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Modular forms and ellipsoidal T-designs

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Abstract
In recent work, Miezaki introduced the notion of a *spherical* T*-design* in $\mathbb{R}^2$, where T is a potentially infinite set. As an example, he offered the $\mathbb{Z}^2$-lattice points with fixed integer norm (a.k.a. shells). These shells are *maximal* spherical T-designs, where $T = \mathbb{Z}^+ \setminus 4\mathbb{Z}^+$. We generalize the notion of a spherical T-design to special ellipses, and extend Miezaki's work to the norm form shells for rings of integers of imaginary quadratic fields with class number 1.

Keywords Modular forms · Combinatorics · Number theory · Spherical t-designs · Hecke eigenforms

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1 Introduction and statement of results

Spherical t-designs were introduced in 1977 by Delsarte, Goethals and Seidel [5], and they have played an important role in algebra, combinatorics, number theory and quantum mechanics (for background see [2–4, 6, 9, 10]). A spherical t-design is a nonempty finite set of points on the unit sphere with the property that the average value of any real polynomial of degree $\leq t$ over this set equals the average value over the sphere. Namely, if S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ centered at the origin, then a finite nonempty subset $X \subset S^{n-1}$ is a spherical t-design if

$$\frac{1}{|X|} \sum_{x \in X} P(x) = \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{S^{n-1}} P(x) \mathrm{d}\sigma(x) \tag{1.1}$$

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for all polynomials $P(x)$ of degree $\leq t$. The right-hand side of (1.1) is the usual surface integral over S^{n-1}. In general, a finite nonempty subset X of $S_{n-1}(r)$, the sphere of radius r centered at the origin, is a spherical t-design if $\frac{1}{r}X$ satisfies (1.1). Since a spherical t-design is also a spherical t'-design for all $t' \leq t$, we say that X has *strength* t if it is the maximum of all such numbers.

Delsarte, Goethals and Seidel developed a very simple criterion for determining spherical t-designs. This criterion involves *homogeneous harmonic* polynomials of bounded degree. A polynomial in n variables is *harmonic* if it is annihilated by the Laplacian operator $\Delta := \sum_{i=1}^{n} \partial^2/\partial x_i^2$, and they showed [5] that $X \subset S^{n-1}$ is a spherical t-design if

$$\sum_{x \in X} P(x) = 0 \tag{1.2}$$

for all homogeneous harmonic polynomials $P(x)$ of nonzero degree $\leq t$. This criterion is a consequence of two results from harmonic analysis. The first result is the mean value property for harmonic functions [1,p. 5], which implies that the integral of a harmonic polynomial over a sphere centered at the origin vanishes, combined with the fact that homogeneous polynomials of fixed degree are spanned by certain harmonic polynomials [1,Theorem 5.7].

In view of this framework, it is natural to ask whether there are generalizations of spherical t-designs to other curves, surfaces and varieties. Here we consider certain *ellipsoids*[1] in dimension two. To be precise, for square-free $D \geq 1$ we define the norm r ellipses

$$C_D(r) := \begin{cases} \{(x, y) \in \mathbb{R}^2 : x^2 + Dy^2 = r\} & \text{if } D \equiv 1, 2 \pmod 4, \\ \{(x, y) \in \mathbb{R}^2 : x^2 + xy + \frac{1+D}{4} y^2 = r\} & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{1.3}$$

Remark These ellipses arise from certain imaginary quadratic orders.

For $D \equiv 1, 2 \pmod 4$, we say that a finite nonempty subset $X \subset C_D(r)$ is an *ellipsoidal t-design* if

$$\frac{1}{|X|} \sum_{(x,y) \in X} P(x, y) = \frac{1}{2\pi\sqrt{D}} \int_{C_D(r)} \frac{P(x, y)}{\sqrt{x^2/D^2 + y^2}} d\sigma(x, y) \tag{1.4}$$

for all polynomials $P(x, y)$ of degree $\leq t$ over $\mathbb{R}$. For $D \equiv 3 \pmod 4$, instead we require

$$\frac{1}{|X|} \sum_{(x,y) \in X} P(x, y) = \frac{\sqrt{D}}{\pi} \int_{C_D(r)} \frac{P(x, y)}{\sqrt{20x^2 + (D^2 + 2D + 5)y^2 + (20 + 4D)xy}} d\sigma(x, y). \tag{1.5}$$

Here the right-hand sides are line integrals. As in the case of spherical t-designs, every ellipsoidal t-design is also an ellipsoidal t'-design for all $t' \leq t$, and the maximum of

[1] We do not use the term *ellipse* to avoid possible confusion that might arise with the term *elliptical*.

all such t's is called the *strength* of X.These definitions coincide with the notion of a spherical t-design when $D = 1$.

