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On sums of sums involving cube-full numbers

Isao Kiuchi[1]

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Abstract
Let f_3 denote the characteristic function of cube-full numbers, and let (n, q) be the greatest common divisor of positive integers n and q. For any positive real numbers x and y, we shall consider several asymptotic formulas for sums of sums of modified cube-full numbers, which is $\sum_{n\le y}\left(\sum_{q\le x}\sum_{d|(n,q)} d f_3(q/d)\right)^k$ with $k = 1, 2$.

Keywords Cube-full numbers · Riemann zeta-function · Divisor function · Asymptotic results on arithmetical functions

Mathematics Subject Classification 11A25 · 11N37 · 11P99

1 Introduction

For any integer $r(\ge 2)$, we call n an r-full integer if $p|n \Rightarrow p^r|n$ and call n an r-free integer if $p|n \Rightarrow p^r \nmid n$, where the letter p denotes a prime number. If $r = 2$ or $r = 3$, we use the terms square-full or cube-full. Let $G(r)$ denote the set of r-full numbers, then we set

$$f_r(n) := \begin{cases} 1 & \text{if } n \in G(r), \\ 0 & \text{if } n \notin G(r). \end{cases}$$

Let $s = \sigma + it$ be the complex variable, and let $\zeta(s)$ be the Riemann zeta-function. Denote the Dirichlet series $F_r(s)$ defined by $F_r(s) := \sum_{n=1}^{\infty} \frac{f_r(n)}{n^s}$. Following (7.3) in Krätzel [7], the representation of $F_r(s)$ is more complicated for $k \ge 3$, and it is known that

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✉ Isao Kiuchi
kiuchi@yamaguchi-u.ac.jp

1 Department of Mathematical Sciences, Faculty of Science, Yamaguchi University, Yoshida 1677-1, Yamaguchi 753-8512, Japan

$$F_r(s) = \prod_{\ell=r}^{2r-1} \frac{\zeta(\ell s)}{\zeta((2r+2)s)} \kappa_{2r+3}(s) \tag{1.1}$$

holds. Here, $c_{2r+3}(n)$ denotes a certain arithmetical function whose associated Dirichlet series $\kappa_{2r+3}(s) := \sum_{n=1}^{\infty} \frac{c_{2r+3}(n)}{n^s}$, which is absolutely convergent for Re $s > \frac{1}{2r+3}$. We define a sum over the r-full numbers by

$$s_q^{(r)}(n) := \sum_{d|(n,q)} d f_r\left(\frac{q}{d}\right), \tag{1.2}$$

where (n, q) denotes the greatest common divisor of integers n and q. For any large positive real numbers x and y, we set the double sums

$$S_k^{(r)}(x, y) := \sum_{n \le y} \left(\sum_{q \le x} s_q^{(r)}(n) \right)^k \qquad (k = 1, 2). \tag{1.3}$$

For $r = 2$, Kiuchi [6] considered the asymptotic formula for the double sum (1.3) concerning square-full numbers, and used the theory of exponent pairs to derive the precise asymptotic formula

$$S_1^{(2)}(x, y) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} xy - \frac{\zeta(4)\zeta(6)}{4\zeta(12)} x^2 + O\left(x^{\frac{1}{2}} y + x y^{\frac{1}{3}} + \frac{x^3}{y}\right), \tag{1.4}$$

where x and y are large real numbers such that $x \ll y \ll x^{\frac{3}{2}}$. When $k = 2$, Kiuchi [6] also showed that

$$S_2^{(2)}(x, y) = \frac{\zeta(2)\zeta^2(3)}{\zeta^2(6)} x^2 y \log x + O\left(x^2 y + x^4\right) \tag{1.5}$$

holds, where x and y are large real numbers such that $y \gg \frac{x^2}{\log x}$. Moreover, he used analytic properties of the Riemann zeta-function to obtain the asymptotic formula

$$\begin{aligned} S_2^{(2)}(x, y) = {} & \frac{\zeta(2)\zeta^2(3)}{\zeta^2(6)} x^2 y \log \frac{x^3}{y} + c_0 x^2 y \\ & + \frac{\zeta(2)\zeta^2(3)}{\zeta^2(6)} \left(2\gamma - 2 + 5\frac{\zeta'(2)}{\zeta(2)} + 9\frac{\zeta'(3)}{\zeta(3)} - 18\frac{\zeta'(6)}{\zeta(6)}\right) x^2 y \\ & + O\left(x^2 y \left(L^5 x^{-\frac{1}{14}} + L^6 y^{-\frac{1}{2}} + L^2 \left(\frac{x}{y}\right)^{\frac{1}{2}} + L^2 \left(\frac{y}{x^2}\right)^{\frac{1}{2}}\right)\right), \end{aligned} \tag{1.6}$$

