



Projections of modular forms on Eisenstein series and its application to Siegel's formula

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Dedicated to Professor Emeritus Kenneth S. Williams on the occasion of his 80th birthday

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Abstract

Let $k \geq 2$ and N be positive integers and let χ be a Dirichlet character modulo N . Let $f(z)$ be a modular form in $M_k(\Gamma_0(N), \chi)$. Then we have a unique decomposition $f(z) = E_f(z) + S_f(z)$, where $E_f(z) \in E_k(\Gamma_0(N), \chi)$ and $S_f(z) \in S_k(\Gamma_0(N), \chi)$. In this paper, we give an explicit formula for $E_f(z)$ in terms of Eisenstein series whose coefficients are sum of divisors function. Then we apply our result to certain families of eta quotients and to representations of positive integers by $2k$ -ary positive definite quadratic forms in order to give an alternative version of Siegel's formula for the weighted average number of representations of an integer by quadratic forms in the same genus. Our formula for the latter is in terms of sum of divisors function and does not involve computation of local densities.

Keywords Dedekind eta function · Theta functions · Eisenstein series · Modular forms · Cusp forms · Fourier coefficients

Mathematics Subject Classification 11F11 · 11F20 · 11F27 · 11E20 · 11E25 · 11F30

1 Introduction and notation

Let \mathbb{N} , \mathbb{N}_0 , \mathbb{Z} , \mathbb{Q} , \mathbb{C} , and \mathbb{H} denote the sets of positive integers, non-negative integers, integers, rational numbers, complex numbers, and upper half-plane of complex numbers, respectively. Throughout the paper, z denotes a complex number in \mathbb{H} , p always denotes a prime number, all divisors considered are positive divisors, q stands for

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$e^{2\pi iz}$, and $\chi_d(n)$ denotes the Kronecker symbol $\left(\frac{d}{n}\right)_K$, where we use the subscript K to avoid confusion with fractions. Let $N \in \mathbb{N}$ and χ be a Dirichlet character modulo N . The space of modular forms of weight k for $\Gamma_0(N)$ with character χ is denoted by $M_k(\Gamma_0(N), \chi)$; and $E_k(\Gamma_0(N), \chi)$, $S_k(\Gamma_0(N), \chi)$ denote its Eisenstein and cusp form subspaces, respectively. Then we have

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi).$$

That is, given $f(z) \in M_k(\Gamma_0(N), \chi)$, we can write

$$f(z) = E_f(z) + S_f(z), \tag{1}$$

where $E_f(z) \in E_k(\Gamma_0(N), \chi)$ and $S_f(z) \in S_k(\Gamma_0(N), \chi)$ are uniquely determined by f . Let ϵ, ψ be primitive Dirichlet characters such that $\epsilon\psi = \chi$ (i.e., $\epsilon(n)\psi(n) = \chi(n)$ for all $n \in \mathbb{Z}$ coprime to N) with conductors say L and M , respectively, and suppose $LM \mid N$. Let d be a positive divisor of N/LM and $2 \leq k \in \mathbb{N}$ be such that $\chi(-1) = (-1)^k$. Let ω be the primitive Dirichlet character corresponding to $\epsilon\bar{\psi}$ and \mathcal{M}_ω be its conductor. We define the Eisenstein series associated with ϵ and ψ by

$$E_k(\epsilon, \psi; dz) := \epsilon(0) + \left(\frac{\mathcal{M}_\omega}{M}\right)^k \left(\frac{W(\bar{\psi})}{W(\omega)}\right) \left(\frac{-2k}{B_{k,\bar{\omega}}}\right) \prod_{p|\text{lcm}(L,M)} \frac{p^k}{p^k - \omega(p)} \times \sum_{n=1}^\infty \sigma_{k-1}(\epsilon, \psi; n) e^{2\pi indz}, \tag{2}$$

where $\bar{\omega}$ is the complex conjugate of ω ,

$$\sigma_{k-1}(\epsilon, \psi; n) := \sum_{1 \leq d|n} \epsilon(n/d)\psi(d)d^{k-1}$$

is the generalized sum of divisors function associated with ϵ and ψ ,

$$W(\psi) := \sum_{a=0}^{M-1} \psi(a)e^{2\pi ia/M}$$

is the Gauss sum of ψ , and $B_{k,\bar{\omega}}$ is the k -th generalized Bernoulli number associated with $\bar{\omega}$ defined by

$$\sum_{k=0}^\infty \frac{B_{k,\bar{\omega}}}{k!} t^k = \sum_{a=1}^{\mathcal{M}_{\bar{\omega}}} \frac{\bar{\omega}(a)t e^{at}}{e^{\mathcal{M}_{\bar{\omega}}t} - 1},$$

see [13, end of pg. 94]. By [5, Corollary 8.5.5] (alternatively [13, Theorem 4.7.1, (7.1.13) and Lemma 7.2.19]), we have

$$E_k(\epsilon, \psi; dz) \in M_k(\Gamma_0(N), \chi) \text{ if } (k, \epsilon, \psi) \neq (2, \chi_1, \chi_1),$$

and

$$E_2(\chi_1, \chi_1; z) - NE_2(\chi_1, \chi_1; Nz) \in M_2(\Gamma_0(N), \chi_1).$$

Remark 1 Let $L(\epsilon\bar{\psi}, k)$ be the Dirichlet L-function defined by

$$L(\epsilon\bar{\psi}, k) := \sum_{n \geq 1} \frac{\epsilon(n)\bar{\psi}(n)}{n^k}.$$

The Eisenstein series we define in (2) is equal to $E_k(Mdz; \epsilon, \bar{\psi})/(2L(\epsilon\bar{\psi}, k))$ in the notation of Theorem 7.1.3 and Theorem 7.2.12 of [13], and it is also equal to $G_k(\psi, \epsilon)(dz)/L(\epsilon\bar{\psi}, k)$ in the notation of Corollary 8.5.5 and Definition 8.5.10 of [5]. Further treatment to obtain the form in (2) is done by using the formula for $L(\epsilon\bar{\psi}, k)$ given in Theorems 3.3.4 and (3.3.14) of [13]. We have normalized the Eisenstein series this way to simplify the constant terms given in (27) and (28). This in return simplifies the notation in Sects. 4 and 5.

Letting $D(N, \mathbb{C})$ to denote the group of Dirichlet characters modulo N , we define

$$\mathcal{E}(k, N, \chi) := \{(\epsilon, \psi) \in D(L, \mathbb{C}) \times D(M, \mathbb{C}) : \epsilon, \psi \text{ primitive,} \\ \epsilon\psi = \chi \text{ and } LM \mid N\}.$$

The set

$$\{E_k(\epsilon, \psi; dz) : (\epsilon, \psi) \in \mathcal{E}(k, N, \chi), d \mid N/LM\}$$

constitutes a basis for $E_k(\Gamma_0(N), \chi)$ whenever $k \geq 2$ and $(k, \chi) \neq (2, \chi_1)$; the set

$$\{E_2(\chi_1, \chi_1; z) - dE_2(\chi_1, \chi_1; dz) : 1 < d \mid N\} \\ \cup \{E_2(Mdz; \epsilon, \psi) : (\epsilon, \psi) \in \mathcal{E}(2, N, \chi_1), (\epsilon, \psi) \neq (\chi_1, \chi_1), d \mid N/LM\}$$

constitutes a basis for $E_2(\Gamma_0(N), \chi_1)$, see [5, Theorems 8.5.17 and 8.5.22], or [22, Proposition 5]. Then we have

$$E_f(z) = \sum_{(\epsilon, \psi) \in \mathcal{E}(k, N, \chi)} \sum_{d \mid N/LM} a_f(\epsilon, \psi, d) E_k(\epsilon, \psi; dz), \tag{3}$$

for some $a_f(\epsilon, \psi, d) \in \mathbb{C}$. When $S_f(z) = 0$, it is easy to determine $a_f(\epsilon, \psi, d)$ by comparing the first few Fourier coefficients of $f(z)$ expanded at $i\infty$ and the first

few Fourier coefficients of the right-hand side of (3) expanded at $i\infty$. However, if $S_f(z) \neq 0$ and an explicit basis for $S_k(\Gamma_0(N), \chi)$ is not known then this method fails. In this paper we solve this problem, in other words we obtain $a_f(\epsilon, \psi, d)$ explicitly (in terms of a finite sum) for all $f \in M_k(\Gamma_0(N), \chi)$, where $k \geq 2$, see Theorem 1. Our treatment is general and its special cases agree with the previously known formulas. Additionally, we give a new treatment of Siegel’s formula for representation numbers of quadratic forms, see Theorem 2.

Let $a \in \mathbb{Z}$ and $c \in \mathbb{N}_0$ be coprime. For an $f(z) \in M_k(\Gamma_0(N), \chi)$, we denote the constant term of $f(z)$ in the Fourier expansion of $f(z)$ at the cusp a/c by

$$[0]_{a/c}f = \lim_{z \rightarrow i\infty} (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right),$$

where $b, d \in \mathbb{Z}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. The value of $[0]_{a/c}f$ does not depend on the choice of b, d . We denote the n th Fourier coefficient of $f(z)$ in the expansion at the cusp $i\infty$ by $[n]f$. Letting $\phi(n)$ denote the Euler totient function, we define an average associated with ψ for the constant terms of Fourier series expansions of modular forms at cusps as follows:

$$[0]_{c,\psi}f := \frac{1}{\phi(c)} \sum_{\substack{a=1, \\ \gcd(a,c)=1}}^c \psi(a)[0]_{a/c}f. \tag{4}$$

We note that working with this average of constant terms at cusps is a new idea which helps studying modular form spaces with non-trivial character, see Sect. 5 for details.

Letting $v_p(n)$ to denote the highest power of p dividing n and $\mu(n)$ be the Möbius function we are ready to state the main theorem.

Theorem 1 (Main Theorem) *Let $f(z) \in M_k(\Gamma_0(N), \chi)$, where $N, k \in \mathbb{N}, k \geq 2, \chi$ is a Dirichlet character modulo N that satisfies $\chi(-1) = (-1)^k$. Let $E_f(z)$ be defined by (1). Then*

$$E_f(z) = \sum_{(\epsilon, \psi) \in \mathcal{E}(k, N, \chi)} \sum_{d|N/LM} a_f(\epsilon, \psi, d) E_k(\epsilon, \psi; dz),$$

where

$$\begin{aligned} a_f(\epsilon, \psi, d) &= \prod_{p|N} \frac{p^k}{p^k - \epsilon(p)\overline{\psi}(p)} \\ &\times \sum_{c \in C_N(\epsilon, \psi)} \mathcal{R}_{k, \epsilon, \psi}(d, c/M) \mathcal{S}_{k, N/LM, \epsilon, \psi}(d, c/M) [0]_{c, \psi}f, \end{aligned}$$

with

$$C_N(\epsilon, \psi) := \{c_1 M : c_1 | N/LM\}, \tag{5}$$

$$\mathcal{R}_{k,\epsilon,\psi}(d, c) := \epsilon \left(\frac{-d}{\gcd(d, c)} \right) \bar{\psi} \left(\frac{c}{\gcd(d, c)} \right) \left(\frac{\gcd(d, c)}{c} \right)^k, \tag{6}$$

and

$$\mathcal{S}_{k,N,\epsilon,\psi}(d, c) := \mu \left(\frac{dc}{\gcd(d, c)^2} \right) \prod_{\substack{p|\gcd(d,c), \\ 0 < v_p(d) = v_p(c) < v_p(N)}} \left(\frac{p^k + \epsilon(p)\bar{\psi}(p)}{p^k} \right). \tag{7}$$

Remark 2 Let $c \mid N$. By Lemma 6, if a/c and a'/c are equivalent cusps of $\Gamma_0(N)$ and $(\epsilon, \psi) \in \mathcal{E}(k, N, \chi)$ with $M \mid c$, then $\psi(a)[0]_{a/c}f = \psi(a')[0]_{a'/c}f$. Therefore, in the applications of Theorem 1 computing $\psi(a)[0]_{a/c}f$ at a set of inequivalent cusps will be sufficient, see [5, Corollary 6.3.23] for a description of such a set.