In analogy to Delsarte, Goethals and Seidel, we have a natural criterion for confirming ellipsoidal t-designs. To this end, we consider the 2-dimensional real vector space

$$H^{\mathbb{R}}_{D,j}[x, y] := \begin{cases} \langle \mathrm{Re}(x + \sqrt{-D}y)^j, \mathrm{Im}(x + \sqrt{-D}y)^j \rangle & \text{if } D \equiv 1, 2 \pmod 4, \\ \langle \mathrm{Re}(x + \frac{1+\sqrt{-D}}{2}y)^j, \mathrm{Im}(x + \frac{1+\sqrt{-D}}{2}y)^j \rangle & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{1.6}$$

In terms of these vector spaces of polynomials, we have the following ellipsoidal t-design criterion.

Theorem 1.1 *A finite nonempty set $X \subset C_D(r)$ is an ellipsoidal t-design if*

$$\sum_{x \in X} P(x, y) = 0$$

for all $P(x, y) \in H^{\mathbb{R}}_{D,j}[x, y]$ for all $0 < j \leq t$.

Remark (1) Observe that if $X \subset S^1$ is a spherical t-design, then $Y = \{(x, y/\sqrt{D}) | (x, y) \in X\} \subset C_D$ (resp. $Y = \{(x + y/\sqrt{D}, 2y/\sqrt{D} | (x, y) \in X\} \subset C_D$) is an ellipsoidal t-design for $D \equiv 1, 2 \pmod 4$ (resp. $D \equiv 3 \pmod 4$). Therefore, the existence of a spherical t-design implies the existence of a corresponding ellipsoidal t-design. In fact, there is a one-to-one correspondence between spherical t-designs and ellipsoidal t-designs. However, the proof of Theorem 1.1 is not a direct consequence because care is required for justifying the role of the vector spaces $H^{\mathbb{R}}_{D,j}[x, y]$.

(2) Since there is one-to-one correspondence between spherical and ellipsoidal t-designs, we get a lower bound [5,p. 2] on the size of ellipsoidal t-design X,

$$|X| \geq t + 1.$$

Recently, Miezaki in [9] introduced a generalization of the notion of spherical t-designs. Instead of restricting to polynomials of degree $\leq t$, he considered harmonic polynomials of degree $j \in T \subset \mathbb{N}$, where T is a potentially infinite set. The main theorem from [9] gives infinitely many spherical T-designs for $T := \mathbb{Z}^+ \setminus 4\mathbb{Z}^+$ in dimension two. Namely, he considered norm r shells, integer points on $x^2 + y^2 = r$ for fixed $r \in \mathbb{Z}^+$. He showed that these r-shells are spherical T-designs. Moreover, these sets have strength T, meaning that (1.2) fails if any multiple of 4 is added to T. His proof makes use of theta functions arising from complex multiplication by $\mathbb{Z}[i]$.

We generalize Miezaki's work to ellipsoidal T-designs. We call $X \subset C_D$ an *ellipsoidal T-design* if the condition in Theorem 1.1 is satisfied for all polynomials in $H^{\mathbb{R}}_{D,j}[x, y]$ with $j \in T$. We say X has strength T if it is maximal among such sets. For each square-free positive integer D, let $\mathcal{O}_D$ be the ring of integers of

$\mathbb{Q}(\sqrt{-D})$. In particular, this means that

$$\mathcal{O}_D = \begin{cases} \mathbb{Z}[\sqrt{-D}] & \text{if } D \equiv 1, 2 \pmod 4, \\ \mathbb{Z}[\frac{1+\sqrt{-D}}{2}] & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{1.7}$$

We consider $D \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$, the square-free positive integers for which $\mathcal{O}_D$ has class number 1. To make this precise, we define the *norm r shells* in $C_D(r)$ by

$$\Lambda_D^r := \mathcal{O}_D \cap C_D(r). \tag{1.8}$$

Generalizing Miezaki's work for $D = 1$, we obtain the following theorem.

Theorem 1.2 *If $D \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$, then every non-empty shell Λ_D^r is an ellipsoidal T_D design with strength T_D, where*

$$T_D := \begin{cases} \mathbb{Z}^+ \setminus 4\mathbb{Z}^+ & \textit{if } D = 1, \\ \mathbb{Z}^+ \setminus 6\mathbb{Z}^+ & \textit{if } D = 3, \\ \mathbb{Z}^+ \setminus 2\mathbb{Z}^+ & \textit{otherwise.} \end{cases}$$

Remark The method used here seems to be well-poised only for the dimension 2 cases. It would be interesting to obtain higher dimensional analogues.