where c_0 is a computable constant, and x and y are large real numbers such that $x \log^4 x \ll y \ll \frac{x^2}{\log^6 x}$. To prove the precise asymptotic formulas (1.4), (1.5) and (1.6), we used the method of proofs of Chan and Kumchev [2] (see also Kiuchi, Minamide and Tanigawa [5], Kühn and Robles [8], Robles [10], Robles and Roy [11]).

For $r = 3$, it is derived from (1.1) that the Dirichlet series for the generating function $f_3(n)$ is

$$F(s) := \sum_{n=1}^{\infty} \frac{f_3(n)}{n^s} = \frac{\zeta(3s)\zeta(4s)\zeta(5s)\kappa(s)}{\zeta(8s)} \tag{1.7}$$

for Re $s > \frac{1}{3}$, where $\kappa(s)$ is the Dirichlet series generated by a certain arithmetical function $c_9(n)$ (see (7.3) in Krätzel [7]), that is $\kappa(s) := \sum_{n=1}^{\infty} \frac{c_9(n)}{n^s}$ which is absolutely convergent for Re $s > \frac{1}{9}$. Moreover, the asymptotic formula for the sum of $f_3(n)$ is also known, and one can see that

$$\begin{aligned}\sum_{n\le x} f_3(n) = \frac{\zeta\left(\frac{4}{3}\right)\zeta\left(\frac{5}{3}\right)\kappa\left(\frac{1}{3}\right)}{\zeta\left(\frac{8}{3}\right)} x^{\frac{1}{3}} + \frac{\zeta\left(\frac{3}{4}\right)\zeta\left(\frac{5}{4}\right)\kappa\left(\frac{1}{4}\right)}{\zeta(2)} x^{\frac{1}{4}} \\ + \frac{\zeta\left(\frac{3}{5}\right)\zeta\left(\frac{4}{5}\right)\kappa\left(\frac{1}{5}\right)}{\zeta\left(\frac{8}{5}\right)} x^{\frac{1}{5}} + \Delta(x)\end{aligned} \tag{1.8}$$

holds with the error term $\Delta(x) = O\left(x^{\frac{1}{8}} \log^4 x\right)$ for any large positive real number x (see section 7.1.3 in Krätzel [7]). In 1988, Balasubramanian et al. [1] showed that $\Delta(x) = \Omega\left(x^{\frac{1}{12}}\sqrt{\log x}\right)$ holds, and the improvement on the estimate of $\Delta(x)$ has been studied by many authors. Under the Riemann hypothesis, Wu [14] obtained that $\Delta(x) = O\left(x^{\frac{97}{804}+\varepsilon}\right)$ holds for any $\varepsilon > 0$. Using (1.7), the Dirichlet series generated by the coefficients $s_q(n)$ is expressed by

$$\sum_{q=1}^{\infty} \frac{s_q^{(3)}(n)}{q^s} = \sigma_{1-s}(n)\frac{\zeta(3s)\zeta(4s)\zeta(5s)\kappa(s)}{\zeta(8s)} \tag{1.9}$$

for Re $s > \frac{1}{3}$, where $\sigma_{1-s}(n) = \sum_{d|n} d^{1-s}$ is the generalized divisor function.

Now, we shall consider several asymptotic formulas of (1.3) concerning cube-full numbers. Our theorems are proved by the same way as in [6], and we shall deduce several interesting formulas for the double sum $S_k^{(3)}(x, y)$. We use the theory of exponent pairs and elementary methods to deal with $S_1^{(3)}(x, y)$. Then the case $k = 1$ implies the following theorem, namely

Theorem 1 *Let x and y be large real numbers such that $x \ll y \ll x^{\frac{5}{3}}$. Then we have*

$$\begin{aligned}S_1^{(3)}(x, y) = \frac{\zeta(3)\zeta(4)\zeta(5)\kappa(1)}{\zeta(8)} xy - \frac{\zeta(6)\zeta(8)\zeta(10)\kappa(2)}{4\zeta(16)} x^2 \\ + O\left(x^{\frac{1}{3}} y + x y^{\frac{1}{3}} + \frac{x^3}{y}\right).\end{aligned} \tag{1.10}$$