Theorem 1 agrees with and extends the previously known formulas. For example, if we let $k \in \mathbb{N}$ even, N squarefree, $\chi = \chi_1$ in Theorem 1, we obtain [3, Theorem 1.1] and if we let $k \in \mathbb{N}$ odd $N \in \{3, 7, 11, 23\}$, $\chi = \chi_{-N}$ in Theorem 1, we obtain [6, (11.20)]. Theorem 1 additionally extends the latter to hold for all primes N that are congruent to 3 modulo 4.

Before we apply Theorem 1 to representation numbers of quadratic forms, we give a snapshot of interesting applications. Since $f(z) - E_f(z)$ is a cusp form, one can use our main theorem to produce cusp forms. At the end of Sect. 3 we use this idea combined with the Modularity Theorem ([9, Theorem 8.8.1]) and consider the elliptic curve $E_{27A} : y^2 + y = x^3 - 7$. Then we use arithmetic properties of Eisenstein series to obtain

$$\begin{aligned} \#E_{27A}(\mathbb{F}_p) &\equiv 0 \pmod{9} \text{ if } p \equiv 1 \pmod{3}, \\ \#E_{27A}(\mathbb{F}_p) &= p + 1 \text{ if } p \equiv 2 \pmod{3}, \end{aligned}$$

where

$$E_{27A}(\mathbb{F}_p) := \{\infty\} \cup \{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p : y^2 + y = x^3 - 7\},$$

with \mathbb{F}_p denoting the finite field of p elements, see Corollary 3.

The Fourier coefficients of special functions (expanded at $i\infty$) have been of huge interest. A very well studied special function is the Dedekind eta function which is defined by

$$\eta(z) := e^{\pi iz/12} \prod_{n \geq 1} (1 - e^{2\pi inz}) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

Quotients of Dedekind eta functions are often referred to as eta quotients. Nathan Fine in his book [10] has given several formulas for Fourier coefficients of eta quotients (expanded at $i\infty$). In his work when the weight of the eta quotient is an integer, the formulas are linear combinations of Eisenstein series defined above. For instance he shows that

$$F(z) := \frac{\eta(2z)\eta(3z)\eta(8z)\eta(12z)}{\eta(z)\eta(24z)} = 1 + \sum_{n \geq 1} \sigma_0(\chi_1, \chi_{-24}; n)q^n,$$

see [10, (32.5)], and acknowledges this equation as being *very beautiful*. We consider the $(2k + 1)$ th power of $F(z)$, that is we consider

$$F^{2k+1}(z) = \frac{\eta^{2k+1}(2z)\eta^{2k+1}(3z)\eta^{2k+1}(8z)\eta^{2k+1}(12z)}{\eta^{2k+1}(z)\eta^{2k+1}(24z)}.$$

Using our main theorem (Theorem 1), we obtain the following analogous formula for $F^{2k+1}(z)$ when $k > 0$:

$$E_{F^{2k+1}}(z) = 1 - \frac{2k + 1}{B_{2k+1, \chi_{-24}}} \sum_{n \geq 1} \left(\sigma_{2k}(\chi_1, \chi_{-24}; n) + (-24)^k \sigma_{2k}(\chi_{-24}, \chi_1; n) \right) q^n,$$

see Corollary 2. Using Theorem 1 one can obtain formulas in this fashion for all holomorphic eta quotients of integral weight $k \geq 2$.

Let $\mathcal{F}(x_1, \dots, x_{2k})$ be a positive definite quadratic form with integer coefficients and $B(\mathcal{F})$ be the matrix associated with \mathcal{F} whose entries are given by

$$B(\mathcal{F})_{i,j} = \left(\frac{\partial^2 \mathcal{F}}{\partial x_i \partial x_j} \right).$$

Then the generating function of the number of representations of a positive integer by the quadratic form \mathcal{F} is

$$\theta_{\mathcal{F}}(z) = \sum_{x \in \mathbb{Z}^{2k}} e^{2\pi iz\mathcal{F}(x)} = \sum_{x \in \mathbb{Z}^{2k}} e^{2\pi izxB(\mathcal{F})x^T/2}.$$

In [19], Siegel gave a formula for the weighted average for representation numbers of positive definite quadratic forms in the same genus. Siegel’s formula is in terms of local densities (for other treatments of Siegel’s formula see [23] and [15, Chapter 3]). In the realm of modular forms, Siegel’s formula corresponds to the Eisenstein part of $\theta_{\mathcal{F}}(z)$, see [1, 17], [18, Remark on p. 110] and [19]. For clarity we note that if \mathcal{F}_1 and \mathcal{F}_2 are in the same genus then $E_{\theta_{\mathcal{F}_1}}(z) = E_{\theta_{\mathcal{F}_2}}(z)$. Below we use Theorem 1 to give an explicit formula for $E_{\theta_{\mathcal{F}}}(z)$, where \mathcal{F} is a $2k$ -ary positive definite quadratic form with integer coefficients. In Sect. 2, we give several applications of our formula including a comparison of our output for the form $\sum_{j=1}^{2k} x_j^2$ with that of Arenas [1, Proposition 1], which uses Siegel’s formula.

By [13, Corollary 4.9.5], we have

$$\theta_{\mathcal{F}}(z) \in M_k(\Gamma_0(N), \chi), \tag{8}$$

where $\chi = ((-1)^k \det(B(\mathcal{F}))/*)_K$ and N is the smallest positive integer such that the matrix $NB(\mathcal{F})^{-1}$ has even diagonal entries. By [21, (10.2)] we have

$$[0]_{a/c}\theta_{\mathcal{F}}(z) = \left(\frac{-i}{c}\right)^k \frac{1}{\sqrt{\det(B(\mathcal{F}))}} \sum_{\substack{x \in \mathbb{Z}^{2k} \\ x \pmod{c}}} e^{2\pi i(\mathcal{F}(x)a/c)}. \tag{9}$$

Putting (2), (8), and (9) in Theorem 1, we obtain the following assertion concerning the representation numbers of $2k$ -ary quadratic forms.

Theorem 2 *Let $\mathcal{F}(x_1, \dots, x_{2k})$ be a positive definite quadratic form with $k \geq 2$; let χ and N be as above and ω be as in (2). Then*

$$[n]E_{\theta_{\mathcal{F}}}(z) = \sum_{(\epsilon, \psi) \in \mathcal{E}(k, N, \chi)} \left(\frac{M_{\omega}}{M}\right)^k \left(\frac{W(\bar{\psi})}{W(\omega)}\right) \left(\frac{-2k}{B_{k, \bar{\omega}}}\right) \prod_{p | \text{lcm}(L, M)} \frac{p^k}{p^k - \omega(p)} \\ \times \sum_{d | N/LM} a_{\theta_{\mathcal{F}}}(\epsilon, \psi, d) \sigma_{k-1}(\epsilon, \psi; n/d), \tag{10}$$

where

$$a_{\theta_{\mathcal{F}}}(\epsilon, \psi, d) = \frac{(-i)^k}{\sqrt{\det(B(\mathcal{F}))}} \prod_{p | N} \frac{p^k}{p^k - \epsilon(p)\bar{\psi}(p)} \\ \times \sum_{c \in C_N(\epsilon, \psi)} \frac{\mathcal{R}_{k, \epsilon, \psi}(d, c/M) \mathcal{S}_{k, N/LM, \epsilon, \psi}(d, c/M)}{c^k \phi(c)} \\ \times \sum_{\substack{a=1, \\ \gcd(a, c)=1}}^c \psi(a) \sum_{\substack{x \in \mathbb{Z}^{2k} \\ x \pmod{c}}} e^{2\pi i(\mathcal{F}(x)a/c)}.$$

The organization of the rest of the paper is as follows. In Sect. 2, we apply Theorem 2 to the representation numbers of diagonal quadratic forms and certain non-diagonal level 2 quadratic forms. A special case of the latter leads to an equation for Ramanujan’s τ -function. In Sect. 3, we apply our Main Theorem to certain families of eta quotients, these applications give extensions of some well-known formulas to higher weight eta quotients. In Sects. 4–6, we prove the main theorem.

2 Applications to representation numbers of certain quadratic forms

To apply (10) to specific quadratic forms we need to compute the quadratic Gauss sum. If \mathcal{F} is a diagonal form, say $\mathcal{F} = \sum_{j=1}^{2k} \alpha_j x_j^2$, then we have

$$\sum_{\substack{x \in \mathbb{Z}^{2k} \\ x \pmod{c}}} e^{2\pi i \mathcal{F}(x)a/c} = \prod_{j=1}^{2k} \gcd(\alpha_j a, c) g\left(\frac{\alpha_j a}{\gcd(\alpha_j a, c)}, \frac{c}{\gcd(\alpha_j a, c)}\right), \tag{11}$$

where, if $\gcd(\alpha, \beta) = 1$,

$$g(\alpha, \beta) = \begin{cases} 0 & \text{if } \beta \equiv 2 \pmod{4}, \\ \left(\frac{\alpha}{\beta}\right)_K \sqrt{\beta} & \text{if } \beta \equiv 1 \pmod{4}, \\ i \left(\frac{\alpha}{\beta}\right)_K \sqrt{\beta} & \text{if } \beta \equiv 3 \pmod{4}, \\ (1+i)\left(\frac{\beta}{\alpha}\right)_K \sqrt{\beta} & \text{if } \beta \equiv 0 \pmod{4} \text{ and } \alpha \equiv 1 \pmod{4}, \\ (1-i)\left(\frac{\beta}{\alpha}\right)_K \sqrt{\beta} & \text{if } \beta \equiv 0 \pmod{4} \text{ and } \alpha \equiv 3 \pmod{4}, \end{cases}$$

see [4, Theorems 1.5.2 and 1.5.4]. Next we apply this result to the form $\mathcal{F} = \sum_{j=1}^{2k} x_j^2$, that is, $\alpha_j = 1$ for all $1 \leq j \leq 2k$. Then we have

$$\sum_{\substack{x \in \mathbb{Z}^{2k} \\ x \pmod{c}}} e^{2\pi i \mathcal{F}(x)/c} = \prod_{i=1}^{2k} g(1, c) = \begin{cases} 1 & \text{if } c = 1, \\ 0 & \text{if } c = 2, \\ (8i)^k & \text{if } c = 4. \end{cases}$$