Example We consider $D = 3$, and $r = 691$. Then we have

$$\begin{aligned} \Lambda_3^{691} = \{&(11, 19), (-11, -19), (19, 11), (-19, -11), \\ &(11, -30), (-11, 30), (30, -19), (-30, 19), \\ &(30, -11), (-30, 11), (19, -30), (-19, 30)\}. \end{aligned}$$

We consider the polynomial $P(x, y) = 2x^2 + 3462xy + 1729y^2 \in H_{3,2}^{\mathbb{R}}[x, y]$, and we find that $\sum_{(x,y)\in\Lambda_3^{691}} P(x, y) = 0$ which shows that Λ_3^{691} is an elliptical 2-design and $2 \in T_3$. On the other hand, Theorem 1.2 implies that Λ_3^{691} is not an ellipsoidal 6-design.To see this we choose $Q(x, y) = 2x^2+6x^5y-15x^4y^2-40x^3y^3-15x^2y^4+6xy^5+2y^6 \in H_{3,6}^{\mathbb{R}}(x, y)$, and we find that $\sum_{(x,y)\in\Lambda_3^{691}} Q(x, y) = -4818834696 \neq 0$.

In Sect. 2 we prove Theorem 1.1, criterion for confirming that a set is an ellipsoidal t-design, and in Section 3 we recall the theory of theta functions arising from complex multiplication, and we prove Theorem 1.2.

2 Criterion for ellipsoidal *t*-design

In this section we prove Theorem 1.1, criterion for confirming ellipsoidal t-designs. Throughout this section we assume that $D \geq 1$ is square-free and $j \geq 1$.

To prove that Theorem 1.1 is indeed a criterion for confirming ellipsoidal t-designs, we first need to show that the spaces $H_{D,k}^{\mathbb{R}}[x, y]$, for $0 < k \leq j$, generate all the polynomials of degree $\leq j$ when restricted to $C_D(r)$. It suffices to show this for $P_j^{\mathbb{R}}[x, y]$, the set of homogeneous polynomials of degree j.

Lemma 2.1 *If $D \geq 1$ is square-free and $j \geq 1$, then the following are true:*
(1) *If $D \equiv 3 \mod 4$, then we have*

$$P_j^{\mathbb{R}}[x, y] = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} (x^2 + Dy^2)^k H_{D,j-2k}^{\mathbb{R}}[x, y].$$

(2) *If $D \equiv 3 \mod 4$, then we have*

$$P_j^{\mathbb{R}}[x, y] = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} \left(x^2 + xy + \frac{1+D}{4} y^2\right)^k H_{D,j-2k}^{\mathbb{R}}[x, y].$$

Proof The lemma is well known for homogeneous harmonic polynomials (for example, see [1,Theorem 5.7]). Namely, if $H_k^{\mathbb{R}}[x, y]$ is the set of homogeneous harmonic polynomials of degree k then

$$P_j^{\mathbb{R}}(x, y) = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} (x^2 + y^2)^k H_{j-2k}^{\mathbb{R}}[x, y].$$

We extend it to general D. It is well known that $H_j^{\mathbb{R}}[x, y] = \langle \mathrm{Re}(x+iy)^j, \mathrm{Im}(x+iy)^j \rangle$, and so if we do the change of variable for $D \equiv 1, 2 \mod 4$ (resp. $D \equiv 3 \mod 4$), $x' = x$,$y' = \sqrt{D}y$ (resp. $x' = x + y/2$,$y' = 2y/\sqrt{D}$), then $H_{j-2}^{\mathbb{R}}(x', y') = \langle \mathrm{Re}(x' + iy')^j, \mathrm{Im}(x' + iy')^j \rangle$ gives

$$P_j^{\mathbb{R}}[x', y'] = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} (x'^2 + y'^2)^k H_{j-2k}^{\mathbb{R}}[x', y'].$$

Therefore, if $D \equiv 1, 2 \mod 4$, then we have

$$P_j^{\mathbb{R}}(x, y) = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} (x^2 + Dy^2)^k H_{D,j-2k}^{\mathbb{R}}[x, y].$$

If $D \equiv 3 \mod 4$, then we have

$$P_j^{\mathbb{R}}(x, y) = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} \left(x^2 + xy + \frac{1+D}{4} y^2\right)^k H_{D,j-2k}^{\mathbb{R}}[x, y].$$

□

We now prove Theorem 1.1.

