{p}, \mathbf{f})}{N(\mathfrak{p})^s} + 1\right)\right).$$

Thus we have

$$\begin{aligned} L^{\#}(s, \mathbf{f}) &= L(s, \mathbf{f}) \sum_{\mathfrak{r}}^{\#} \prod_{\mathfrak{p}|\mathfrak{r}} S_{\mathfrak{p}}(s) \\ &= L(s, \mathbf{f}) \prod_{\mathfrak{p}} \left(1 + S_{\mathfrak{p}}(s)\right). \end{aligned} \tag{3.6}$$

The product on the right hand side of (3.6) is absolutely convergent for $\mathrm{Re}(s) > 1/2$ and therefore we can see that $L^{\#}(s, \mathbf{f})$ has an analytic continuation to the plane $\mathrm{Re}(s) > 1/2$. □

Remark 3.1 By performing similar calculations as in the previous proof it is possible to show that $D_{\mathfrak{r}}(s)/L(s, \mathbf{f})$ is an entire function for all integral ideals $\mathfrak{r}$.

Lemma 3.3 *Let* $\mathbf{f}$ *be same as in Theorem 1.1. Then*

$$\sum_{\substack{\mathfrak{m} \\ N(\mathfrak{m})\le X}}^{\#} C(\mathfrak{m},\mathbf{f}) \ll_\epsilon X^{1/2+\epsilon}.$$

Proof From Lemma 3.2 we know that $L^\#(s,\mathbf{f})$ has an analytic continuation to the half plane $\mathrm{Re}(s) > 1/2$. Furthermore, from the known properties of $L(s,\mathbf{f})$ we can see that $L^\#(s,\mathbf{f})$ has polynomial growth in $\mathrm{Im}(s)$ for a given $1/2 < \mathrm{Re}(s) < 1$. From the standard inverse Mellin transform we can see that

$$\frac{1}{2\pi i}\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} L^\#(s,\mathbf{f})\Gamma(s)X^s\,ds = \sum_{n=1}^{\infty} B(n,\mathbf{f})e^{-n/X}, \tag{3.7}$$

where we have set

$$B(n,\mathbf{f}) := \sum_{\substack{\mathfrak{m} \\ N(\mathfrak{m})=n}}^{\#} C(\mathfrak{m},\mathbf{f}).$$

We shift the line of integration in (3.7) to $\mathrm{Re}(s) = 1/2+\epsilon$. Since $L^\#(s,\mathbf{f})$ has at worst polynomial growth in $\mathrm{Im}(s)$ inside the critical strip, this is justified by the exponential decay of the Γ function. Furthermore, since the integrand is analytic in the given region, we do not pick up any residues. Therefore, we see that

$$\sum_{n=1}^{\infty} B(n,\mathbf{f})e^{-n/X} \ll_\epsilon X^{1/2+\epsilon}. \tag{3.8}$$

Since $B(n,\mathbf{f}) \ll_\epsilon n^\epsilon$ for every $\epsilon > 0$, we can see that $\sum_{n\ge X} B(n,\mathbf{f})e^{-n/X} = O(1)$, for large enough X. Finally this gives us $\sum_{\substack{\mathfrak{m} \\ N(\mathfrak{m})\le X}}^{\#} C(\mathfrak{m},\mathbf{f}) \ll_\epsilon X^{1/2+\epsilon}$. □

We shall also need the following proposition from [1].

Proposition 3.1 [R. Agnihotri, K. Chakraborty] *Let* f *be a positive and smooth function supported on* $[\frac{1}{2},1]$. *Let* $\mathbf{f},\mathbf{g} \in S_{\boldsymbol{k}}(\mathfrak{n})$ *be primitive cusp forms. Then for any* $\frac{1}{2} < c < 1$ *and for every* $\epsilon > 0$ *the following assertions hold:*

(1) *If* $\mathbf{f}=\mathbf{g}$, *then there exists a constant* $A(\mathbf{f},f) > 0$ *such that,*

$$\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_F \\ (\mathfrak{m},\mathfrak{n})=\mathcal{O}_F}}^{\#} |C(\mathfrak{m},\mathbf{f})|^2 f\Big(\frac{N(\mathfrak{m})}{X}\Big) = A(\mathbf{f},f)X + O\big(X^c k_0^{n(1-c)+\epsilon}N(\mathfrak{n})^{2(\frac{1-c}{2}+\epsilon)}\big).$$

(2) *If* $\mathbf{f} \neq \mathbf{g}$,

$$\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_F \\ (\mathfrak{m},\mathfrak{n})=\mathcal{O}_F}}^{\#} C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g})f\Big(\frac{N(\mathfrak{m})}{X}\Big) = O\big(X^c k_0^{n(1-c)+\epsilon} N(\mathfrak{n})^{2(\frac{1-c}{2}+\epsilon)}\big).$$

The implied constants depend only on ϵ *and* F.