Thus, by Theorem 2 when k is even we have

$$E_{\theta_{\mathcal{F}}}(z) = 1 - \frac{2k}{(2k-1)B_{k,\chi_1}} \sum_{n \geq 1} \left((-i)^k \sigma(\chi_1, \chi_1; n) - (i^k + 1) \sigma(\chi_1, \chi_1; n/2) + 2^k \sigma(\chi_1, \chi_1; n/4) \right) q^n \tag{12}$$

and when k is odd we have

$$E_{\theta_{\mathcal{F}}}(z) = 1 - \frac{2k}{B_{k,\chi_{-4}}} \sum_{n \geq 1} \left(\sigma(\chi_1, \chi_{-4}; n) + (2i)^{k-1} \sigma(\chi_{-4}, \chi_1; n) \right) q^n. \tag{13}$$

Formulas (12) and (13) agree with Ramanujan’s statements [16, (131)–(134)], which was first proven by Mordell in [14]. In [1, Proposition 1] Arenas uses Siegel’s formula to compute $E_{\theta_{\mathcal{F}}}(z)$ and obtains (12) and (13) in the same form. Now, we turn our attention to another diagonal form. Let

$$\mathcal{F}(a, b; p) = \sum_{i=1}^a x_i^2 + \sum_{i=1}^b py_i^2.$$

In [7] Cooper, Kane, and Ye found formulas for the representation numbers of $\mathcal{F}(k, k; p)$, where $p = 3, 7, 11$, or 23 . Their result relies on the existence of a Hauptmodul in the levels considered. Inspired by their results, in [2], we derived formulas

for the representation numbers of $\mathcal{F}(2a, 2b; p)$ where $a, b \in \mathbb{N}_0$ and p is an odd prime. These results are considered as analogs of the Ramanujan–Mordell formula and specialized version of Theorem 2 agrees with these results. Below we give formulas in all the remaining cases, that is, we find formulas for representation numbers of $\mathcal{F}(a, b; p)$ where $a, b \equiv 1 \pmod{2}$ and p an odd prime.

Corollary 1 *Let $a, b \geq 1$ be odd integers such that $a + b \geq 4$. Set $k = (a + b)/2$ and $\mathfrak{p} = \chi_{-4}(p)p$. Then for any odd prime p , whenever $(-1)^k = \chi_{-4}(p)$ we have*

$$E_{\theta_{\mathcal{F}(a,b;p)}}(z) = 1 + \sum_{n=1}^{\infty} \frac{2k (a_1\sigma_{k-1}(\chi_1, \chi_{\mathfrak{p}}; n) + a_2\sigma_{k-1}(\chi_1, \chi_{\mathfrak{p}}; n/2) + a_32^k\sigma_{k-1}(\chi_1, \chi_{\mathfrak{p}}; n/4))}{(2^k - \chi_{\mathfrak{p}}(2))B_{k,\chi_{\mathfrak{p}}}} q^n + \sum_{n=1}^{\infty} \frac{p^{(a-1)/2}2k (a_4\sigma_{k-1}(\chi_{\mathfrak{p}}, \chi_1; n) + a_5\sigma_{k-1}(\chi_{\mathfrak{p}}, \chi_1; n/2) + a_62^k\sigma_{k-1}(\chi_{\mathfrak{p}}, \chi_1; n/4))}{(2^k - \chi_{\mathfrak{p}}(2))B_{k,\chi_{\mathfrak{p}}}} q^n,$$

and whenever $p \equiv (-1)^{k+1} \pmod{4}$ we have

$$E_{\theta_{\mathcal{F}(a,b;p)}}(z) = 1 - \sum_{n=1}^{\infty} \frac{2k (b_1\sigma_{k-1}(\chi_1, \chi_{-4\mathfrak{p}}; n) + b_22^k\sigma_{k-1}(\chi_{-4}, \chi_{\mathfrak{p}}; n))}{B_{k,\chi_{-4\mathfrak{p}}}} q^n - \sum_{n=1}^{\infty} \frac{p^{(a-1)/2}2k (b_3\sigma_{k-1}(\chi_{\mathfrak{p}}, \chi_{-4}; n) + b_42^k\sigma_{k-1}(\chi_{-4\mathfrak{p}}, \chi_1; n))}{B_{k,\chi_{-4\mathfrak{p}}}} q^n,$$

where

$$a_1 = \begin{cases} (-1)^{k/2} & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{(k+a+2)/2} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$a_2 = \begin{cases} (-1)^{k/2+1} - \chi_{\mathfrak{p}}(2) & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{(k+a)/2} - \chi_{\mathfrak{p}}(2) & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$a_3 = 1,$$

$$a_4 = \begin{cases} (-1)^{k/2} & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{(k-1)/2} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$a_5 = \begin{cases} (-1)^{k/2+1}\chi_{\mathfrak{p}}(2) - 1 & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{(b+1)/2} + (-1)^{(k+1)/2}\chi_{\mathfrak{p}}(2) & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$a_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{(b-1)/2} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$b_1 = 1,$$

$$b_2 = \begin{cases} (-1)^{(k-1)/2}/2 & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{(k+a-1)/2}/2 & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$b_3 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{(b+1)/2} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$b_4 = \begin{cases} (-1)^{(k-1)/2}/2 & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{k/2}/2 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Using (11) and Theorem 2 one can obtain results similar to Corollary 1 for any diagonal form. Next we consider the non-diagonal form

$$\mathcal{F}_k = \sum_{m=1}^k \sum_{\substack{1 \leq i \leq j \leq 4, \\ (i,j) \neq (1,2)}} x_{i,m} x_{j,m}.$$

We obtain

$$N = 2, \quad \det(B(\mathcal{F}_k)) = 2^{2k} \text{ and } \sum_{\substack{x \in \mathbb{Z}^{2k}, \\ x \pmod{c}}} e^{2\pi i \mathcal{F}_k(x)/c} = \begin{cases} 1 & \text{if } c = 1, \\ (-8)^k & \text{if } c = 2. \end{cases}$$

Thus, $\theta_{\mathcal{F}_k} \in M_{2k}(\Gamma_0(2), \chi_1)$, hence by Theorem 2 we have

$$[n]E_{\theta_{\mathcal{F}_k}}(z) = \frac{-4k}{((-2)^k + 1)B_{2k,\chi_1}} \left(\sigma_{2k-1}(\chi_1, \chi_1; n) + (-2)^k \sigma_{2k-1}(\chi_1, \chi_1; n/2) \right). \tag{14}$$

When $k = 6$ we compute the first few coefficients of the cusp part of $\theta_{\mathcal{F}_6}$:

$$\theta_{\mathcal{F}_6}(z) - E_{\theta_{\mathcal{F}_6}}(z) = \frac{2^6 3^4 19}{691} q + \frac{2^6 3^4 19}{691} (2^6 - 24) q^2 + \frac{2^6 3^4 19}{691} 252 q^3 + O(q^4) \in S_{12}(\Gamma_0(2), \chi_1).$$

The Fourier coefficients of $\eta^{24}(z)$ are called Ramanujan’s τ -function and first few terms are given as follows:

$$\eta^{24}(z) = \sum_{n \geq 1} \tau(n) q^n = q - 24q^2 + 252q^3 + O(q^4). \tag{15}$$

It is well known that $\eta^{24}(z)$ and $\eta^{24}(2z) \in S_{12}(\Gamma_0(2), \chi_1)$ and thus by Sturm’s Theorem [5, Corollary 5.6.14] we obtain

$$\theta_{\mathcal{F}_6}(z) - E_{\theta_{\mathcal{F}_6}}(z) = \frac{2^6 3^4 19}{691} (\eta^{24}(z) + 2^6 \eta^{24}(2z)). \tag{16}$$

If we compare n th coefficient of both sides of (16) we get

$$\begin{aligned}
 [n]\theta_{\mathcal{F}_6}(z) &= \frac{2^4 3^2 7}{691} (\sigma_{11}(\chi_1, \chi_1; n) + 2^6 \sigma_{11}(\chi_1, \chi_1; n/2)) \\
 &= \frac{2^6 3^4 19}{691} (\tau(n) + 2^6 \tau(n/2)).
 \end{aligned}$$

Since

$$[n]\theta_{\mathcal{F}_6}(z) \in \mathbb{N}_0 \quad \text{for all } n \in \mathbb{N}_0 \quad \text{and} \quad 2^4 3^2 7 \equiv -2^6 3^4 19 \pmod{691}$$

it is not hard to deduce the well-known congruence relation

$$\tau(n) \equiv \sigma_{11}(\chi_1, \chi_1; n) \pmod{691}.$$

3 Applications to eta quotients

In this section, we give further applications of Theorem 1. Recall that the Dedekind eta function is defined by

$$\eta(z) = e^{\pi iz/12} \prod_{n \geq 1} (1 - e^{2\pi inz}).$$

Let $k \in \mathbb{N}$. We define

$$f_k(z) := \frac{\eta^{2k+1}(2z)\eta^{2k+1}(3z)\eta^{2k+1}(8z)\eta^{2k+1}(12z)}{\eta^{2k+1}(z)\eta^{2k+1}(24z)}, \tag{17}$$

$$g_k(z) := \frac{\eta^{6k-5}(3z)\eta^{6k-4}(4z)}{\eta^{2k-3}(z)\eta^{2k-2}(2z)\eta^{2k-4}(6z)\eta^{2k}(12z)}, \tag{18}$$

$$h_k(z) := \frac{\eta^{6k-4}(9z)\eta^3(27z)}{\eta^{2k-1}(3z)}. \tag{19}$$

In Corollary 2 below, we obtain formulas concerning $f_k(z)$, $g_k(z)$, and $h_k(z)$. Similar formulas can be obtained via Theorem 1 for all integer weight holomorphic eta quotients.