Proof of Theorem 1.1 Lemma 2.1 shows that the set of polynomials when restricted to C_D are generated by the spaces $H^{\mathbb{R}}_{D,j}[x, y]$ since $x^2 + Dy^2 = r$ (resp., $x^2 + xy + \frac{1+D}{4}y^2 = r$) on $C_D(r)$. Therefore, it suffices to show that if $P(x, y) \in H^{\mathbb{R}}_{D,j}[x, y]$, then the following are true:

(1) If $D \equiv 1, 2 \mod 4$, then we have

$$\int_{C_D(r)} \frac{P(x, y)}{\sqrt{x^2/D^2 + y^2}} \mathrm{d}\sigma(x, y) = 0.$$

(2) If $D \equiv 3 \mod 4$, then we have

$$\int_{C_D(r)} \frac{P(x, y)}{\sqrt{20x^2 + (D^2 + 2D + 5)y^2 + (20 + 4D)xy}} \mathrm{d}\sigma(x, y) = 0.$$

As $H^{\mathbb{R}}_{D,j}[x, y]$ is a vector space, it is enough to show these claims for basis vectors. Since $X \subset C_D(r)$ is an ellipsoidal t-design if and only if $\frac{1}{r} \subset C_D(1)$ is an ellipsoidal t-design, it's enough to consider $r = 1$. For $D \equiv 1, 2 \pmod 4$, $H^{\mathbb{R}}_{D,j}[x, y] = \langle \mathrm{Re}(x+\sqrt{-D}y)^j, \mathrm{Im}(x + \sqrt{-D}y)^j\rangle$. By the parametrization of $C_D(1) : x^2 + Dy^2 = 1$ as $\gamma := \{(\cos\theta, \sin\theta/\sqrt{D}) | 0 \le \theta \le 2\pi\}$, we have

$$\begin{aligned}
&\int_{C_D(1)} \frac{\mathrm{Re}(x + \sqrt{-D}y)^j}{\sqrt{x^2/D^2 + y^2}} \mathrm{d}\sigma(x, y)\\
&= \int_0^{2\pi} \frac{\mathrm{Re}(\cos\theta + \sqrt{-D}(\sin\theta/\sqrt{D}))^j}{\sqrt{\cos\theta^2/D^2 + \sin\theta^2/D}} \sqrt{\sin\theta^2 + \cos\theta^2/D}\mathrm{d}\theta\\
&= \sqrt{D} \int_0^{2\pi} \mathrm{Re}(\cos\theta + i \sin\theta)^j \mathrm{d}\theta\\
&= \sqrt{D} \int_{S^1} \mathrm{Re}(x + iy)^j \mathrm{d}z = 0.
\end{aligned}$$

Since $\mathrm{Re}(x + iy)^j$ is harmonic, the last integral over S^1 is 0.

A similar argument shows that

$$\int_{C_D(1)} \frac{\mathrm{Im}(x + \sqrt{-D}y)^j}{\sqrt{x^2/D^2 + y^2}} \mathrm{d}\sigma(x, y) = 0.$$

If $D \equiv 3 \pmod 4$, $H^{\mathbb{R}}_{D,j}[x, y] = \langle \mathrm{Re}(x + \frac{1+\sqrt{-D}}{2}y)^j, \mathrm{Im}(x + \frac{1+\sqrt{-D}}{2}y)^j\rangle$. By the parametrization of $C_D(1) : x^2 + xy + \frac{1+D}{4}y^2 = 1$ as $\gamma := \{(\cos\theta -$

$\sin\theta/\sqrt{D}, 2\sin\theta/\sqrt{D}) : 0 \le \theta \le 2\pi\}$, we have

$$\begin{aligned}
&\int_{C_D(1)} \frac{\mathrm{Re}(x+(1+\sqrt{-D})y/2)^j}{\sqrt{20x^2+(D^2+2D+5)y^2+(20+4D)xy}}\mathrm{d}\sigma(x,y)\\
&= \int_0^{2\pi} \frac{\mathrm{Re}(\cos\theta - \sin\theta/\sqrt{D} + (1+\sqrt{-D}\sin\theta/\sqrt{D})^j}{\sqrt{4D\sin\theta^2+20\cos\theta^2+8\sqrt{D}\sin\theta\cos\theta}}\\
&\quad \times \sqrt{\sin\theta^2+5\cos\theta^2/D+2\sin\theta\cos\theta/\sqrt{D}}\mathrm{d}\theta\\
&= \frac{1}{2\sqrt{D}}\int_0^{2\pi}\mathrm{Re}(\cos\theta+i\sin\theta)^j\mathrm{d}\theta = \frac{1}{2\sqrt{D}}\int_{S^1}\mathrm{Re}(x+iy)^j\mathrm{d}z = 0.
\end{aligned}$$

A similar argument shows that

$$\int_{C_D(1)} \frac{P(x)}{\sqrt{20x^2+(D^2+2D+5)y^2+(20+4D)xy}}\mathrm{d}\sigma(x,y) = 0.$$

□

3 Ellipsoidal T-designs

Here we prove Theorem 1.2, the construction of ellipsoidal T-designs arising from the ring of integers of imaginary quadratic fields with class number 1. We use the theory of theta functions with complex multiplication. Throughout, we shall assume that $D \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$.