It follows from (1.10) that

$$\frac{1}{xy}\sum_{n\le y}\sum_{q\le x} s_q^{(3)}(n) = \frac{\zeta(3)\zeta(4)\zeta(5)\kappa(1)}{\zeta(8)} - \frac{\zeta(6)\zeta(8)\zeta(10)\kappa(2)}{4\zeta(16)}\frac{x}{y} + O\left(x^{-\frac{2}{3}} + y^{-\frac{2}{3}} + \frac{x^2}{y^2}\right)$$

holds. This is described by saying that the average order of $s_q^{(3)}(n)$ is $\frac{\zeta(3)\zeta(4)\zeta(5)\kappa(1)}{\zeta(8)}$ under q and n satisfying the condition $q \ll n \ll q^{\frac{5}{3}}$.

Remark 1.1 It would be an interesting problem to investigate the asymptotic behaviour of $S_1^{(3)}(x, y)$ under the condition $y \ll x$. However, this would require a different method.

For $k = 2$, there are two quite different methods to deal with this function $S_2^{(3)}(x, y)$. We utilize an elementary lattice point counting argument to obtain the formula (1.11) below, and use the generating Dirichlet series and the properties of the Riemann zeta-function to prove (1.12) below, which we state as

Theorem 2 *Let x and y be large real numbers such that* $y \gg \frac{x^2}{\log x}$. *Then we have*

$$S_2^{(3)}(x, y) = \frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta(2)\zeta^2(8)} x^2 y \log x + O\left(x^2 y + x^4\right). \tag{1.11}$$

Similarly, as in Theorem 1, we use (1.11) to get

$$\frac{1}{y}\sum_{n\le y}\left(\sum_{q\le x} s_q^{(3)}(n)\right)^2 = \frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta(2)\zeta^2(8)} x^2 \log x + O\left(x^2 + \frac{x^4}{y}\right).$$

This is described by saying that the average order of $s_q^{(3)}(n)$ is

$$\frac{\zeta(3)\zeta(4)\zeta(5)\kappa(1)}{\sqrt{\zeta(2)}\zeta(8)}\sqrt{\log q}$$

under q and n satisfying the condition $n \gg \frac{q^2}{\log q}$. We utilize the generating Dirichlet series and the properties of the Riemann zeta-function to prove (1.12) below, which we state as

Theorem 3 *Let x and y be large real numbers such that* $x \log^6 x \ll y \ll \frac{x^2}{\log^4 x}$. *Then we have*

$$S_2^{(3)}(x, y) = \frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta(2)\zeta^2(8)}\left(\log\frac{x^3}{y} + c_1\right) x^2 y + \eta x^2 y + O\left(x^2 y L^2\left(L^3 x^{-\frac{1}{3}} + L^4 y^{-\frac{1}{2}} + L\left(\frac{x}{y}\right)^{\frac{1}{2}} + \left(\frac{y}{x^2}\right)^{\frac{1}{2}}\right)\right), \tag{1.12}$$

where η is a computable constant, which is defined by (5.9) below, and the constant c_1 is given by

$$c_1 = 2\gamma - 2 + 9\frac{\zeta'(3)}{\zeta(3)} + 12\frac{\zeta'(4)}{\zeta(4)} + 15\frac{\zeta'(5)}{\zeta(5)} - 24\frac{\zeta'(8)}{\zeta(8)} - \frac{\zeta'(2)}{\zeta(2)} + 3\frac{\kappa'(1)}{\kappa(1)}.$$

Remark 1.2 It would be an interesting problem to investigate the asymptotic behaviour of $S_2^{(3)}(x, y)$ under the condition $y \ll x \log^6 x$. However, this would require a different method.