From Proposition 3.1, we deduce that

$$X \ll_{\epsilon,F} \sum_{N(\mathfrak{m})\leq X}{}^{\#} C^2(\mathfrak{m},\mathbf{f}). \tag{3.9}$$

Proof of Theorem 1.1 In order to prove the theorem we first state a result of Meher and Murty [12].

Theorem 3.1 [Meher-Murty] *Let* $a(n)_{(n\geq 1)}$ *be a sequence of real numbers satisfying* $a(n) = O(n^\alpha)$ *such that*

$$\sum_{n\leq X} a(n) \ll X^\beta$$

and

$$\sum_{n\leq X} a(n)^2 = cX + O(X^\gamma)$$

where α, β, γ *and* c *are non-negative constants. If* $\alpha+\beta < 1$*, then for any* r *satisfying* $\max\{\alpha+\beta,\gamma\} < r < 1$*, the sequence* $a(n)(n \geq 1)$ *has at least one sign change for* $n \in (X, X+X^r]$*. In particular, the sequence* $a(n)$ *has infinitely many sign changes and the number of sign changes for* $n \leq X$ *is* $\gg X^{1-r}$ *for sufficiently large* X.

Since the Fourier coefficients are not indexed by natural numbers, we cannot apply Theorem 3.1 directly. Nevertheless, we can modify the proof of Theorem 3.1 for our purposes.

Without loss of generality assume the contrary that finitely many of $C(\mathfrak{m},\mathbf{f})$ are negative. Therefore, for large enough X, we have $C(\mathfrak{m},\mathbf{f}) > 0$ whenever $N(\mathfrak{m}) \in (X, X+X^r]$ for some $1 > r > 1/2$. Therefore we have

$$\sum_{N(\mathfrak{m})\in(X,X+X^r]} C^2(\mathfrak{m},\mathbf{f}) \ll X^\epsilon \sum_{N(\mathfrak{m})\in(X,X+X^r]} C(\mathfrak{m},\mathbf{f}) \ll X^{1/2+\epsilon}. \tag{3.10}$$

The first inequality follows from the Ramanujan bound and the second inequality follows from Lemma 3.3.

On the other hand, since $r > 1/2$ we have

$$X^r \ll \sum_{N(\mathfrak{m})\in(X,X+X^r]} C^2(\mathfrak{m},\mathbf{f})$$

from (3.9), thus giving us a contradiction. The quantitative assertion of Theorem 1.1 follows easily from here. This completes the proof. □

A slight generalization

In Theorem 1.1 the requirement that $\mathbf{f}$ be a *primitive* form can be relaxed when the level $\mathfrak{n}$ is square-free as follows. Suppose that $\mathbf{f} \in S_k(\mathfrak{n})$ is an old form and that $\mathfrak{n}$ is square-free. We observe that there are new forms $\mathbf{f}_{(j,i)}$ of lower levels $\mathfrak{q}_i$ and constants $\alpha_{(j,i)}$ such that

$$\mathbf{f} = \sum_{\mathfrak{q}_i | \mathfrak{n}} \sum_j \alpha_{(j,i)} \mathbf{f}_{(j,i)} | B_{\mathfrak{q}_i}.$$

The coefficients satisfy

$$\begin{aligned} C(\mathfrak{m}, \mathbf{f}) &= \sum_{\mathfrak{q}_i | \mathfrak{n}} \sum_j \alpha_{(j,i)} C(\mathfrak{m}, \mathbf{f}_{(j,i)} | B_{\mathfrak{q}_i}) \\ &= \sum_{\mathfrak{q}_i | \mathfrak{n}} \sum_j \alpha_{(j,i)} C(\mathfrak{m}\mathfrak{q}_i^{-1}, \mathbf{f}_{(j,i)}). \end{aligned} \tag{3.11}$$

Without loss of generality, we can assume that $\mathbf{f}_{(j,i)}$s are primitive and therefore $C(\mathfrak{m}, \mathbf{f}_{(j,i)})$'s are real. Starting from Eq. (3.11) it is possible to perform similar calculations as above and arrive at an analogue of Theorem 1.1 in this case. The assumption that $\mathfrak{n}$ is square-free will be used in finding lower bounds similar to that of Proposition 3.1 (see [1]).