3.1 Theta functions

Given an n-dimensional lattice Λ and a polynomial $P(x)$ of degree j in n variables, the theta function of $P(x)$ over the lattice Λ is defined by the Fourier series (note $q := e^{2\pi i z}$)

$$\Theta(\Lambda, P; z) := \sum_{x\in\Lambda} P(x)q^{N(x)} = \Theta(\Lambda, P; z) = \sum_{n=0}^{\infty} a(\Lambda, P, n)q^n, \tag{3.1}$$

where $N(x)$ is the standard norm in $\mathbb{R}^n$. The theta functions for $\Lambda_D = \mathcal{O}_D$ play an important role in the study of ellipsoidal T-designs. Namely, if $\Theta(\Lambda_D, P; z) = \sum_{r=0}^{\infty} a(\Lambda_D, P, r)q^r$, then

$$a(\Lambda_D, P, r) = \sum_{(x,y)\in\Lambda_D^r} P(x, y). \tag{3.2}$$

The theta function $\Theta(\Lambda_D, P; z) \in \mathcal{M}_k(\Gamma_0(4D), \chi)$, the space of holomorphic modular forms with weight $k = j+1$ and nebentypus $\chi(A) = (\frac{-D}{d})$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ [7,Theorem 10.8]. Moreover, $\Theta(\Lambda_D, P; z)$ is a cusp form when $j > 0$.

To ease the study of these theta function, it is convenient to introduce the following the polynomials for each $j \geq 1$:

$$R_{D,j}(x, y) := \begin{cases} \mathrm{Re}(x + \sqrt{-D}y)^j & \text{if } D \equiv 1, 2 \pmod 4, \\ \mathrm{Re}(x + \frac{1+\sqrt{-D}}{2}y)^j & \text{if } D \equiv 3 \pmod 4, \end{cases} \tag{3.3}$$

and

$$I_{D,j}(x, y) := \begin{cases} \mathrm{Im}(x + \sqrt{-D}y)^j & \text{if } D \equiv 1, 2 \pmod 4, \\ \mathrm{Im}(x + \frac{1+\sqrt{-D}}{2}y)^j & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{3.4}$$

By definition, we have that $H^{\mathbb{R}}_{D,j}[x, y] = \langle R_{D,j}(x, y), I_{D,j}(x, y)\rangle$. In particular, $\Theta(\Lambda_D, R_{D,j}; z)$ and $\Theta(\Lambda_D, I_{D,j}; z)$ are cusp forms. Theorem 1.1 together with the discussion above gives the following lemma which transforms the problem of determining ellipsoidal T-designs into the vanishing of certain coefficients of special theta functions.

Lemma 3.1 *The norm r shell $\Lambda^r_D = \Lambda_D \cap C_D(r)$ is an ellipsoidal T-design if and only if $a(\Lambda_D, R_{D,j}, r) = 0$ and $a(\Lambda_D, I_{D,j}, r) = 0$ for all $j \in T$.*

We require some standard facts from the theory of newforms. Since $\mathcal{O}_D$ has class number 1, each *Hecke character* mod $\mathcal{O}_D$ is defined by its values on principal ideals. Let $(\alpha) \subset \mathcal{O}_D$ be a principal ideal. Let u_D be the number of units in $\mathcal{O}_D$, namely

$$u_D := \begin{cases} 4 & \text{if } D = 1, \\ 6 & \text{if } D = 3, \\ 2 & \text{otherwise.} \end{cases} \tag{3.5}$$

For each positive $j_D \equiv 0 \pmod{u_D}$, define Hecke characters mod $\mathcal{O}_D$ by:

$$\zeta_{j_D}((\alpha)) = \left(\frac{\alpha}{|\alpha|}\right)^{j_D}$$

Then by [8,Theorem 4.8.2], we have the following well known lemma about the modular form

$$f_{j_D}(\zeta_{j_D}; z) := \begin{cases} \Theta(\Lambda_D, (x + \sqrt{-D}y)^j; z) & \text{if } D \equiv 1, 2 \pmod 4, \\ \Theta\Big(\Lambda_D, \Big(x + \frac{1+\sqrt{-D}}{2}y\Big)^j; z\Big) & \text{if } D \equiv 3 \pmod 4) \end{cases}$$

Lemma 3.2 *Assuming the notations above, we have*

$$f_{j_D}(\zeta_{j_D}; z) = \sum_{(\alpha)\subset\mathcal{O}_D} \zeta_{j_D}((\alpha)) N(\alpha)^{j/2} q^{N(\alpha)} \in \mathcal{S}_{k_D}(\Gamma_0(N), \chi),$$

the space of cusp forms of weight $k_D = j_D + 1$ *with nebentypus* $\chi \pmod N$. *Here* $N := |\Delta_{\mathcal{O}_D}|$, *the absolute value of the discriminant of* $\mathcal{O}_D$. *Moreover,* $f_{j_D}(\zeta_{j_D}; z)$ *is a newform.*

3.2 Other propositions and lemmas

Recall that $\Lambda_D^r = C_D(r) \cap \mathcal{O}_D$. Using well known facts about the positive definite binary quadratic forms corresponding to class number 1 norm forms, we have the following lemma.