Corollary 2 *Let $k \geq 1$ and let $f_k(z)$, $g_k(z)$, and $h_k(z)$ be defined by (17), (18), and (19), respectively. Then we have*

$$E_{f_k}(z) = 1 - \frac{4k + 2}{B_{2k+1, \chi_{-24}}} \sum_{n \geq 1} \left(\sigma_{2k}(\chi_1, \chi_{-24}; n) + (-24)^k \sigma_{2k}(\chi_{-24}, \chi_1; n) \right) q^n,$$

$$E_{g_k}(z) = 1 - \frac{4k}{B_{2k, \chi_{12}}} \sum_{n \geq 1} \sigma_{2k-1}(\chi_1, \chi_{12}; n) q^n,$$

and

$$E_{h_k}(z) = \frac{-4k}{B_{2k, \chi_1}} \sum_{n \geq 1} \left(\sum_{d|9} a_d \sigma_{2k-1}(\chi_1, \chi_1; n/d) + b_1 \sigma_{2k-1}(\chi_{-3}, \chi_{-3}; n) \right) q^n, \quad (20)$$

where

$$\begin{aligned} a_1 &= \frac{(-1)^k - \cos((k+4)\pi/3)}{3^{3k+1}(3^{2k}-1)}, \\ a_3 &= \frac{(-1)^{k+1} + (3^{2k}+1)\cos((k+4)\pi/3)}{3^{3k+1}(3^{2k}-1)}, \\ a_9 &= \frac{-\cos((k+4)\pi/3)}{3^{k+1}(3^{2k}-1)}, \\ b_1 &= \frac{\sqrt{3}\sin((k+4)\pi/3)}{3^{3k+1}(3^{2k}-1)}. \end{aligned}$$

Proof We use [5, Proposition 5.9.2] to determine

$$\begin{aligned} f_k(z) &\in M_{2k+1}(\Gamma_0(24), \chi_{-24}), \\ g_k(z) &\in M_{2k}(\Gamma_0(12), \chi_{12}), \\ h_k(z) &\in M_{2k}(\Gamma_0(27), \chi_1). \end{aligned}$$

We evaluate the constant terms of $f_k(z)$, $g_k(z)$, $h_k(z)$ at the relevant cusps using [12, Proposition 2.1]. We do this with the help of some SAGE functions we have written; the code is provided in the Appendix A. From these we compute

$$\begin{aligned} [0]_{1/1} f_k &= -\frac{i^{2k+1}\sqrt{6}}{3^{k+1}2^{3k+2}}, \\ [0]_{a/c} f_k &= 0, \text{ for } a/c = 1/2, 1/3, 1/4, 1/6, 1/8, 1/12, \\ [0]_{1/24} f_k &= 1, \end{aligned}$$

see Appendix A for details. We determine the set of tuples of characters as

$$\mathcal{E}(2k+1, 24, \chi_{-24}) = \{(\chi_1, \chi_{-24}), (\chi_{-3}, \chi_8), (\chi_8, \chi_{-3}), (\chi_{-24}, \chi_1)\}.$$

Thus, we have

$$E_{f_k}(z) = \sum_{(\epsilon, \psi) \in \mathcal{E}(2k+1, 24, \chi_{-24})} a_{f_k}(\epsilon, \psi, 1) E_{2k+1}(\epsilon, \psi; z).$$

Now, we compute

$$\begin{aligned}
 a_{f_k}(\chi_1, \chi_{-24}, 1) &= \left(\prod_{p|24} \frac{p^k}{p^k - \chi_1(p)\chi_{-24}(p)} \right) \\
 &\quad \times \mathcal{R}_{k, \chi_1, \chi_{-24}}(1, 1)\mathcal{S}_{k, 1, \chi_1, \chi_{-24}}(1, 1)[0]_{24, \chi_{-24}} f_k \\
 &= \mathcal{R}_{k, \chi_1, \chi_{-24}}(1, 1)\mathcal{S}_{k, 1, \chi_1, \chi_{-24}}(1, 1)[0]_{24, \chi_{-24}} f_k \\
 &= [0]_{24, \chi_{-24}} f_k.
 \end{aligned} \tag{21}$$

Additionally, we have

$$[0]_{24, \chi_{-24}} f_k = \frac{1}{\phi(24)} \sum_{\substack{a=1, \\ \gcd(a, 24)=1}}^{24} \chi_{-24}(a)[0]_{a/24} f_k = \chi_{-24}(1)[0]_{1/24} f_k = 1. \tag{22}$$

Combining (21) and (22) we have $a_{f_k}(\chi_1, \chi_{-24}, 1) = 1$.

The rest of the coefficients are obtained similarly. □

Now, we turn our attention to special cases of these formulas. The dimension of $S_2(\Gamma_0(12), \chi_{12})$ is 0, so we obtain an exact formula for g_1 , i.e., we have $g_1(z) = E_{g_1}(z)$. When $k = 1$, (20) specializes to

$$\begin{aligned}
 E_{h_1}(z) &= \sum_{n \geq 1} \left(\frac{1}{18}\sigma_1(\chi_1, \chi_1; n) - \frac{2}{9}\sigma_1(\chi_1, \chi_1; n/3) \right. \\
 &\quad \left. + \frac{1}{6}\sigma_1(\chi_1, \chi_1; n/9) + \frac{1}{18}\sigma_1(\chi_{-3}, \chi_{-3}; n) \right) q^n.
 \end{aligned}$$

Clearly $h_1(z) - E_{h_1}(z)$ is a cusp form, and if we normalize $h_1(z) - E_{h_1}(z)$ so that the coefficient of q is 1, we obtain the newform $\mathcal{N}_{27}(z)$ in $S_2(\Gamma_0(27), \chi_1)$, that is, we have

$$\begin{aligned}
 \mathcal{N}_{27}(z) &= -9 \frac{\eta^2(9z)\eta^3(27z)}{\eta(3z)} + \sum_{n \geq 1} \left(\frac{1}{2}\sigma_1(\chi_1, \chi_1; n) - 2\sigma_1(\chi_1, \chi_1; n/3) \right. \\
 &\quad \left. + \frac{3}{2}\sigma_1(\chi_1, \chi_1; n/9) + \frac{1}{2}\sigma_1(\chi_{-3}, \chi_{-3}; n) \right) q^n.
 \end{aligned}$$

By [8, Table 1] this newform is associated to the elliptic curve

$$E_{27A} : y^2 + y = x^3 - 7.$$

Recall that in Sect. 1 we defined

$$E_{27A}(\mathbb{F}_p) = \{\infty\} \cup \{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p : y^2 + y = x^3 - 7\},$$

where \mathbb{F}_p is the finite field of p elements. Then by the Modularity Theorem, see [9, Theorem 8.8.1], we have

$$\#E_{27A}(\mathbb{F}_p) = (p + 1) - [p]\mathcal{N}_{27}(z) \text{ for all } p \neq 3.$$

Thus, for all $p \neq 3$ we have

$$\#E_{27A}(\mathbb{F}_p) = 9[p] \frac{\eta^2(9z)\eta^3(27z)}{\eta(3z)} + (p + 1) \left(\frac{1 - \chi_{-3}(p)}{2} \right).$$

Since $[p] \frac{\eta^2(9z)\eta^3(27z)}{\eta(3z)} \in \mathbb{Z}$ for all $p \in \mathbb{N}$ and $[p] \frac{\eta^2(9z)\eta^3(27z)}{\eta(3z)} = 0$ when $p \equiv 2 \pmod{3}$, we obtain the following statement.

Corollary 3 *We have*

$$\begin{aligned} \#E_{27A}(\mathbb{F}_p) &\equiv 0 \pmod{9} \text{ if } p \equiv 1 \pmod{3}, \\ \#E_{27A}(\mathbb{F}_p) &= p + 1 \text{ if } p \equiv 2 \pmod{3}. \end{aligned}$$

4 Orthogonal relations

In this section, we prove some orthogonal relations involving the functions $\mathcal{R}_{k,\epsilon,\psi}(d, c)$ and $\mathcal{S}_{k,N,\epsilon,\psi}(d, c)$ defined in (6) and (7), respectively. These orthogonal relations concern the constant terms of the Eisenstein series and give the means to determine $a_f(\epsilon, \psi, d)$ of Theorem 1. Throughout the section we assume $k, N \in \mathbb{N}$ and ϵ, ψ are primitive Dirichlet characters with conductors L, M , respectively, such that $LM \mid N$.

Lemma 1 *Let $p \mid N$ be a prime and let $t \mid N/p^v$, where $v = v_p(N)$. Then, for $0 \leq i \leq v$, we have*

$$\mathcal{S}_{k,N,\epsilon,\psi}(t \cdot p^i, d) = \mathcal{S}_{k,p^v,\epsilon,\psi}(p^i, p^{v_p(d)}) \mathcal{S}_{k,N/p^v,\epsilon,\psi}(t, d/p^{v_p(d)}).$$

Proof Since $t \mid N/p^v$ we have $\gcd(t, p) = 1$. Using the multiplicative properties of the Möbius function we obtain

$$\begin{aligned} &\mathcal{S}_{k,N,\epsilon,\psi}(t \cdot p^i, d) \\ &= \mu \left(\frac{t \cdot p^i \cdot d}{\gcd(t \cdot p^i, d)^2} \right) \prod_{\substack{p_2 \mid \gcd(t \cdot p^i, d), \\ 0 < v_{p_2}(t \cdot p^i) = v_{p_2}(d) < v_{p_2}(N)}}} \left(\frac{p_2^k + \epsilon(p_2)\overline{\psi}(p_2)}{p_2^k} \right) \\ &= \mu \left(\frac{p^i \cdot p^{v_p(d)}}{\gcd(p^i, p^{v_p(d)})^2} \right) \prod_{\substack{p_2 \mid \gcd(p^i, p^{v_p(d)}), \\ 0 < v_{p_2}(p^i) = v_{p_2}(p^{v_p(d)}) < v_{p_2}(N)}}} \left(\frac{p_2^k + \epsilon(p_2)\overline{\psi}(p_2)}{p_2^k} \right) \end{aligned}$$

$$\begin{aligned} & \times \mu \left(\frac{t \cdot d / p^{v_p(d)}}{\gcd(t, d / p^{v_p(d)})^2} \right) \prod_{\substack{p_2 | \gcd(t, d / p^{v_p(d)}), \\ 0 < v_{p_2}(t) = v_{p_2}(d / p^{v_p(d)}) < v_{p_2}(N / p^{v_p(d)})}} \left(\frac{p_2^k + \epsilon(p_2) \bar{\psi}(p_2)}{p_2^k} \right) \\ & = \mathcal{S}_{k, p^v, \epsilon, \psi}(p^i, p^{v_p(d)}) \mathcal{S}_{k, N / p^v, \epsilon, \psi}(t, d / p^{v_p(d)}). \end{aligned}$$

□

Lemma 2 *If $\gcd(t, p^i) = 1$, then we have*

$$\begin{aligned} \mathcal{R}_{k, \epsilon, \psi}(c, t \cdot p^i) &= \epsilon(-1) \mathcal{R}_{k, \epsilon, \psi}(p^{v_p(c)}, p^i) \mathcal{R}_{k, \epsilon, \psi}(c / p^{v_p(c)}, t), \\ \mathcal{R}_{k, \epsilon, \psi}(t \cdot p^i, d) &= \epsilon(-1) \mathcal{R}_{k, \epsilon, \psi}(p^i, p^{v_p(d)}) \mathcal{R}_{k, \epsilon, \psi}(t, d / p^{v_p(d)}). \end{aligned}$$

Proof By elementary manipulations we obtain

$$\begin{aligned} \mathcal{R}_{k, \epsilon, \psi}(c, t \cdot p^i) &= \epsilon \left(\frac{-c}{\gcd(c, t \cdot p^i)} \right) \bar{\psi} \left(\frac{t \cdot p^i}{\gcd(c, t \cdot p^i)} \right) \left(\frac{\gcd(c, t \cdot p^i)}{t \cdot p^i} \right)^k \\ &= \epsilon \left(\frac{-c / p^{v_c}}{\gcd(c / p^{v_c}, t)} \right) \bar{\psi} \left(\frac{t}{\gcd(c / p^{v_p(c)}, t)} \right) \left(\frac{\gcd(c / p^{v_p(c)}, t)}{t} \right)^k \\ &\quad \times \epsilon \left(\frac{p^{v_p(c)}}{\gcd(p^{v_p(c)}, p^i)} \right) \bar{\psi} \left(\frac{p^i}{\gcd(p^{v_p(c)}, p^i)} \right) \left(\frac{\gcd(p^{v_c}, p^i)}{p^i} \right)^k \\ &= \epsilon(-1) \mathcal{R}_{k, \epsilon, \psi}(p^{v_p(c)}, p^i) \mathcal{R}_{k, \epsilon, \psi}(c / p^{v_p(c)}, t). \end{aligned}$$

Proof of the second equation is similar.