Remark 3.2 Removing the square-free assumption on $\mathfrak{n}$ should be possible after some tedious calculations along the lines of [1]. The authors believe that the calculations should be straight forward and do not require any serious mathematical ideas.

Remark 3.3 The methods of this section seem adaptable to study sign changes in power free coefficients, although in those cases, the only interesting problem is the density of sign changes as the existence of infinitely many sign changes follows from the present work.

Remark 3.4 Suppose Φ is a real-valued Hecke character modulo the level $\mathfrak{n}$. The above methods can be adapted to study correlations between $C(\mathfrak{m}, \mathbf{f})$ and $\Phi([\mathfrak{m}])$. One but needs to replace $\mathbf{f}$ with the twist $\mathbf{f}|_\Phi$, and use the relation $C(\mathfrak{m}, \mathbf{f}|_\Phi) = \Phi(\mathfrak{m}) C(\mathfrak{m}, \mathbf{f})$.

Remark 3.5 To the best of the authors' knowledge, sign change for square-free coefficients has not been studied for the case of classical modular forms. Our methods will yield results in that setting too.

4 Sign change in arithmetic progressions

In this section, we shall prove Theorem 1.2. As we have seen, in the case of Hilbert modular forms, the Fourier coefficients are parameterized by the integral ideals of the totally real field F.

Suppose we choose an F modulus $\mathfrak{m}$ coprime with the level $\mathfrak{n}$ of $\mathbf{f}$. In this section we shall study the sign changes in $C(\mathfrak{b}, \mathbf{f})$ as integral ideals $\mathfrak{b}$ vary in a fixed class inside $\mathfrak{R}^+_{\mathfrak{m}}$, the strict ray class group of F for $\mathfrak{m}$. To that end we fix the F modulus $\mathfrak{m}$.

If two integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ lie in the same ideal class inside $\mathfrak{R}^+_{\mathfrak{m}}$, we shall denote this by $\mathfrak{a} \equiv \mathfrak{b}(\mathfrak{m})$. First, we obtain an expression for the indicator function of an ideal class inside $\mathfrak{R}^+_{\mathfrak{m}}$. This is achieved using character theory as follows. For a given (fractional) ideals $\mathfrak{a}$, $\mathfrak{b}$ coprime to $\mathfrak{m}$, define an ideal function

$$\delta_{\mathfrak{a}}(\mathfrak{b}) = \begin{cases} 1 & \mathfrak{b} \equiv \mathfrak{a}(\mathfrak{m}), \\ 0 & \text{otherwise}. \end{cases} \tag{4.1}$$

Since $\mathfrak{R}^+_{\mathfrak{m}}$ is a finite abelian group, we have from the orthogonality of characters [2] that

$$\delta_{\mathfrak{a}}(\mathfrak{b}) = \frac{1}{|\mathfrak{R}^+_{\mathfrak{m}}|} \sum_{\Phi} \Phi([\mathfrak{b}])\overline{\Phi([\mathfrak{a}])}, \tag{4.2}$$

where the sum runs over the Hecke characters of $\mathfrak{R}^+_{\mathfrak{m}}$.

Let $\tilde{\Phi}$ denote the idelic version of the Hecke character Φ. From the work of Shemanske [19] we can see that if $\mathbf{f} \in \mathcal{M}_{k}(\mathfrak{n}, \Psi)$, then the twist of $\mathbf{f}$ by Φ, $\mathbf{f}\mid_{\Phi}$ (see [19] for the definition) lies inside the space $\mathcal{M}_{k}(\mathfrak{n}\mathfrak{m}^2, \Psi\Phi^2)$. Further, Hecke eigenforms are mapped to Hecke eigenforms under such character twists. Also, the Fourier coefficients of $\mathbf{f}$ and $\mathbf{f}\mid_{\Phi}$ are related as

$$C(\mathfrak{a}, \mathbf{f}\mid_{\Phi}) = \Phi^*(\mathfrak{a})C(\mathfrak{a}, \mathbf{f}). \tag{4.3}$$

Here Φ^* is defined as

$$\Phi^*(\mathfrak{a}) = \begin{cases} \tilde{\Phi}(\mathfrak{a}) & (\mathfrak{a}, \mathfrak{m}) = \mathcal{O}_F, \\ 0 & \text{otherwise}. \end{cases}$$

From here, it is clear that $\Phi^* = \Phi$.