Lemma 3.3 *Suppose* r *is a positive integer. Then* Λ_D^r *is nonempty if and only if* $ord_p(r)$ *is even for every prime* $p \nmid r$ *for which* Λ_D^p *is nonempty.*

Rewriting (3.2), we have

$$a(\Lambda_D, P, r) = \sum_{(x,y)\in\Lambda_D^r} P(x, y). \tag{3.6}$$

Lemma 3.1 implies that Λ_D^r is an ellipsoidal T-design if and only if $a(\Lambda_D, R_{D,j}, r)$ and $a(\Lambda_D, I_{D,j}, r)$ vanish for all $j \in T$. Since Λ_D^r is antipodal (*i.e.* $-\Lambda_D^r = \Lambda_D^r$ for all r), $a(\Lambda_D, R_{D,j}, r)$ and $a(\Lambda_D, I_{D,j}, r)$ are 0 for all $j \in \mathbb{Z}^+ \setminus 2\mathbb{Z}^+$. Therefore, we have that following proposition.

Proposition 3.1 *Suppose* $r \in \mathbb{Z}^+$ *such that* Λ_D^r *is nonempty. Then* Λ_D^r *is an ellipsoidal* $\mathbb{Z}^+ \setminus 2\mathbb{Z}^+$*-design.*

Our objective is to find maximal set T_D for which Λ_D^r is ellipsoidal T-design. By proposition above we have that $\mathbb{Z}^+ \setminus 2\mathbb{Z}^+ \subset T_D$. So we only look for all even j which can be in T_D.

Proposition 3.2 *Suppose* $j \equiv 0 \pmod 2$, *and* $r \in \mathbb{Z}^+$. *Then the following are true:*

(1) *We have that* $a(\Lambda_D, I_{D,j}, r) = 0$.

(2) *We have that* $a(\Lambda_D, R_{D,j}, r) = \begin{cases} \sum_{(x_0,y_0)\in\Lambda_D^r} (x + \sqrt{-D}y)^j & \text{if } D \equiv 1, 2 \pmod 4, \\ \sum_{(x_0,y_0)\in\Lambda_D^r} \left(x + \frac{1+\sqrt{-D}}{2} y\right)^j & \text{if } D \equiv 3 \pmod 4 \end{cases}$

Proof Part *(2)* is an obvious consequence of part *(1)*. So it is enough to prove part*(1)*. The idea is to show that points in Λ_D^r occur in pairs on which value of $I_{D,j}$ cancel. If $D \equiv 1, 2 \pmod 4$, then $I_{D,j} = \mathrm{Im}(x + \sqrt{-D}y)^j$. In this case $(a, b), (a, -b) \in \Lambda_D^r$ such that $I_{D,j}(a, b) + I_{D,j}(a, -b) = 0$. This is true because each term of $I_{D,j}(x, y)$ has odd power in both the variables x, y. If $D \equiv 3 \pmod 4$, then $I_{D,j} = \mathrm{Im}((x + \frac{1}{2}y) + \frac{\sqrt{-D}}{2}y)^j$. In this case $(a, b), (a+b, -b) \in \Lambda_D^j$ such that $I_{D,j}(a, b) + I_{D,j}(a + b, -b) = 0$. This is because each term of $I_{D,j}(x, y)$ has odd power in $x + y/2, y$. $\square$

We notice that if $(x_0, y_0) \in \mathcal{O}_D$, then we have

$$\sum_{\alpha_D\in\mathcal{O}_D:|\alpha_D|=1} R_{D,j}(\alpha_D(x_0, y_0)) = R_{D,j}(x_0, y_0) \sum_{\alpha_D\in\mathcal{O}_D:|\alpha_D|=1} \alpha_D^j. \tag{3.7}$$

Proposition 3.3 *If $r \geq 1$, $1 \leq j \not\equiv 0 \pmod{u_D}$, and Λ_D^r nonempty, then $a(\Lambda_D^r, R_{D,j}, r) = 0$*