2 Some lemmas

To prove our theorems, we first prepare several lemmas. Let $\psi(x) = x - [x] - \frac{1}{2}$ denote the first periodic Bernoulli function. In the proof of Theorem 1, we need an upper bound of the sum

$$\sum_{n \in I} \psi\left(\frac{y}{n}\right).$$

An efficient way to estimate these ψ-sums is to apply the theory of exponent pairs: *An exponent pair (κ, λ) is a pair of numbers $0 \le \kappa \le \frac{1}{2} \le \lambda \le 1$ such that*

$$\sum_{n \in I} e^{2\pi i f(n)} \ll A^{\kappa} N^{\lambda}$$

holds, where $I \subset (N, 2N]$ and $A \ll |f'(u)| \ll A$ for $u \in I$. For the precise definition and its properties, the reader should consult Graham and Kolesnik [3] and Ivić [4]. Now applying Lemma 4.3 in [3] with $f(n) = \frac{y}{n}$, we have

Lemma 2.1 *Let (κ, λ) be an exponent pair. If I is a subinterval in $(N, 2N]$, we have*

$$\sum_{n \in I} \psi\left(\frac{y}{n}\right) \ll y^{\frac{\kappa}{\kappa+1}} N^{\frac{\lambda-\kappa}{\kappa+1}} + \frac{N^2}{y}.$$

In particular, if we take the exponent pair $(\kappa, \lambda) = \left(\frac{1}{2}, \frac{1}{2}\right)$, we get

$$\sum_{n \in I} \psi\left(\frac{y}{n}\right) \ll y^{\frac{1}{3}} + \frac{N^2}{y}. \tag{2.1}$$

The proofs of Theorem 3 need the following lemmas, namely

Lemma 2.2 *Suppose that the Dirichlet series* $\alpha(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *absolutely converges for* $\operatorname{Re} s > \sigma_a$. *If* $\sigma_0 > \max(0, \sigma_a)$ *and* $x > 0,\ T > 0$, *then*

$$\sum_{n\le x}{}' a_n = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s)\frac{x^s}{s}ds + R,$$

where

$$R \ll \sum_{\substack{\frac{x}{2}<n<2x\\ n\neq x}} |a_n| \min\left(1, \frac{x}{T|x-n|}\right) + \frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma_0}},$$

and $\sum{}'$ *indicates that the last term is to be halved if* x *is an integer.*

Proof This is Perron's famous formula (see Theorem 5.2 and Corollary 5.3 in Montgomery and Vaughan [9]). □

Lemma 2.3 *Let* $G(s_1, s_2; y)$ *be a sum function defined by*

$$G(s_1, s_2; y) = \sum_{n\le y}{}' \sigma_{1-s_1}(n)\sigma_{1-s_2}(n) \tag{2.2}$$

and $L = \log y$. *Then we have*

$$G(s_1, s_2; y) = \sum_{j=1}^{4} R_j(s_1, s_2; y) + O\left(yL^6\left(y^{-\frac{1}{2}} + \frac{1}{T}\right)\right) \tag{2.3}$$

for $\operatorname{Re} s_j \ge 1/2$ *and* $|\operatorname{Im} s_j| \le T\ (j = 1, 2)$, *where*

$$\begin{aligned}
R_1(s_1, s_2; y) &= y\frac{\zeta(s_1)\zeta(s_2)\zeta(s_1+s_2-1)}{\zeta(s_1+s_2)},\\
R_2(s_1, s_2; y) &= y^{2-s_1}\frac{\zeta(2-s_1)\zeta(1-s_1+s_2)\zeta(s_2)}{(2-s_1)\zeta(2-s_1+s_2)},\\
R_3(s_1, s_2; y) &= y^{2-s_2}\frac{\zeta(2-s_2)\zeta(1+s_1-s_2)\zeta(s_1)}{(2-s_2)\zeta(2+s_1-s_2)},\\
R_4(s_1, s_2; y) &= y^{3-s_1-s_2}\frac{\zeta(3-s_1-s_2)\zeta(2-s_2)\zeta(2-s_1)}{(3-s_1-s_2)\zeta(4-s_1-s_2)},
\end{aligned}$$

where $\sum{}'$ *indicates that the last term is to be halved if* y *is an integer.*

Proof The proof of this lemma follows from (4.12) in Chan and Kumchev [2]. □

Lemma 2.4 *For $t \geq t_0 > 0$ uniformly in σ, we have*

$$\zeta(\sigma + it) = \begin{cases} t^{\frac{1}{6}(3-4\sigma)} \log t & \left(0 \leq \sigma \leq \frac{1}{2}\right), \\ t^{\frac{1}{3}(1-\sigma)} \log t & \left(\frac{1}{2} \leq \sigma \leq 1\right), \\ \log t & (1 \leq \sigma < 2), \\ 1 & (\sigma \geq 2). \end{cases}$$