Given a fractional ideal $\mathfrak{a}$ coprime to $\mathfrak{m}$, define

$$\mathbf{g}_{\mathfrak{a}} = \sum_{\Phi} \overline{\Phi([\mathfrak{a}])}\mathbf{f}\mid_{\Phi}, \tag{4.4}$$

where the sum runs over the characters of $\mathfrak{R}_{\mathfrak{m}}^{+}$. From previous discussions it is clear that the Fourier coefficients of $\mathbf{g}_{\mathfrak{a}}$ satisfy

$$C(\mathfrak{b}, \mathbf{g}_{\mathfrak{a}}) = \begin{cases} C(\mathfrak{b}, \mathbf{f}) & \text{if } [\mathfrak{a}] = [\mathfrak{b}] \text{ inside } \mathfrak{R}_{\mathfrak{m}}^{+}, \\ 0 & \text{otherwise,} \end{cases}$$

for all integral ideals $\mathfrak{b}$. Therefore when we consider the L-function associated to $\mathbf{g}_{\mathfrak{a}}$, say $L(s, \mathbf{g}_{\mathfrak{a}})$, we can see that

$$L(s, \mathbf{g}_{\mathfrak{a}}) = \sum_{\Phi} \overline{\Phi([\mathfrak{a}])} L(s, \mathbf{f}\,|_{\Phi}).$$

The above equation is initially true for some half plane and by analytic continuation is valid everywhere in the complex plane. Furthermore, since we have chosen $\mathbf{f}$ to be a cusp form, $L(s, \mathbf{g}_{\mathfrak{a}})$ is entire. We record this as a lemma for future reference.

Lemma 4.1 *For every integral ideal* $\mathfrak{a}$ *coprime to* $\mathfrak{m}$*, the* L*-series* $L(s, \boldsymbol{g}_{\mathfrak{a}})$ *is entire.*

In order to prove the existence of sign changes, we appeal to a classical theorem of Landau, but in order to do that successfully, we need to show that $L(s, \mathbf{g}_{\mathfrak{a}})$ has a finite abscissa of convergence.

For the remaining of the Section we fix an equivalence class $[\mathfrak{a}]$ inside $\mathfrak{R}_{\mathfrak{m}}^{+}$. The arguments in the remainder of this Section is inspired from Sect. 2 of [18].

Recall that $\mathbb{A}_F^{\times}$ denotes the group of ideles of F, and for any $t \in \mathbb{A}_F^{\times}$, $t\mathcal{O}_F$ denotes the fractional ideal of F associated with t in the natural way. As before, we let $h_{\mathfrak{m}} = |\mathfrak{R}_{\mathfrak{m}}^{+}|$ and choose $\{t_1, t_2, \ldots, t_{h_{\mathfrak{m}}}\}$ such that $\{t_i\mathcal{O}_F\}_{i=1}^{h_{\mathfrak{m}}}$ forms a complete set of representatives for the classes of $\mathfrak{R}_{\mathfrak{m}}^{+}$. Let $\mathfrak{t}_i$ denote $t_i\mathcal{O}_F$ for $1 \leq i \leq h_{\mathfrak{m}}$.

It is clear that for every integral ideal $\mathfrak{b}$ coprime to $\mathfrak{m}$, there is a unique λ with $1 \leq \lambda \leq h_{\mathfrak{m}}$ and a totally positive element $\xi \in \mathfrak{t}_{\lambda}$ such that $\mathfrak{b} = \xi\mathfrak{t}_{\lambda}^{-1}$. Now recall the definition of Fourier coefficients of a Hilbert modular form,

$$C(\mathfrak{b}, \mathbf{f}) = N(\mathfrak{t}_{\lambda})^{-k_0/2} a_{\lambda}(\xi)\xi^{(k_0\mathbf{1}-\boldsymbol{k})/2},$$