Proof The idea is that if $(x_0, y_0) \in \Lambda_D^r$ then $\alpha_D(x_0, y_0) \in \Lambda_D^r$ where α_D is a unit in $\mathcal{O}_D$. Therefore enough to show that the sum in RHS of (3.7) is 0. For $D = 1$, number of units in $\mathcal{O}_D$, $u_D = 4$ which are $\{1, -1, i, -i\}$. We have $1^j + (-1)^j + i^j + (-i)^j = 0$. For $D = 3$, number of units in $\mathcal{O}_D$, $u_D = 6$ which are $\{\pm 1, \frac{\pm 1 \pm \sqrt{-3}}{2}\}$. A brute force calculation shows the result. For other D, the number of units in $\mathcal{O}_D$, $u_D = 2$ which are $\{1, -1\}$. For all j odd, $(1)^j + (-1)^j = 0$ □

From here on we will only consider the theta function $\Theta\Big(\Lambda_D, \frac{1}{u_D} R_{D,j}; z\Big)$ so let's give its coefficients a shorthand.

$$\Theta\Big(\Lambda_D, \frac{1}{u_D} R_{D,j}; z\Big) = \sum_{r=0}^{\infty} a(D, j, r) q^r. \tag{3.8}$$

Proposition 3.2 together with Lemma 3.2 give us that if $j \equiv 0 \pmod{u_D}$, then the theta function $\Theta\Big(\Lambda_D, \frac{1}{u_D} R_{D,j}; z\Big) \in \mathcal{S}_{j+1}(\Gamma_0(N), \chi)$ is a Hecke eigenform. So we have the following lemma.

Lemma 3.4 *Suppose $j \in u_D\mathbb{Z}^+$. Then the following is true:*
(1) *If* $\gcd(r_1, r_2) = 1$ *then*

$$a(D, j, r_1 r_2) = a(D, j, r_1) a(D, j, r_2).$$

(2) *For p prime and $\alpha > 0$, we have*

$$a(D, j, p^\alpha) = a(D, j, p) a(D, j, p^{\alpha-1}) - \chi(p) p^j a(D, j, p^{\alpha-2}).$$

(3) *For p prime and $\alpha > 0$, we have*

$$a(D, j, p^\alpha) = a(D, j, p)^\alpha \pmod{p}.$$

Suppose p be a prime such that Λ_D^p be nonempty. Let $(x_p, y_p) \in \Lambda_D^p$ and $j \equiv 0 \pmod{u_D}$. When $p = D$ then it ramifies in $\mathcal{O}_D$ and there are exactly u_D points in Λ_D^p. From (3.7) we have $a(D, j, p) = R_{D,j}(x_p, y_p)$. If $p \neq D$ then it's unramified and we get exactly $2u_D$ solutions. In this case $a(D, j, p) = 2R_{D,j}(x_p, y_p)$.

Lemma 3.5 *Suppose $j \in u_D\mathbb{Z}^+$ and p be an odd prime such that Λ_D^p is nonempty. Let $(x_p, y_p) \in \Lambda_D^p$ then $R_{D,j}(x_p, y_p) \not\equiv 0 \pmod{p}$. In particular, $a(D, j, p)$ is non-zero.*

Proof We will consider two cases, $D \equiv 1, 2 \pmod 4$ and $D \equiv 3 \pmod 4$. Proof is essentially same in both the cases.

If $D \equiv 1, 2 \pmod 4$ then $p = x_p^2 + Dy_p^2$, in particular $x_p \not\equiv 0 \pmod p$. we consider the binomial expansion

$$\begin{aligned} R_{D,j}(x_p, y_p) &= \mathrm{Re}(x_p + \sqrt{-D}y_p)^j \\ &= \frac{1}{2}\sum_{n=0}^{j/2}\binom{j}{2n}x_p^{j-2n}(-1)^n(Dy_p^2)^n \\ &= \frac{1}{2}\sum_{n=0}^{j/2}\binom{j}{2n}x_p^{j-2n}(-1)^n(p - x_p^2)^n \\ &\equiv \frac{1}{2}x_p^j\sum_{n=0}^{j/2}\binom{j}{2n} \equiv 2^{j-2}x_p^j \not\equiv 0 \pmod p \end{aligned}$$

If $D \equiv 1, 2 \pmod 4$ then $p = (x_p + y_p/2)^2 + Dy_p^2/4$, in particular $x_p + y_p/2 \not\equiv 0 \pmod p$. we consider the binomial expansion

$$\begin{aligned} R_{D,j}(x_p, y_p) &= \mathrm{Re}\Big(x_p + y_p/2 + \sqrt{-D}y_p/2\Big)^j \\ &= \frac{1}{2}\sum_{n=0}^{j/2}\binom{j}{2n}\Big(x_p + \frac{y_p}{2}\Big)^{j-2n}(-1)^n\Big(\frac{Dy_p^2}{4}\Big)^n \\ &= \frac{1}{2}\sum_{n=0}^{j/2}\binom{j}{2n}\Big(x_p + \frac{y_p}{2}\Big)^{j-2n}(-1)^n\Big(p - \Big(x_p + \frac{y_p}{2}\Big)^2\Big)^n \\ &\equiv \frac{1}{2}\Big(x_p + \frac{y_p}{2}\Big)^j\sum_{n=0}^{j/2}\binom{j}{2n} \equiv 2^{j-2}\Big(x_p + \frac{y_p}{2}\Big)^j \not\equiv 0 \pmod p \end{aligned}$$