Proof The proof of this lemma follows from Theorem II.3.8 in Tenenbaum [12], and Ivić [4]. Also see Titchmarsh [13]. □

3 Proof of Theorem 1

We use (1.2) and (1.3) and change the order of summation to obtain

$$\begin{aligned} S_1^{(3)}(x, y) &= \sum_{n \leq y} \sum_{q \leq x} s_q^{(3)}(n) \\ &= y \sum_{dk \leq x} f_3(k) - \frac{1}{2} \sum_{dk \leq x} d f_3(k) - \sum_{dk \leq x} d f_3(k) \psi\left(\frac{y}{d}\right) \\ &=: S_{1,1}^{(3)}(x, y) - S_{1,2}^{(3)}(x, y) - S_{1,3}^{(3)}(x, y). \end{aligned} \tag{3.1}$$

We consider the first term on the right of (3.1). We use (1.7) to get

$$\sum_{k \leq x} \frac{f_3(k)}{k} = \frac{\zeta(3)\zeta(4)\zeta(5)\kappa(1)}{\zeta(8)} + O\left(x^{-\frac{2}{3}}\right). \tag{3.2}$$

We obtain from (1.7) and the above

$$\begin{aligned} S_{1,1}^{(3)}(x, y) &= yx \sum_{k \leq x} \frac{f_3(k)}{k} + O\left(y \sum_{k \leq x} f_3(k)\right) \\ &= \frac{\zeta(3)\zeta(4)\zeta(5)\kappa(1)}{\zeta(8)} xy + O\left(x^{\frac{1}{3}} y\right). \end{aligned} \tag{3.3}$$

Similarly, we have

$$\begin{aligned} S_{1,2}^{(3)}(x, y) &= \frac{1}{2} \sum_{k \leq x} f_3(k) \left(\frac{x^2}{2k^2} + O\left(\frac{x}{k}\right)\right) \\ &= \frac{\zeta(6)\zeta(8)\zeta(10)\kappa(2)}{4\zeta(16)} x^2 + O(x). \end{aligned} \tag{3.4}$$

To estimate $S_{1,3}(x, y)$, we use the theory of exponent pairs. Let $N_j = N_{j,k} = \left(\frac{x}{k}\right) 2^{-j}$. Then we have

$$S_{1,3}^{(3)}(x, y) = \sum_{k \le x} f_3(k) \sum_{d \le \frac{x}{k}} d\psi\left(\frac{y}{d}\right)$$
$$\ll \sum_{k \le x} f_3(k) \sum_{j=0}^{\infty} N_j \sup_I \left| \sum_{d \in I} \psi\left(\frac{y}{d}\right) \right|,$$

where the sup is over all subintervals I in $(N_j, 2N_j]$. From (2.1) of Lemma 2.1 and (3.2), we have

$$S_{1,3}^{(3)}(x, y) \ll \sum_{k \le x} f_3(k) \sum_{j=0}^{\infty} \left\{ N_j y^{\frac{1}{3}} + \frac{N_j^3}{y} \right\}$$
$$\ll \sum_{k \le x} \frac{f_3(k)}{k} \cdot xy^{1/3} + \sum_{k \le x} \frac{f_3(k)}{k^3} \cdot \frac{x^3}{y}$$
$$\ll xy^{1/3} + \frac{x^3}{y}. \tag{3.5}$$

Substituting (3.3), (3.4) and (3.5) into (3.1), we get the assertion of Theorem 1.

□

4 Proof of Theorem 2

From (1.2) and (1.3), we have

$$S_2^{(3)}(x, y) = \sum_{n \le y} \left(\sum_{\substack{dk \le x \\ d|n}} df_3(k) \right)^2$$
$$= \sum_{d_1 k_1 \le x} d_1 f_3(k_1) \sum_{d_2 k_2 \le x} d_2 f_3(k_2) \sum_{\substack{n \le y \\ d_1|n, d_2|n}} 1$$
$$= \sum_{d_1 k_1 \le x} \sum_{d_2 k_2 \le x} d_1 d_2 f_3(k_1)\ f_3(k_2) \left[\frac{y}{[d_1, d_2]} \right]$$
$$= y \sum_{d_1 k_1 \le x} \sum_{d_2 k_2 \le x} (d_1, d_2) f_3(k_1)\ f_3(k_2) + O(E), \tag{4.1}$$

where $[d_1, d_2]$ denotes the least common multiple of d_1 and d_2. We use (1.7) to get

$$
\begin{aligned}
E &:= \sum_{d_1k_1\le x}\sum_{d_2k_2\le x} d_1d_2 f_3(k_1)f_3(k_2)\\
&\ll x^2\sum_{k_1\le x}\frac{f_3(k_1)}{k_1^2}\cdot x^2\sum_{k_2\le x}\frac{f_3(k_2)}{k_2^2}\ll x^4.
\end{aligned}
$$