where we have chosen ξ and λ such that $\mathfrak{b} = \xi\mathfrak{t}_{\lambda}^{-1}$ as before and $\mathbf{1} = (1, 1, \ldots, 1)$. Without loss of generality, assume that $[\mathfrak{t}_1] = [\mathfrak{a}]$, denote $a_1(\xi)$ as $a(\xi)$ and $\mathfrak{t}_1$ as $\mathfrak{t}$. It follows that

$$L(s, \mathbf{g}_{\mathfrak{a}}) = \sum_{\mathfrak{b} \neq 0} \frac{C(\mathfrak{b}, \mathbf{g}_{\mathfrak{a}})}{N(\mathfrak{b})^s} = N(\mathfrak{t})^{-k_0/2} \sum_{\substack{\xi \in \mathfrak{t}^{-1} \\ 0 \ll \xi}} \frac{a(\xi)\xi^{(k_0\mathbf{1}-\boldsymbol{k})/2}}{N(\mathfrak{b})^s}. \tag{4.5}$$

To show that $L(s, \mathbf{g}_{\mathfrak{a}})$ has finite abscissa of convergence, we will show that $\sum_{\substack{\mathfrak{b} \in [\mathfrak{a}] \\ \mathfrak{b} \subset \mathcal{O}_F}} |C(\mathfrak{b}, \mathbf{f})|$ is not finite. We now consider the equivalence class of $\mathfrak{t}_1$ in the narrow class group and denote it by $[\mathfrak{t}_1]$. Note that if $\mathbf{f}|_{\Phi} \in S_{\boldsymbol{k}}(\mathfrak{n}\mathfrak{m}^2, \Psi\Phi^2)$, then by the Shimura correspondence [18] we have an h-tuple $((f|_\Phi)_1, (f|_\Phi)_2, \ldots, (f|_\Phi)_h)$ of classical Hilbert modular forms such that

$$C(\mathfrak{b}, \mathbf{f}|_\Phi) = \begin{cases} N(\mathfrak{b})^{\frac{k_0}{2}} a_\lambda(\xi)\xi^{\frac{-\boldsymbol{k}}{2}} & \text{if } \mathfrak{b} = \xi\mathfrak{t}_\lambda^{-1} \subset \mathcal{O}_F, \\ 0 & \text{otherwise} \end{cases}, \tag{4.6}$$

and

$$(f|_\Phi)_\lambda(z) = \sum_{0 \ll \xi \in \mathfrak{t}_\lambda} a_\lambda(\xi) \exp(2\pi i \operatorname{Tr}(\xi z)) \in S_{\boldsymbol{k}}(\Gamma(\mathfrak{t}_\lambda, \mathfrak{n}\mathfrak{m}^2), \psi\phi^2) \subset M_{\boldsymbol{k}}(\Gamma_{N(\mathfrak{n}\mathfrak{m}^2)}). \tag{4.7}$$

Consider the sum $\sum_\Phi \overline{\Phi[\mathfrak{a}]}(f|_\Phi)_1$ which is a classical Hilbert modular form in the space $M_{\boldsymbol{k}}(\Gamma_{N(\mathfrak{n}\mathfrak{m}^2)})$. Now using (4.6) and (4.7), we have