□

Theorem 3 *If $c, d \mid N$, then*

$$\sum_{t|N} \mathcal{S}_{k, N, \epsilon, \psi}(c, t) \mathcal{R}_{k, \epsilon, \psi}(c, t) \mathcal{R}_{k, \epsilon, \psi}(t, d) = \begin{cases} 0 & \text{if } c \neq d, \\ \prod_{p|N} \frac{p^k - \epsilon(p) \bar{\psi}(p)}{p^k} & \text{if } c = d. \end{cases}$$

Proof Let $p \mid N$ be prime and $v_p(N) = v$. We use Lemmas 1 and 2 to obtain

$$\begin{aligned} & \sum_{t|N} \mathcal{S}_{k, N, \epsilon, \psi}(c, t) \mathcal{R}_{k, \epsilon, \psi}(c, t) \mathcal{R}_{k, \epsilon, \psi}(t, d) \\ &= \sum_{0 \leq i \leq v} \sum_{t|N/p^v} \mathcal{S}_{k, N, \epsilon, \psi}(c, t \cdot p^i) \mathcal{R}_{k, \epsilon, \psi}(c, t \cdot p^i) \mathcal{R}_{k, \epsilon, \psi}(t \cdot p^i, d) \\ &= \sum_{0 \leq i \leq v} \sum_{t|N/p^v} \mathcal{S}_{k, p^v, \epsilon, \psi}(p^{v_p(c)}, p^i) \mathcal{S}_{k, N / p^v, \epsilon, \psi}(c / p^{v_p(c)}, t) \\ &\quad \times \epsilon(-1) \mathcal{R}_{k, \epsilon, \psi}(p^{v_p(c)}, p^i) \mathcal{R}_{k, \epsilon, \psi}(c / p^{v_p(c)}, t) \\ &\quad \times \epsilon(-1) \mathcal{R}_{k, \epsilon, \psi}(p^i, p^{v_p(d)}) \mathcal{R}_{k, \epsilon, \psi}(t, d / p^{v_p(d)}) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{0 \leq i \leq v} \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i) \mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^i) \mathcal{R}_{k,\epsilon,\psi}(p^i, p^{v_p(d)}) \\
 &\times \sum_{t|N/p^v} \mathcal{S}_{k,N/p^v,\epsilon,\psi}(c/p^{v_p(c)}, t) \mathcal{R}_{k,\epsilon,\psi}(c/p^{v_p(c)}, t) \mathcal{R}_{k,\epsilon,\psi}(t, c/p^{v_p(d)}).
 \end{aligned}$$

Using this recursively we obtain

$$\begin{aligned}
 &\sum_{t|N} \mathcal{S}_{k,N,\epsilon,\psi}(c, t) \mathcal{R}_{k,\epsilon,\psi}(c, t) \mathcal{R}_{k,\epsilon,\psi}(t, d) \\
 &= \prod_{p|N} \sum_{0 \leq i \leq v_p} \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i) \mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^i) \mathcal{R}_{k,\epsilon,\psi}(p^i, p^{v_p(d)}). \tag{23}
 \end{aligned}$$

Now, we prove for all $p \mid N$ we have

$$\begin{aligned}
 &\sum_{0 \leq i \leq v} \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i) \mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^i) \mathcal{R}_{k,\epsilon,\psi}(p^i, p^{v_p(d)}) \\
 &= \begin{cases} 0 & \text{if } v_p(c) \neq v_p(d), \\ \frac{p^k - \epsilon(p)\bar{\psi}(p)}{p^k} & \text{if } v_p(c) = v_p(d). \end{cases} \tag{24}
 \end{aligned}$$

We first note that

$$\mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i) = \begin{cases} 0 & \text{if } |v_p(c) - i| > 1, \\ \frac{p^k + \epsilon(p)\bar{\psi}(p)}{p^k} & \text{if } i = v_p(c) \text{ and } v > v_p(c) > 0, \\ 1 & \text{if } i = v_p(c) \text{ and } v_p(c) = v, \\ 1 & \text{if } i = v_p(c) \text{ and } v_p(c) = 0, \\ -1 & \text{if } i = v_p(c) - 1 \text{ and } v_p(c) > 0, \\ -1 & \text{if } i = v_p(c) + 1 \text{ and } v_p(c) < v, \end{cases} \tag{25}$$

and

$$\mathcal{R}_{k,\epsilon,\psi}(p^i, p^j) = \begin{cases} \epsilon(-1) & \text{if } i = j, \\ \epsilon(-1)\bar{\psi}(p^{j-i}) \left(\frac{1}{p^{j-i}}\right)^k & \text{if } i < j, \\ \epsilon(-p^{i-j}) & \text{if } i > j. \end{cases} \tag{26}$$

The cases

- (Case 1) $0 < v_p(c) < v_p(d) \leq v$,
- (Case 2) $0 = v_p(c) < v_p(d) \leq v$,
- (Case 3) $v > v_p(c) > v_p(d) \geq 0$,
- (Case 4) $v = v_p(c) > v_p(d) \geq 0$,
- (Case 5) $0 < v_p(c) = v_p(d) < v$,

- (Case 6) $0 = v_p(c) = v_p(d)$,
- (Case 7) $v = v_p(c) = v_p(d)$,

need to be handled separately, which is done below.

Case 1 If $0 < v_p(c) < v_p(d) \leq v$, then by employing (25) for all i such that $|v_p(c) - i| > 1$ we have

$$\mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^i, p^{v_p(d)}) = 0.$$

Therefore, we have

$$\begin{aligned} &\sum_{0 \leq i \leq v} \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^i, p^{v_p(d)}) \\ &= \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)-1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)-1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)-1}, p^{v_p(d)}) \\ &\quad + \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(d)}) \\ &\quad + \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)+1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)+1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)+1}, p^{v_p(d)}), \end{aligned}$$

which, by (25) and (26), equals to

$$\begin{aligned} &= -1 \cdot \epsilon(-p) \cdot \epsilon(-1)\overline{\psi}(p^{v_p(d)-v_p(c)+1}) \left(\frac{1}{p^{v_p(d)-v_p(c)+1}}\right)^k \\ &\quad + \frac{p^k + \epsilon(p)\overline{\psi}(p)}{p^k} \cdot \epsilon(-1) \cdot \epsilon(-1)\overline{\psi}(p^{v_p(d)-v_p(c)}) \left(\frac{1}{p^{v_p(d)-v_p(c)}}\right)^k \\ &\quad + (-1) \cdot \epsilon(-1)\overline{\psi}(p) \left(\frac{1}{p}\right)^k \cdot \epsilon(-1)\overline{\psi}(p^{v_p(d)-v_p(c)-1}) \left(\frac{1}{p^{v_p(d)-v_p(c)-1}}\right)^k. \end{aligned}$$

By using multiplicative properties of Dirichlet characters we conclude that this expression is equal to 0.

Case 2 If $0 = v_p(c) < v_p(d) \leq v$, then by employing (25) and (26) we have

$$\begin{aligned} &\sum_{0 \leq i \leq v} \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^i, p^{v_p(d)}) \\ &= \mathcal{S}_{k,p^v,\epsilon,\psi}(1, 1)\mathcal{R}_{k,\epsilon,\psi}(1, 1)\mathcal{R}_{k,\epsilon,\psi}(1, p^{v_p(d)}) \\ &\quad + \mathcal{S}_{k,p^v,\epsilon,\psi}(1, p)\mathcal{R}_{k,\epsilon,\psi}(1, p)\mathcal{R}_{k,\epsilon,\psi}(p, p^{v_p(d)}) \\ &= \overline{\psi}(p^{v_p(d)}) \left(\frac{1}{p^{v_p(d)}}\right)^k - \overline{\psi}(p^{v_p(d)}) \left(\frac{1}{p^{v_p(d)}}\right)^k, \end{aligned}$$

which equals to 0.

Case 3 If $v > v_p(c) > v_p(d) \geq 0$, then by employing (25) and (26) we have

$$\sum_{0 \leq i \leq v} \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^i, p^{v_p(d)})$$

$$\begin{aligned}
 &= \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)-1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)-1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)-1}, p^{v_p(d)}) \\
 &\quad + \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(d)}) \\
 &\quad + \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)+1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)+1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)+1}, p^{v_p(d)}) \\
 &= -\epsilon(-p)\epsilon(-p^{v_p(c)-1-v_p(d)}) + \frac{p^k + \epsilon(p)\bar{\psi}(p)}{p^k} \cdot \epsilon(-1) \cdot \epsilon(-p^{v_p(c)-v_p(d)}) \\
 &\quad - \epsilon(-1)\bar{\psi}(p)\frac{1}{p^k} \cdot \epsilon(-p^{v_p(c)+1-v_p(d)}).
 \end{aligned}$$

By using multiplicative properties of Dirichlet characters we conclude that this expression is equal to 0.

Case 4 If $v = v_p(c) > v_p(d) \geq 0$, then by employing (25) and (26) we have

$$\begin{aligned}
 &\sum_{0 \leq i \leq v} \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^i, p^{v_p(d)}) \\
 &= \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)-1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)-1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)-1}, p^{v_p(d)}) \\
 &\quad + \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(d)}) \\
 &= -\epsilon(p^{v_p(c)-v_p(d)}) + \epsilon(p^{v_p(c)-v_p(d)}) \\
 &= 0.
 \end{aligned}$$

Case 5 If $0 < v_p(c) = v_p(d) < v$, then by employing (25), (26) and multiplicative properties of Dirichlet characters we have

$$\begin{aligned}
 &\sum_{0 \leq i \leq v} \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^i, p^{v_p(d)}) \\
 &= \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)-1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)-1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)-1}, p^{v_p(c)}) \\
 &\quad + \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)}) \\
 &\quad + \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)+1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^{v_p(c)+1})\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)+1}, p^{v_p(c)}) \\
 &= -\epsilon(-p) \cdot \epsilon(-1)\bar{\psi}(p)\frac{1}{p^k} + \frac{p^k + \epsilon(p)\bar{\psi}(p)}{p^k} \cdot \epsilon(-1) \cdot \epsilon(-1) \\
 &\quad - \epsilon(-1)\bar{\psi}(p)\frac{1}{p^k} \cdot \epsilon(-p) \\
 &= \frac{p^k - \epsilon(p)\bar{\psi}(p)}{p^k}.
 \end{aligned}$$

Case 6 If $0 = v_p(c) = v_p(d)$, then by employing (25), (26) and multiplicative properties of Dirichlet characters we have

$$\begin{aligned}
 &\sum_{0 \leq i \leq v} \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^i, p^{v_p(d)}) \\
 &= \mathcal{S}_{k,p^v,\epsilon,\psi}(1, 1)\mathcal{R}_{k,\epsilon,\psi}(1, 1)\mathcal{R}_{k,\epsilon,\psi}(1, 1)
 \end{aligned}$$

$$\begin{aligned}
 &+ \mathcal{S}_{k,p^v,\epsilon,\psi}(1,p)\mathcal{R}_{k,\epsilon,\psi}(1,p)\mathcal{R}_{k,\epsilon,\psi}(p,1) \\
 &= 1 \cdot \epsilon(-1) \cdot \epsilon(-1) - \epsilon(-1)\overline{\psi}(p)\frac{1}{p^k} \cdot \epsilon(-p) \\
 &= \frac{p^k - \epsilon(p)\overline{\psi}(p)}{p^k}.
 \end{aligned}$$