To evaluate the main term of (4.1), we use

$$
\sum_{mk\le x} f_3(k)=\frac{\zeta(3)\zeta(4)\zeta(5)\kappa(1)}{\zeta(8)}x+O\left(x^{1/3}\right), \tag{4.2}
$$

which follows from (1.7) and (3.2). Using the Gauss identity $\sum_{d|n}\phi(d)=n$, (4.2) and $\sum_{d\le x}\frac{\phi(d)}{d^2}=\frac{1}{\zeta(2)}\log x+O(1)$, we have

$$
\begin{aligned}
&\sum_{d_1k_1\le x}\sum_{d_2k_2\le x}(d_1,d_2)f_3(k_1)f_3(k_2)=\sum_{d\le x}\phi(d)\sum_{dl_1k_1\le x}\sum_{dl_2k_2\le x}f_3(k_1)f_3(k_2)\\
&=\sum_{d\le x}\phi(d)\left(\sum_{mk\le x/d}f_3(k)\right)^2\\
&=\frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta^2(8)}x^2\sum_{d\le x}\frac{\phi(d)}{d^2}+O\left(x^{4/3}\sum_{d\le x}\frac{\phi(d)}{d^{4/3}}\right)\\
&=\frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta(2)\zeta^2(8)}x^2\log x+O\left(x^2\right).
\end{aligned}
$$

Hence, we have

$$
S_2^{(3)}(x,y)=\frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta(2)\zeta^2(8)}x^2y\log x+O\left(x^2y+x^4\right).
$$

This completes the proof of Theorem 2. □

5 Proof of Theorem 3

In this section, we assume that $1\le y\le x^M$ for some constant M. Without loss of generality we can assume that $x,\ y\in\mathbb{Z}+\frac{1}{2}$. We apply Lemma 2.2 with (1.9), then

$$
\sum_{q\le x}s_q^{(3)}(n)=\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\sigma_{1-s}(n)\frac{\zeta(3s)\zeta(4s)\zeta(5s)\kappa(s)}{\zeta(8s)}\frac{x^s}{s}ds+E_1(x,n) \tag{5.1}
$$

with $\alpha = 1 + \frac{1}{\log x}$ and T being a real parameter at our disposal, where $E_1(x, n)$ is the error term given by

$$E_1(x, n) \ll \frac{x}{T} \sum_{q=1}^{\infty} \frac{s_q^{(3)}(n)}{q} \ll \frac{x}{T} \sigma_0(n)$$

by using (1.9). Let $\alpha_1 = 1 + \frac{1}{\log x}$ and $\alpha_2 = 1 + \frac{2}{\log x}$. Applying (5.1) with $\alpha = \alpha_j$ $(j = 1, 2)$ we have

$$\left(\sum_{q \le x} s_q^{(3)}(n) \right)^2 = \frac{1}{(2\pi i)^2} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \int_{\alpha_2 - iT}^{\alpha_2 + iT} F(s_1, s_2, n) ds_2 ds_1 + E_2(x, n), \quad (5.2)$$

where

$$F(s_1, s_2, n) = \sigma_{1-s_1}(n) \sigma_{1-s_2}(n) \times \frac{\zeta(3s_1)\zeta(4s_1)\zeta(5s_1)\zeta(3s_2)\zeta(4s_2)\zeta(5s_2)\kappa(s_1)\kappa(s_2)}{\zeta(8s_1)\zeta(8s_2)} \frac{x^{s_1+s_2}}{s_1 s_2}$$

and

$$E_2(x, n) = E_1(x, n) \left(\frac{1}{2\pi i} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \sigma_{1-s_1}(n) \frac{\zeta(3s_1)\zeta(4s_1)\zeta(5s_1)\kappa(s_1)}{\zeta(8s_1)} \frac{x^{s_1}}{s_1} ds_1 + \frac{1}{2\pi i} \int_{\alpha_2 - iT}^{\alpha_2 + iT} \sigma_{1-s_2}(n) \frac{\zeta(3s_2)\zeta(4s_2)\zeta(5s_2)\kappa(s_2)}{\zeta(8s_2)} \frac{x^{s_2}}{s_2} ds_2 + E_1(x, n) \right).$$