$$\begin{aligned}
\sum_\Phi \overline{\Phi[\mathfrak{a}]}(f|_\Phi)_1 &= \sum_\Phi \overline{\Phi[\mathfrak{a}]} \sum_{0 \ll \xi \in \mathfrak{t}_1} a_\lambda(\xi) \exp(2\pi i \operatorname{Tr}(\xi z))) \\
&= \sum_\Phi \overline{\Phi[\mathfrak{a}]} \sum_{\substack{\xi \in \mathfrak{t}_1 \\ \mathfrak{b} = \xi\mathfrak{t}_1{}^{-1} \subset \mathcal{O}_F}} C(\mathfrak{b}, \mathbf{f}|_\Phi) N(\mathfrak{b})^{\frac{-k_0}{2}} \xi^{\frac{\boldsymbol{k}}{2}} \exp(2\pi i \operatorname{Tr}(\xi z)) \\
&= \sum_\Phi \overline{\Phi[\mathfrak{a}]} \sum_{\substack{\xi \in \mathfrak{t}_1 \\ \mathfrak{b} = \xi\mathfrak{t}_1^{-1} \subset \mathcal{O}_F}} \Phi^*(\mathfrak{b}) C(\mathfrak{b}, \mathbf{f}) N(\mathfrak{b})^{\frac{-k_0}{2}} \xi^{\frac{\boldsymbol{k}}{2}} \exp(2\pi i \operatorname{Tr}(\xi z)) \\
&= \sum_{\substack{\xi \in \mathfrak{t}_1 \\ \mathfrak{b} = \xi\mathfrak{t}_1^{-1} \subset \mathcal{O}_F}} C(\mathfrak{b}, \mathbf{f}) N(\mathfrak{b})^{\frac{-k_0}{2}} \xi^{\frac{\boldsymbol{k}}{2}} \exp(2\pi i \operatorname{Tr}(\xi z)) \sum_\Phi \overline{\Phi[\mathfrak{a}]}\Phi([\mathfrak{b}]) \\
&= \sum_{\substack{\xi \in \mathfrak{t}_1 \\ \mathfrak{b} = \xi\mathfrak{t}_1^{-1} \subset \mathcal{O}_F \\ (\mathfrak{b}, \mathfrak{m}) = 1}} C(\mathfrak{b}, \mathbf{f}) N(\mathfrak{b})^{\frac{-k_0}{2}} \xi^{\frac{\boldsymbol{k}}{2}} \exp(2\pi i \operatorname{Tr}(\xi z)) \\
&= \sum_{\substack{\mathfrak{b} \in [\mathfrak{a}] \in \mathfrak{R}_\mathfrak{m}^+ \\ \mathfrak{b} \subset \mathcal{O}_F}} C(\mathfrak{b}, \mathbf{f}) N(\xi\mathfrak{t}_1^{-1})^{\frac{-k_0}{2}} \xi^{\frac{\boldsymbol{k}}{2}} \exp(2\pi i \operatorname{Tr}(\xi z)).
\end{aligned}$$

In the last two equalities we have used the facts, $C(\mathfrak{b}, \mathbf{f}) = 0$ whenever $\mathfrak{b} \not\subseteq \mathcal{O}_F$ and

$$\sum_\Phi \overline{\Phi([\mathfrak{a}])}\Phi([\mathfrak{b}]) = \begin{cases} 1 & \mathfrak{b} \equiv \mathfrak{a} \pmod{\mathfrak{m}} \text{ and } (\mathfrak{b}, \mathfrak{m}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $N(\xi\mathfrak{t}_1^{-1})^{\frac{-k_0}{2}} \xi^{\frac{\boldsymbol{k}}{2}} = N(\mathfrak{t}_1^{-1})^{\frac{k_0}{2}} \xi^{\frac{(\boldsymbol{k} - k_0 \cdot \mathbf{1})}{2}}$, where $\mathbf{1} = (1, 1, \ldots, 1)$. Therefore the sum

$$\sum_{\substack{\mathfrak{b}\in[\mathfrak{a}]\in\mathfrak{R}_{\mathfrak{m}}^{+}\\ \mathfrak{b}\subset\mathcal{O}_F}} \left|C(\mathfrak{b},\mathbf{f})N(\xi\mathfrak{t}_1^{-1})^{\frac{-k_0}{2}}\xi^{\frac{\boldsymbol{k}}{2}}\right| = \sum_{\substack{\mathfrak{b}\in[\mathfrak{a}]\in\mathfrak{R}_{\mathfrak{m}}^{+}\\ \mathfrak{b}\subset\mathcal{O}_F}} \left|C(\mathfrak{b},\mathbf{f})N(\mathfrak{t}_1^{-1})^{\frac{k_0}{2}}\xi^{\frac{(\boldsymbol{k}-k_0\cdot\mathbf{1})}{2}}\right|$$

$$\leq N(\mathfrak{t}_1^{-1})^{\frac{k_0}{2}} \sum_{\substack{\mathfrak{b}\in[\mathfrak{a}]\in\mathfrak{R}_{\mathfrak{m}}^{+}\\ \mathfrak{b}\subset\mathcal{O}_F}} \left|C(\mathfrak{b},\mathbf{f})\right|.$$