Case 7 If $v = v_p(c) = v_p(d)$, then by employing (25), (26) and multiplicative properties of Dirichlet characters we have

$$\begin{aligned}
 &\sum_{0 \leq i \leq v} \mathcal{S}_{k,p^v,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^{v_p(c)}, p^i)\mathcal{R}_{k,\epsilon,\psi}(p^i, p^{v_p(d)}) \\
 &= \mathcal{S}_{k,p^v,\epsilon,\psi}(p^v, p^{v-1})\mathcal{R}_{k,\epsilon,\psi}(p^v, p^{v-1})\mathcal{R}_{k,\epsilon,\psi}(p^{v-1}, p^v) \\
 &\quad + \mathcal{S}_{k,p^v,\epsilon,\psi}(p^v, p^v)\mathcal{R}_{k,\epsilon,\psi}(p^v, p^v)\mathcal{R}_{k,\epsilon,\psi}(p^v, p^v) \\
 &= -1 \cdot \epsilon(-p) \cdot \epsilon(-1)\overline{\psi}(p)\frac{1}{p^k} + 1 \cdot \epsilon(-1) \cdot \epsilon(-1) \\
 &= \frac{p^k - \epsilon(p)\overline{\psi}(p)}{p^k}.
 \end{aligned}$$

This yields (24). Finally, if $c \neq d$, then there exists a prime $p \mid N$ such that $v_p(c) \neq v_p(d)$. Hence by (24) the product in (23) is 0. If $c = d$ then for all prime divisors p of N we have $v_p(c) = v_p(d)$. Therefore, by (23) and (24) we have the desired result. \square

5 Constant terms of expansions of Eisenstein series at the cusps

Recall that $E_k(\epsilon, \psi; dz)$ is defined by (2) and we have

$$E_k(\epsilon, \psi; dz) \in E_k(\Gamma_0(N), \chi) \quad \text{when } (k, \epsilon, \psi) \neq (2, \chi_1, \chi_1),$$

and

$$L_d(z) := E_2(\chi_1, \chi_1; z) - dE_2(\chi_1, \chi_1; dz) \in E_2(\Gamma_0(N), \chi_1).$$

The constant terms of Eisenstein series in the expansion at the cusp a/c with $\gcd(a, c) = 1$ are given by

$$\begin{aligned}
 [0]_{a/c}E_k(\epsilon, \psi; dz) &= \overline{\psi}(a)\mathcal{R}_{k,\epsilon,\psi}(c, Md) \text{ when } (k, \epsilon, \psi) \neq (2, \chi_1, \chi_1) \text{ and} \quad (27) \\
 [0]_{a/c}L_d(z) &= \mathcal{R}_{2,\chi_1,\chi_1}(c, 1) - d\mathcal{R}_{2,\chi_1,\chi_1}(c, d), \quad (28)
 \end{aligned}$$

where $\mathcal{R}_{k,\epsilon,\psi}(c, t)$ is defined by (6). For (27) see [3, (6.2)], [5, Proposition 8.5.6 and Ex. 8.7 (i) on pg. 308]. The formula (28) is proved later in this section.

The structure of the terms $[0]_{a/c}E_k(\epsilon, \psi; dz)$ is complicated and difficult to work with. We observe that taking the average $[0]_{c,\psi}E_k(\epsilon, \psi; dz)$ gives constant terms a

very nice structure which is easier to work with (see (38)). Throughout the section we assume $k, N \in \mathbb{N}$, ϵ and ψ are primitive Dirichlet characters with conductors L and M , respectively, such that $LM \mid N$ and $(k, \epsilon, \psi) \neq (2, \chi_1, \chi_1)$.

Lemma 3 *Let $c \mid N$ and $LMd \mid N$. If $M \nmid c$, or $M \mid c$ and $L \nmid N/c$, then*

$$[0]_{a/c} E_k(\epsilon, \psi; dz) = 0.$$

Proof First, assuming $M \nmid c$, we see that $M \nmid \gcd(Md, c)$. Thus, $\gcd\left(\frac{Md}{\gcd(Md, c)}, M\right) \mid M$, which implies $\overline{\psi}\left(\frac{Md}{\gcd(Md, c)}\right) = 0$ (since the conductor of ψ is M). Therefore, the result follows from (27).

Second, we assume $M \mid c$ and $L \nmid N/c$. Setting $c_1 = c/M$, we have $c_1 \mid N/M$. Since $L \nmid N/c$, we have $c_1 \nmid N/LM$. Since $\frac{(N/M)}{L} \in \mathbb{Z}$, $\frac{(N/M)}{c_1} \in \mathbb{Z}$ and $\frac{(N/M)/c_1}{L} \notin \mathbb{Z}$, we have $\gcd(c_1, L) \neq 1$. Additionally, there exists a prime p dividing c_1 such that

$$v_p(c_1) > v_p(N) - v_p(M) - v_p(L). \tag{29}$$

Since $c_1 \mid N/M$, for all $p \mid c_1$ we have

$$v_p(c_1) \leq v_p(N) - v_p(M). \tag{30}$$

By (29) and (30) we have

$$v_p(N) - v_p(M) > v_p(N) - v_p(M) - v_p(L).$$

Therefore,

$$v_p(L) > 0. \tag{31}$$

Since $d \mid N/LM$, we have

$$v_p(N) - v_p(M) - v_p(L) \geq v_p(d) \geq 0. \tag{32}$$

Inequalities (29) and (32) together imply $v_p(c_1) > v_p(d)$. Employing (31) we have

$$p \mid \gcd\left(\frac{p^{v_p(c_1)}}{\gcd(p^{v_p(c_1)}, p^{v_p(d)}), p^{v_p(L)}\right).$$

That is,

$$p \mid \gcd\left(\frac{c_1}{\gcd(c_1, d)}, L\right).$$

This implies $\epsilon\left(\frac{c}{\gcd(Md, c)}\right) = 0$ (since the conductor of ϵ is L). Therefore, the result follows from (27). □

Next we need the orthogonality of characters. The following lemma is a result of standard Schur orthogonality relations for the characters on the unit group $(\mathbb{Z}/c\mathbb{Z})^\times$ (see [5, Proposition 3.4.2]).

Lemma 4 *Let $c \in \mathbb{N}$, and let ψ_1, ψ_2 be two primitive Dirichlet characters with conductors M_1 and M_2 , respectively. If both M_1 and M_2 divide c , then we have*

$$\sum_{\substack{a=1, \\ \gcd(a,c)=1}}^c \overline{\psi_1(a)}\psi_2(a) = \begin{cases} 0 & \text{if } \psi_1 \neq \psi_2, \\ \phi(c) & \text{if } \psi_1 = \psi_2. \end{cases}$$

Before we prove the main result of this section we prove (28).

Lemma 5 *If $\gcd(a, c) = 1$, then we have*

$$[0]_{a/c}L_d(z) = \mathcal{R}_{2,\chi_1,\chi_1}(c, 1) - d\mathcal{R}_{2,\chi_1,\chi_1}(c, d).$$

Proof Since $\gcd(a, c) = 1$, there exist $\beta, \gamma \in \mathbb{Z}$ such that $A = \begin{bmatrix} a & \beta \\ c & \gamma \end{bmatrix} \in SL_2(\mathbb{Z})$. Then by [12, (1.21)] we have

$$E_2(\chi_1, \chi_1; A(z)) = (cz + \gamma)^2 E_2(\chi_1, \chi_1; z) - \frac{6ic}{\pi}(cz + \gamma), \tag{33}$$

where $A(z)$ is the usual linear fractional transformation. Let $e = \frac{ad}{\gcd(c,ad)}$ and $g = \frac{c}{\gcd(c,ad)}$. Since $\gcd(e, g) = 1$ there exist f, h such that $\begin{bmatrix} e & f \\ g & h \end{bmatrix} \in SL_2(\mathbb{Z})$. Hence we have

$$\begin{aligned} & E_2(\chi_1, \chi_1; dA(z);) \\ &= E_2\left(\chi_1, \chi_1; \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} ahd - cf & \beta hd - \gamma f \\ -agd + ce & -\beta gd + \gamma e \end{bmatrix} (z)\right) \\ &= E_2\left(\chi_1, \chi_1; \begin{bmatrix} \frac{ad}{\gcd(c,ad)} & f \\ \frac{c}{\gcd(c,ad)} & h \end{bmatrix} \begin{bmatrix} ahd - cf & \beta hd - \gamma f \\ 0 & \frac{d}{\gcd(c,ad)} \end{bmatrix} (z)\right) \\ &= \left(\frac{\gcd(c, d)}{d}\right)^2 (cz + \gamma)^2 E_2\left(\chi_1, \chi_1; \begin{bmatrix} ahd - cf & \beta hd - \gamma f \\ 0 & \frac{d}{\gcd(c,ad)} \end{bmatrix} (z)\right) \\ &\quad - \frac{6ic}{\pi d}(cz + \gamma), \end{aligned}$$

where in the last line we used (33). Thus, we obtain

$$\begin{aligned} [0]_{a/c}L_d(z) &= [0]_{a/c}(E_2(\chi_1, \chi_1; z) - dE_2(\chi_1, \chi_1; dz)) = \frac{d - \gcd(c, d)^2}{d} \\ &= \mathcal{R}_{2,\chi_1,\chi_1}(c, 1) - d\mathcal{R}_{2,\chi_1,\chi_1}(c, d). \end{aligned}$$

□

Theorem 4 *If $c \mid N$ and $(\epsilon_1, \psi_1), (\epsilon_2, \psi_2) \in \{(\epsilon, \psi) \in \mathcal{E}(k, N, \chi) : M \mid c\}$, then we have*

$$[0]_{c, \psi_2} E_k(\epsilon_1, \psi_1; dz) = \begin{cases} [0]_{1/c} E_k(\epsilon_2, \psi_2; dz) & \text{if } \psi_1 = \psi_2, \\ 0 & \text{otherwise.} \end{cases}$$

If $c \mid N$ and $(\epsilon_2, \psi_2) \in \{(\epsilon, \psi) \in \mathcal{E}(2, N, \chi_1) : M \mid c\}$, then we have

$$[0]_{c, \psi_2} L_d(z) = \begin{cases} [0]_{1/c} L_d(z) & \text{if } \psi_2 = \chi_1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof If $(k, \epsilon, \psi) \neq (2, \chi_1, \chi_1)$ by (27) we have

$$\begin{aligned} [0]_{c, \psi_2} E(\epsilon_1, \psi_1; dz) &= \frac{1}{\phi(c)} \sum_{\substack{a=1, \\ \gcd(a, c)=1}}^c \psi_2(a) \overline{\psi_1(a)} \mathcal{R}_{k, \epsilon_1, \psi_1}(c, M_1 d) \\ &= [0]_{1/c} E_k(\epsilon_1, \psi_1; dz) \frac{1}{\phi(c)} \sum_{\substack{a=1, \\ \gcd(a, c)=1}}^c \psi_2(a) \overline{\psi_1(a)}. \end{aligned}$$

Therefore, by Lemma 4 we obtain the first part of the statement. Proof of the second part is similar. \square

6 Proof of the main theorem

Recall that $E_k(\epsilon, \psi; dz)$ is defined by (2) and the set

$$\{E_k(\epsilon, \psi; dz) : (\epsilon, \psi) \in \mathcal{E}(k, N, \chi), d \mid N/LM\} \quad (34)$$

constitutes a basis for $E_k(\Gamma_0(N), \chi)$ whenever $(k, \chi) \neq (2, \chi_1)$ and the set

$$\begin{aligned} &\{E_2(\chi_1, \chi_1; z) - dE_2(\chi_1, \chi_1; dz) : 1 < d \mid N\} \\ &\cup \{E_2(\epsilon, \psi; dz) : (\epsilon, \psi) \in \mathcal{E}(2, N, \chi_1), (\epsilon, \psi) \neq (\chi_1, \chi_1), d \mid N/LM\} \quad (35) \end{aligned}$$

constitutes a basis for $E_2(\Gamma_0(N), \chi_1)$, see [5, Theorems 8.5.17 and 8.5.22], or [22, Proposition 5].