It follows that

$$E_2(x, n) \ll \frac{x^2}{T} \sigma_0(n)^2 \log T.$$

Summing (5.2) over n and using the estimate $\sum_{n \le y} \sigma_0(n)^2 \ll y \log^3 y$, we get

$$S_2^{(3)}(x, y) = \frac{1}{(2\pi i)^2} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \int_{\alpha_2 - iT}^{\alpha_2 + iT} G(s_1, s_2; y) \times \frac{\zeta(3s_1)\zeta(4s_1)\zeta(5s_1)\kappa(s_1)\zeta(3s_2)\zeta(4s_2)\zeta(5s_2)\kappa(s_2)}{\zeta(8s_1)\zeta(8s_2)} \times \frac{x^{s_1+s_2}}{s_1 s_2} ds_2\, ds_1 + O\left(\frac{x^2 y L^4}{T} \right), \quad (5.3)$$

where $G(s_1, s_2; y) := \sum_{n \le y} \sigma_{1-s_1}(n) \sigma_{1-s_2}(n)$ and $L = \log(Txy)$.

Now we shall evaluate the integrals in appearing in (5.3). Substituting (2.3) into (5.3), we have

$$S_2^{(3)}(x,y)=\sum_{j=1}^{4}S_{2,j}^{(3)}(x,y)+O\left(x^2yL^8\left(\frac{1}{T}+y^{-1/2}\right)\right), \tag{5.4}$$

where

$$\begin{aligned}S_{2,j}^{(3)}(x,y)=&\frac{1}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-iT}^{\alpha_2+iT}R_j(s_1,s_2;y)\\&\times\frac{\zeta(3s_1)\zeta(4s_1)\zeta(5s_1)\kappa(s_1)\zeta(3s_2)\zeta(4s_2)\zeta(5s_2)\kappa(s_2)}{\zeta(8s_1)\zeta(8s_2)}\\&\times\frac{x^{s_1+s_2}}{s_1s_2}ds_2\,ds_1.\end{aligned}$$

Note that we substitute $T=x$ into the error term on the right-hand side of (5.4) to get

$$\ll x^2yL^8\left(x^{-1}+y^{-1/2}\right). \tag{5.5}$$

5.1 Evaluation of $S_{2,1}^{(3)}(x,y)$

Let $\alpha_1=1+\frac{1}{\log x}$ and $\alpha_2=1+\frac{2}{\log x}$. From the definition of $R_1(s_1,s_2,y)$, we get

$$\begin{aligned}S_{2,1}^{(3)}(x,y)=&\frac{y}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-iT}^{\alpha_2+iT}\frac{\zeta(s_1)\zeta(s_2)\zeta(s_1+s_2-1)}{\zeta(s_1+s_2)}\\&\times\frac{\zeta(3s_1)\zeta(4s_1)\zeta(5s_1)\kappa(s_1)\zeta(3s_2)\zeta(4s_2)\zeta(5s_2)\kappa(s_2)}{\zeta(8s_1)\zeta(8s_2)}\frac{x^{s_1+s_2}}{s_1s_2}ds_2ds_1.\end{aligned} \tag{5.6}$$

Let $\Gamma(\alpha,\beta,T)$ denote the contour consisting of the line segments $[\alpha-iT,\beta-iT]$, $[\beta-iT,\beta+iT]$ and $[\beta+iT,\alpha+iT]$. In (5.6), we move the integration with respect to s_2 to $\Gamma(\alpha_2,\frac{1}{2}+\frac{1}{\log x},T)$. We denote the integrals over the horizontal line segments by $J_{1,1}$ and $J_{1,3}$, and the integral over the vertical line segment by $J_{1,2}$, respectively. Then using the estimate $\int_1^T|\zeta(\alpha_1+it)|dt\ll T$ and Lemma 2.4, we have

$$\begin{aligned}&J_{1,1},J_{1,3}\\&\quad\ll\frac{xyL}{T}\int_{-T}^{T}\frac{|\zeta(\alpha_1+it_1)|}{1+|t_1|}dt_1\\&\qquad\times\int_{\frac{1}{2}+\frac{1}{\log x}}^{\alpha_2}|\zeta(\sigma_2+iT)\zeta(\alpha_1+\sigma_2-1+i(t_1+T))|x^{\sigma_2}\,d\sigma_2\end{aligned}$$