□

Proposition 3.4 *For prime 2, Λ_D^2 is nonempty only for $D = 1, 2, 7$. In this case $a(D, j, 2)$ does not vanish for all $j \in 2\mathbb{Z}^+$. Moreover, we have that $a(7, j, 2) \equiv 1 \pmod 2$*

Proof For $D = 1, 2$, $2|\Delta_{\mathcal{O}_D}(= -4D)$ so the ideal (2) is ramified in $\mathcal{O}_D$, in particular there are elements of norm 2. For $D \in \{3, 7, 11, 19, 43, 67, 163\}$, $2 \nmid \Delta_{\mathcal{O}_D}(= -D)$. So the ideal (2) is unramified in $\mathcal{O}_D$. Here we need to check whether 2 splits or not. We have the condition that 2 splits if and only if $-D \equiv 1 \pmod 8$. Only $D = 7$ satisfies the condition.

A brute force calculation shows that $a(1, j, 2) = (1 + i)^j \neq 0$, $a(2, j, 2) = i^j 2^{j+1} \neq 0$, and $a(7, j, 2) = 4\mathrm{Re}\Big(\frac{1+\sqrt{-7}}{2}\Big)^j \neq 0$.

We prove that $a(7, j, 2) \equiv 1 \pmod 2$ using induction on even j. First, note that $a(7, 2, 2) = -3 \equiv 1 \pmod 2$. Now we assume that $a(7, j, 2) \equiv 1 \pmod 2$, which implies that $\mathrm{Re}\Big(\frac{1+\sqrt{-7}}{2}\Big)^j = (2k + 1)/2$ for some k. The norm of $\Big(\frac{1+\sqrt{-7}}{2}\Big)^j$ is even, so we get that $\mathrm{Im}\left(\frac{1+\sqrt{-7}}{2}\right)^j = \sqrt{7}(2k'+1)/2$ for some k'. An easy calculation shows that $a(7, j+2, 2) = -3\mathrm{Re}\left(\frac{1+\sqrt{-7}}{2}\right)^j - \sqrt{7}\mathrm{Im}\left(\frac{1+\sqrt{-7}}{2}\right)^j \equiv 1 \pmod 2$. □

3.3 Proof of Theorem 1.2

Propositions 3.1, 3.2 and 3.3 together imply that $a(\Lambda_D, R_{D,j}, r)$ and $a(\Lambda_D, I_{D,j}, r)$ vanish for all $j \not\equiv 0 \pmod{u_D}$, which implies that every nonempty shell Λ_D^r is an ellipsoidal T_D-design (remember that $T_D = \mathbb{Z}^+ \setminus u_D\mathbb{Z}^+$).

Now we prove the maximality of T_D. We show that $a(D, j, r) \neq 0$ (note that $a(D, j, r) = \frac{1}{u_D}a(\Lambda_D, R_{D,j}, r)$) for all $j \notin T_D$ and Λ_D^r nonempty. By Lemma 3.4, enough to take r to be a prime power. Suppose p be a prime and $\alpha \geq 1$ be such that $\Lambda_D^{p^a} \neq \phi$. There are two cases possible, either Λ_D^p is empty or it is not. First suppose Λ_D^p is nonempty. If p is 2 then $a(D, j, 2) \neq 0$ by Proposition 3.4. By part*(2)* of Lemma 3.4, we have that $a(D, j, 2^\alpha) = a(D, j, 2)^\alpha \neq 0$ for $D = 1, 2$ since $\chi(2) = 0$. When $D = 7$ then part*(3)* of Lemma 3.4, we have $a(7, j, 2^\alpha) \neq 0$. If p is an odd prime, then Lemma 3.5 implies that $a(D, j, p) \neq 0$. Now using part*(3)* of Lemma 3.4 again, we have $a(D, j, p^\alpha) \neq 0$. Suppose Λ_D^p is empty then $a(D, j, p) = 0$ and Lemma 3.3 implies α is even. Now by part*(2)* of Lemma 3.5, we get $a(D, j, p^\alpha) = p^{j\alpha/2} \neq 0$ (note that this case includes 2 too). So we get that $a(D, j, p^\alpha) \neq 0$ whenever $\Lambda_D^{p^\alpha}$ is nonempty.

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