To show that the sum $\sum_{\substack{\mathfrak{b}\in[\mathfrak{a}]\in\mathfrak{R}_{\mathfrak{m}}^{+}\\ \mathfrak{b}\subset\mathcal{O}_F}} \left|C(\mathfrak{b},\mathbf{f})\right|$ is not finite it is enough to show that left hand side of the above inequality is not finite. Note that $\sum_{\Phi}\overline{\Phi[\mathfrak{a}]}(f|_{\Phi})_1$ is a classical Hilbert modular form and $C(\mathfrak{b},\mathbf{f})N(\xi\mathfrak{t}_1^{-1})^{\frac{-k_0}{2}}\xi^{\frac{\boldsymbol{k}}{2}}$ are its Fourier coefficients where $\xi\in\mathfrak{t}_1$. We will show for a general classical Hilbert modular form $g(\neq 0)\in M_{\boldsymbol{k}}(\Gamma_{N(\mathfrak{n}\mathfrak{m}^2)})$, the sum of absolute value of Fourier coefficients diverges. Suppose g has following Fourier expansion

$$g(\mathbf{z}) = \sum_{\substack{\xi\in\mathfrak{t}^{-1}\\ 0\ll\xi}} a(\xi)\exp(Tr(\xi\mathbf{z}))$$

and suppose that

$$\sum_{\substack{\xi\in\mathfrak{t}^{-1}\\ 0\ll\xi}} |a(\xi)| < \infty.$$

Then we have

$$|g(\mathbf{z})| \leq K < \infty, \tag{4.8}$$

for all $\mathbf{z}\in\mathcal{H}^n$. Suppose that $\mathfrak{n}\mathfrak{m}^2\cap\mathbb{Z}=\kappa\mathbb{Z}$. From (4.4) we observe that for every $\gamma_n=\begin{pmatrix}1 & 0\\ n\kappa & 1\end{pmatrix}\in\Gamma_\kappa$ (c.f. (2.2)), we have

$$g(\gamma_n(\mathbf{z})) = (n\kappa\mathbf{z}+1)^{\boldsymbol{k}}g(\mathbf{z}).$$

Now observing (4.8) we see that

$$|g(\mathbf{z})| = |n\kappa\mathbf{z}+1|^{-\boldsymbol{k}}|g(\gamma(\mathbf{z}))| \leq K|n\kappa\mathbf{z}+1|^{-\boldsymbol{k}}.$$

If we let $n\to\infty$, we see that $g\equiv 0$. In other words we can conclude that if $\mathbf{g}\neq 0$, then

$$\sum_{\substack{\xi\in\mathfrak{t}^{-1}\\ 0\ll\xi}} |a(\xi)| \to \infty.$$

Now the following lemma is immediate.

Lemma 4.2 *If* $\mathbf{g}_{\mathfrak{a}} \neq 0$*, then the L series* $L(s, \mathbf{g}_{\mathfrak{a}})$ *has a finite abscissa of convergence. In other words,* $L(s, \mathbf{g}_{\mathfrak{a}})$ *is not absolutely convergent everywhere.*

Proof of Theorem 1.2 In order to prove Theorem 1.2 we first state the following famous result of Landau (see [16]).

Theorem 4.1 [Landau] *Suppose* $L(s) = \sum_{n \in \mathbb{N}} a_n n^{-s}$ *is absolutely convergent on some half plane and suppose that* $a_n \geq 0$ *for all but finitely many n. Then either* $L(s)$ *is absolutely convergent everywhere or* $L(s)$ *has a singularity at the abscissa of convergence.*

We apply Theorem 4.1 for the L series $L(s, \mathbf{g}_{\mathfrak{a}})$. From Lemma 4.1, $L(s, \mathbf{g}_{\mathfrak{a}})$ is entire and from Lemma 4.2, $L(s, \mathbf{g}_{\mathfrak{a}})$ is not absolutely convergent everywhere. Therefore the only possibility is that $L(s, \mathbf{g}_{\mathfrak{a}})$ does not satisfy the assumptions of the theorem. In other words, we can conclude that $B'(n_1) < 0$ and $B'(n_2) > 0$ for infinitely many choices of $n_1, n_2 \in \mathbb{N}$, where we have defined $B'(\kappa)$ as

$$B'(\kappa) = \sum_{\substack{[\mathfrak{b}]=[\mathfrak{a}] \\ N(\mathfrak{b})=\kappa}} C(\mathfrak{b}, \mathbf{g}_{\mathfrak{a}}).$$

Now, Theorem 1.2 follows. □

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