Now, we prove the main theorem whenever $(k, \chi) \neq (2, \chi_1)$. Let $f(z) \in M_k(\Gamma_0(N), \chi)$ where $N, k \in \mathbb{N}$, $k \geq 2$ and $(k, \chi) \neq (2, \chi_1)$. By (34) we have

$$E_f(z) = \sum_{(\epsilon, \psi) \in \mathcal{E}(k, N, \chi)} \sum_{d \mid N/LM} a_f(\epsilon, \psi, d) E_k(\epsilon, \psi; dz), \quad (36)$$

for some $a_f(\epsilon, \psi, d) \in \mathbb{C}$. Our strategy for the proof is, using the interplay between the constant terms of Eisenstein series, to create sets of linear equations (see (38)) and to solve those sets of linear equations for $a_f(\epsilon, \psi, d)$ using Theorem 3.

By (1) we have $f(z) = E_f(z) + S_f(z)$, where $E_f(z) \in E_k(\Gamma_0(N), \chi)$ and $S_f(z) \in S_k(\Gamma_0(N), \chi)$ are unique. Since by definition $S_f(z)$ vanishes at all cusps, we have $[0]_{a/c}f(z) = [0]_{a/c}E_f(z)$. Therefore, by (36) for each $c \mid N$ and $a \in \mathbb{Z}$ such that $\gcd(a, c) = 1$, we obtain

$$[0]_{a/c}f(z) = \sum_{(\epsilon, \psi) \in \mathcal{E}(k, N, \chi)} \sum_{d \mid N/LM} a_f(\epsilon, \psi, d)[0]_{a/c}E_k(\epsilon, \psi; dz).$$

Let $(\epsilon_2, \psi_2) \in \mathcal{E}(k, N, \chi)$, and let the conductors of ϵ_2 and ψ_2 be L_2 and M_2 , respectively. Note that for each ψ_2 there is a unique ϵ_2 such that $(\epsilon_2, \psi_2) \in \mathcal{E}(k, N, \chi)$. If we average the constant terms with ψ_2 using (4), then for all $c \mid N$ we obtain

$$[0]_{c, \psi_2}f(z) = \sum_{(\epsilon, \psi) \in \mathcal{E}(k, N, \chi)} \sum_{d \mid N/LM} a_f(\epsilon, \psi, d)[0]_{c, \psi_2}E_k(\epsilon, \psi; dz).$$

Our goal here is to isolate a set of linear equations from which we can determine $a_f(\epsilon_2, \psi_2, d)$ for all $d \mid N/L_2M_2$. By Lemma 3 we have $[0]_{c, \psi_2}E_k(\epsilon_2, \psi_2; dz) = 0$, if $c \mid N$ is such that $M_2 \mid c$, or $M_2 \nmid c$ and $L_2 \mid N/c$. Therefore, from now on we restrict c to be in $C_N(\epsilon_2, \psi_2)$, see (5) for definition. By applying Lemma 3 one more time we have $[0]_{c, \psi_2}E_k(\epsilon, \psi; dz) = 0$ if $M \nmid c$. Therefore, for all $c \in C_N(\epsilon_2, \psi_2)$ we have

$$[0]_{c, \psi_2}f(z) = \sum_{\substack{(\epsilon, \psi) \in \mathcal{E}(k, N, \chi) \\ M \mid c}} \sum_{d \mid N/LM} a_f(\epsilon, \psi, d)[0]_{c, \psi_2}E_k(\epsilon, \psi; dz). \tag{37}$$

Recall that for each ψ_2 there is a unique $(\epsilon_2, \psi_2) \in \mathcal{E}(k, N, \chi)$. Additionally, for all $c \in C_N(\epsilon_2, \psi_2)$ we have $(\epsilon_2, \psi_2) \in \{(\epsilon, \psi) \in \mathcal{E}(k, N, \chi) : M \mid c\}$. Therefore, for all $c \in C_N(\epsilon_2, \psi_2)$ we have

$$[0]_{c, \psi_2}f(z) = \sum_{\substack{(\epsilon, \psi) \in \mathcal{E}(k, N, \chi) \\ (\epsilon, \psi) \neq (\epsilon_2, \psi_2) \\ M \mid c}} \sum_{d \mid N/LM} a_f(\epsilon, \psi, d)[0]_{c, \psi_2}E_k(\epsilon, \psi; dz) + \sum_{d \mid N/L_2M_2} a_f(\epsilon_2, \psi_2, d)[0]_{c, \psi_2}E_k(\epsilon_2, \psi_2; dz).$$

From this, using Theorem 4, we obtain

$$[0]_{c, \psi_2}f(z) = \sum_{d \mid N/L_2M_2} a_f(\epsilon_2, \psi_2, d)[0]_{1/c}E_k(\epsilon_2, \psi_2; dz).$$

Since $M_2 \mid c$ we have

$$[0]_{1/c} E_k(\epsilon_2, \psi_2; dz) = \mathcal{R}_{k, \epsilon_2, \psi_2}(c, M_2 d) = \mathcal{R}_{k, \epsilon_2, \psi_2}(c/M_2, d).$$

Hence for all $c \in C_N(\epsilon_2, \psi_2)$ we have

$$[0]_{c, \psi_2} f(z) = \sum_{d \mid N/L_2 M_2} a_f(\epsilon_2, \psi_2, d) \mathcal{R}_{k, \epsilon_2, \psi_2}(c/M_2, d). \tag{38}$$

Below we solve the equations coming from (38) for $a_f(\epsilon_2, \psi_2, d)$ using Theorem 3. For $d_2 \mid N/L_2 M_2$ we consider the sum

$$\sum_{c \in C_N(\epsilon_2, \psi_2)} \mathcal{R}_{k, \epsilon_2, \psi_2}(d_2, c/M_2) \mathcal{S}_{k, N/L_2 M_2, \epsilon_2, \psi_2}(d_2, c/M_2) [0]_{c, \psi_2} f(z), \tag{39}$$

which, by (38), equals to

$$\begin{aligned} &= \sum_{c \in C_N(\epsilon_2, \psi_2)} \mathcal{R}_{k, \epsilon_2, \psi_2}(d_2, c/M_2) \mathcal{S}_{k, N/L_2 M_2, \epsilon_2, \psi_2}(d_2, c/M_2) \\ &\quad \times \sum_{d \mid N/L_2 M_2} a_f(\epsilon_2, \psi_2, d) \mathcal{R}_{k, \epsilon_2, \psi_2}(c/M_2, d). \end{aligned} \tag{40}$$

Rearranging the terms of (40) we obtain

$$\begin{aligned} &\sum_{c \in C_N(\epsilon_2, \psi_2)} \mathcal{R}_{k, \epsilon_2, \psi_2}(d_2, c/M_2) \mathcal{S}_{k, N/L_2 M_2, \epsilon_2, \psi_2}(d_2, c/M_2) [0]_{c, \psi_2} f(z) \\ &= \sum_{d \mid N/L_2 M_2} a_f(\epsilon_2, \psi_2, d) \\ &\quad \times \sum_{c \in C_N(\epsilon_2, \psi_2)} \mathcal{R}_{k, \epsilon_2, \psi_2}(d_2, c/M_2) \mathcal{S}_{k, N/L_2 M_2, \epsilon_2, \psi_2}(d_2, c/M_2) \mathcal{R}_{k, \epsilon_2, \psi_2}(c/M_2, d). \end{aligned} \tag{41}$$

Recall that $C_N(\epsilon_2, \psi_2)$ is defined by (5) and is a set equivalent to the set

$$\{c : M_2 \mid c, c/M_2 \mid N/L_2 M_2\},$$

i.e., c/M_2 runs through all the divisors of $N/L_2 M_2$ as c runs through all the elements of $C_N(\epsilon_2, \psi_2)$. In Theorem 3 we use this and we replace N by $N/L_2 M_2$, t by c/M_2 , c by d_2 and d by d to obtain

$$\sum_{c \in C_N(\epsilon_2, \psi_2)} \mathcal{R}_{k, \epsilon_2, \psi_2}(d_2, c/M_2) \mathcal{S}_{k, N/L_2 M_2, \epsilon_2, \psi_2}(d_2, c/M_2) \mathcal{R}_{k, \epsilon_2, \psi_2}(c/M_2, d)$$

$$= \begin{cases} \prod_{p|N/L_2M_2} \frac{p^k - \epsilon_2(p)\overline{\psi_2}(p)}{p^k} & \text{if } d = d_2, \\ 0 & \text{if } d \neq d_2. \end{cases} \tag{42}$$

Therefore, from (41) and (42) we obtain

$$\begin{aligned} & \sum_{c \in C_N(\epsilon_2, \psi_2)} \mathcal{R}_{k, \epsilon_2, \psi_2}(d_2, c/M_2) \mathcal{S}_{k, N/L_2M_2, \epsilon_2, \psi_2}(d_2, c/M_2) [0]_{c, \psi_2} f(z) \\ &= a_f(\epsilon_2, \psi_2, d_2) \prod_{p|N/L_2M_2} \frac{p^k - \epsilon_2(p)\overline{\psi_2}(p)}{p^k}. \end{aligned}$$

Since $p \mid L_2M_2$ implies $\epsilon_2(p)\overline{\psi_2}(p) = 0$ we have

$$\begin{aligned} a_f(\epsilon_2, \psi_2, d_2) &= \prod_{p|N} \frac{p^k}{p^k - \epsilon_2(p)\overline{\psi_2}(p)} \\ &\times \sum_{c \in C_N(\epsilon_2, \psi_2)} \mathcal{R}_{k, \epsilon_2, \psi_2}(d_2, c/M_2) \mathcal{S}_{k, N/L_2M_2, \epsilon_2, \psi_2}(d_2, c/M_2) [0]_{c, \psi_2} f. \end{aligned}$$

This completes the proof of Theorem 1 when $(k, \chi) \neq (2, \chi_1)$.

Now, if $(k, \chi) = (2, \chi_1)$, then a basis of $E_2(\Gamma_0(N), \chi_1)$ is given by (35). Using Lemma 5, Theorem 4, and arguments similar to the first part of this proof we obtain

$$E_f(z) = \sum_{1 < d|N} c_f(\chi_1, \chi_1, d) L_d(z) + \sum_{\substack{(\epsilon, \psi) \in \mathcal{E}(2, N, \chi) \\ (\epsilon, \psi) \neq (\chi_1, \chi_1)}} \sum_{d|N/LM} a_f(\epsilon, \psi, d) E_2(\epsilon, \psi; dz),$$

where $a_f(\epsilon, \psi, d)$ is as above (with $k = 2$) and

$$\begin{aligned} c_f(\chi_1, \chi_1, d) &= -\frac{1}{d} \prod_{p|N} \frac{p^2}{p^2 - 1} \sum_{c|N} \mathcal{R}_{2, \chi_1, \chi_1}(d, c) \mathcal{S}_{2, N, \chi_1, \chi_1}(d, c) [0]_{c, \chi_1} f \\ &= -\frac{1}{d} a_f(\chi_1, \chi_1, d). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{1 < d|N} c_f(\chi_1, \chi_1, d) L_d(z) &= \sum_{1 < d|N} c_f(\chi_1, \chi_1, d) (E_2(\chi_1, \chi_1; z) - dE_2(\chi_1, \chi_1; dz)) \\ &= \sum_{1 < d|N} c_f(\chi_1, \chi_1, d) E_2(\chi_1, \chi_1; z) \\ &\quad + \sum_{1 < d|N} a_f(\chi_1, \chi_1, d) E_2(\chi_1, \chi_1; dz) \end{aligned}$$

$$= \sum_{d|N} a_f(\chi_1, \chi_1, d) E_2(\chi_1, \chi_1; dz),$$

since

$$\begin{aligned} a_f(\chi_1, \chi_1, 1) \prod_{p|N} \frac{p^2 - 1}{p^2} &= \sum_{c|N} \mathcal{R}_{2, \chi_1, \chi_1}(1, c) \mathcal{S}_{2, N, \chi_1, \chi_1}(1, c) [0]_{c, \chi_1} f \\ &= \sum_{c|N} \frac{\mu(c)}{c^2} \sum_{1 < d|N} c_f(\chi_1, \chi_1, d) \frac{d - \gcd(d, c)^2}{d} \\ &= \sum_{1 < d|N} c_f(\chi_1, \chi_1, d) \sum_{c|N} \frac{\mu(c)}{c^2} \frac{d - \gcd(d, c)^2}{d} \\ &= \prod_{p|N} \frac{p^2 - 1}{p^2} \sum_{1 < d|N} c_f(\chi_1, \chi_1, d), \end{aligned}$$

i.e., $\sum_{1 < d|N} c_f(\chi_1, \chi_1, d) = a_f(\chi_1, \chi_1, 1)$. This completes the proof of the Main Theorem.

At last we prove a lemma which is useful in reducing the number of constant term computations in the applications of Theorem 1.

Lemma 6 *Let $f(z) \in M_k(\Gamma_0(N), \chi)$ and $c \mid N$. Let a/c and a'/c be equivalent cusps of $\Gamma_0(N)$. If $(\epsilon, \psi) \in \mathcal{E}(k, N, \chi)$ with $M \mid c$, then we have*

$$\psi(a)[0]_{a/c} f = \psi(a')[0]_{a'/c} f.$$

Proof If a/c and a'/c are equivalent cusps of $\Gamma_0(N)$, then there exists a matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \Gamma_0(N)$ such that

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a' & b' \\ c & d' \end{bmatrix}. \tag{43}$$

Then using transformation properties of modular forms we have

$$\begin{aligned} \psi(a')[0]_{a'/c} f &= \psi(a') \lim_{z \rightarrow i\infty} (cz + d')^{-k} f\left(\frac{a'z + b'}{cz + d'}\right) \\ &= \psi(a') \lim_{z \rightarrow i\infty} (cz + d')^{-k} \chi(\delta) \left(\gamma \frac{az + b}{cz + d} + \delta\right)^k f\left(\frac{az + b}{cz + d}\right) \\ &= \psi(a') \chi(\delta) \lim_{z \rightarrow i\infty} (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) \\ &= \psi(a') \chi(\delta) [0]_{a/c} f. \end{aligned}$$

We have $M \mid c$ and by (43) we have $a' = \alpha a + \beta c$, thus $\psi(a') = \psi(\alpha) \psi(a)$. Since $M \mid c, c \mid N$, and $N \mid \gamma$, we have $M \mid \gamma$, therefore, we have $1 = \psi(1) = \psi(\alpha\delta - \gamma\beta)$

which implies $\psi(\alpha) = \overline{\psi}(\delta)$. Putting these together, we obtain

$$\psi(a')\chi(\delta) = \psi(a)\overline{\psi}(\delta)\chi(\delta).$$

Since $\gcd(\delta, N) = 1$ we have $\overline{\psi}(\delta)\chi(\delta) = \epsilon(\delta)$. Now, we prove $\epsilon(\delta) = 1$ which finishes the proof. Recall that $LM \mid N$, therefore, $c \mid M$ implies $L \mid N/c$, i.e., $L \mid \gamma/c$. From (43) we have $\delta = 1 - a\gamma/c$, thus, since $\gcd(a, c) = 1$ and $L \mid \gamma/c$, we have $\epsilon(\delta) = \epsilon(1 - a\gamma/c) = \epsilon(1) = 1$. □

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Appendix A The SAGE functions for computing the constant terms of eta quotients at a given cusp

Let $r_d \in \mathbb{Z}$, not all zeros, $N \in \mathbb{N}$ and define

$$f(z) = \prod_{d|N} \eta^{r_d}(dz).$$

Assuming $f(z)$ to be a modular form the following SAGE functions (written using version 9.1 of the software [20]) help computing $[0]_{a/c}f$, the constant term of $f(z)$ at the cusp a/c .

```
def v_eta1(a,b,c,d):
    if c%2==1:
        return kronecker_symbol(d,abs(c))
    if c%2==0:
        return kronecker_symbol(c,abs(d))

def v_eta2(a,b,c,d):
    if c%2==1:
        return 1
    if c%2==0:
        return (-1)^(1/4*(sgn(c)-1)*(sgn(d)-1))

def v_eta3(a,b,c,d):
    if c%2==1:
        return (1/24 * ((a+d) * c-b * d * (c^2-1)-3 * c))
    if c%2==0:
        return (1/24 * ((a+d) * c-b * d * (c^2-1)+3 * d-3 * c * d))

def L_constr(m,d,c): #Proposition 2.1 of [12]
    x1= m * d/gcd(c,m)
```

```

u1=-c/gcd(c,m)
y1=0
v1=0
for i1 in range(-abs(x1*u1),abs(x1*u1)):
    if gcd(i1 ,x1)==1 and
    (1+i1*u1)%x1==0 and
    ((1+i1*u1)/x1)%2==1:
        y1=i1
        v1=(1+i1*u1)/x1
        return [x1,y1,u1,v1]
    break

def A_find(d,c): #finds a suitable matrix
    for b in range(abs(d*c)):
        if gcd(b,d)==1 and
        (1+b*c)%d==0:
            a=(1+b*c)/d
            return [a,b,c,d]
    break

def f_c_of_eta(m,d,c): #Constant term of the Dedekind eta function at -d/c
A = A_find(d,c)
L = L_constr(m,d,c)
a = A[0]
b = A[1]
c = A[2]
d = A[3]
x = L[0]
y = L[1]
u = L[2]
v = L[3]
vv = -m *b *v -y *a
OP1 = v_eta1(x,y,u,v)
OP2 = v_eta2(x,y,u,v)
OP3 = v_eta3(x,y,u,v)
OP4 = (1/24 /m *vv*gcd(c,m))
OP5 = (gcd(c,m)/m)^(1/2)
return [OP1,OP2,OP3,OP4,OP5]

def first_coeff_of_eta_q(N,etaq,a,c): #Computes the constant term of
#the eta quotient [r_1,...,r_d,...,r_N] at cusp a/c
d=-a
divs=divisors(N)
L=len(etaq)
if sum(1/24/divs[i2]*(gcd(c,divs[i2]))^2*etaq[i2] for i2 in range(L))>0:
    return 0 #does the vanishing order analysis
else:
    VV1=prod((f_c_of_eta(divs[i1],d,c)[0])^(etaq[i1]) for i1 in range(L))
    VV2=prod((f_c_of_eta(divs[i1],d,c)[1])^(etaq[i1]) for i1 in range(L))

```

```

SS1=sum((f_c_of_eta(divs[i1],d,c)[2])*(etaq[i1]) for i1 in range(L))
SS2=sum((f_c_of_eta(divs[i1],d,c)[3])*(etaq[i1]) for i1 in range(L))
VV3=prod((f_c_of_eta(divs[i1],d,c)[4])^(etaq[i1]) for i1 in range(L))
VV4=e^(2*pi*I*(SS1+SS2))
kk=sum(r for r in etaq)/2
return (-1)^kk*VV1*VV2*VV3*VV4
    
```

By Lemma 6 it will be sufficient to compute the constant terms of the eta quotient $f_k(z)$ defined by (17) at a set of inequivalent cusps of $\Gamma_0(24)$, which is done below with the help of this code. The set

$$\{1/1, 1/2, 1/3, 1/4, 1/6, 1/8, 1/12, 1/24\}$$

gives a complete set of inequivalent cusps of $\Gamma_0(24)$, see [5, Corollary 6.3.23]. Note that if k is fixed then the code can handle the vanishing order analysis. For instance the output for the code

```

k=3
print(first_coeff_of_eta_q(24,[-2*k-1,2*k+1,2*k+1,0,0,2*k+1,2*k+1,-2*k-1],1,2))
    
```

will be 0. However, here we are working with a general k , and therefore, the order analysis has to be done manually. When $k \geq 1$, the vanishing order of $f_k(z)$ is greater than 0 at cusps $\{1/2, 1/3, 1/4, 1/6, 1/8, 1/12\}$. Thus, we have

$$[0]_{1/2}f_k = 0, [0]_{1/3}f_k = 0, [0]_{1/4}f_k = 0, [0]_{1/6}f_k = 0, [0]_{1/8}f_k = 0, [0]_{1/12}f_k = 0.$$

To compute $[0]_{1/1}f_k$ and $[0]_{1/24}f_k$, we run the following code:

```

k=var('k')
assume(k, 'integer')
eta=[-2*k-1,2*k+1,2*k+1,0,0,2*k+1,2*k+1,-2*k-1]
print(first_coeff_of_eta_q(24,eta,1,1).simplify())
print(first_coeff_of_eta_q(24,eta,1,24).simplify())
    
```

The output will be

$$-I*6^{(k + 1/2)}*3^{(-2*k - 1)}*2^{(-4*k - 2)}*(-1)^k$$

Simplifying these we obtain

$$[0]_{1/1}f_k = -\frac{i^{2k+1}\sqrt{6}}{3^{k+1}2^{3k+2}}, [0]_{1/24}f_k = 1.$$

Putting everything together, for all $k \geq 1$ we have

$$\begin{aligned}
 [0]_{1/1}f_k &= -\frac{i^{2k+1}\sqrt{6}}{3^{k+1}2^{3k+2}}, [0]_{1/2}f_k = 0, [0]_{1/3}f_k = 0, [0]_{1/4}f_k = 0, \\
 [0]_{1/6}f_k &= 0, [0]_{1/8}f_k = 0, [0]_{1/12}f_k = 0, [0]_{1/24}f_k = 1.
 \end{aligned}$$

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