$$\ll \frac{xyL^3}{T}\int_{-T}^{T}\frac{|\zeta(\alpha_1+it_1)|}{1+|t_1|}dt_1\int_{\frac{1}{2}+\frac{1}{\log x}}^{\alpha_2} T^{\frac{2}{3}(1-\sigma_2)}x^{\sigma_2}\,d\sigma_2$$
$$\ll \frac{x^2yL^4}{T^{2/3}}\left(x^{-1/2}+T^{-1/3}\right).$$

For the integral along the vertical line we have

$$J_{1,2} \ll yx^{\frac{3}{2}}L$$
$$\times \int_{-T}^{T}\int_{-T}^{T}\frac{|\zeta(\alpha_1+it_1)\zeta\left(\frac{1}{2}+\frac{1}{\log x}+it_2\right)\zeta(\alpha_1+\frac{1}{\log x}-\frac{1}{2}+i(t_1+t_2))|}{(1+|t_1|)(1+|t_2|)}dt_1\,dt_2$$
$$\ll yx^{\frac{3}{2}}L^2\int_{-2T}^{2T}\left|\zeta\left(\frac{1}{2}+\frac{1}{\log x}+iu\right)\right|\int_{-T}^{T}\frac{|\zeta(\frac{1}{2}+\frac{2}{\log x}+it)|}{(1+|t|)(1+|t-u|)}dt\,du.$$

Hence we use the estimate

$$\int_{-T}^{T}\frac{|\zeta(\frac{1}{2}+it)|^2}{(1+|t|)(1+|t-u|)}dt \ll \frac{|u|^{\frac{1}{3}}}{1+|u|}$$

(see p.161 in [5]) and the Cauchy–Schwarz inequality to get

$$J_{1,2} \ll yx^{\frac{3}{2}}L^3\int_{-2T}^{2T}\left|\zeta\left(\frac{1}{2}+\frac{1}{\log x}+iu\right)\right|\frac{|u|^{\frac{1}{3}}}{1+|u|}du$$
$$\ll yx^{\frac{3}{2}}T^{\frac{1}{3}}L^5. \tag{5.7}$$

It remains to evaluate the residues of the poles of the integrand when we move the line of integration to $\Gamma(\alpha_2, \frac{1}{2}+\frac{1}{\log x}, T)$. There exists a simple pole at $s_2 = 2 - s_1$ with residue

$$\frac{\zeta(s_1)\zeta(2-s_1)\zeta(3s_1)\zeta(4s_1)\zeta(5s_1)\zeta(6-3s_1)\zeta(8-4s_1)\zeta(10-5s_1)\kappa(s_1)\kappa(2-s_1)}{\zeta(2)\zeta(8s_1)\zeta(16-8s_1)s_1(2-s_1)}\,x^2$$
$$=: H_1(s_1)x^2,$$

and also a simple pole at $s_2 = 1$ with residue

$$\frac{\zeta(3)\zeta(4)\zeta(5)\kappa(1)}{\zeta(8)}\cdot\frac{\zeta^2(s_1)\zeta(3s_1)\zeta(4s_1)\zeta(5s_1)\kappa(s_1)}{\zeta(8s_1)\zeta(s_1+1)s_1}\,x^{s_1+1} =: H_2(s_1)x^{s_1+1}.$$

The contributions to $S_{2,1}^{(3)}(x,y)$ from these residues are

$$\frac{x^2y}{2\pi i}\int_{\alpha_1-iT}^{\alpha_1+iT}H_1(s_1)ds_1+\frac{xy}{2\pi i}\int_{\alpha_1-iT}^{\alpha_1+iT}H_2(s_1)x^{s_1}\,ds_1$$
$$:= I_1 + I_2,\ \text{say}.$$

For I_1, moving the line of integration to $\Gamma(\alpha_1, \frac{5}{4}, T)$, we have

$$I_1 = \frac{x^2 y}{2\pi i}\int_{\frac{5}{4}-i\infty}^{\frac{5}{4}+i\infty} H_1(s_1)ds_1 + O\left(x^2 y \int_T^\infty \left|H_1\left(\frac{5}{4}+it_1\right)\right| dt_1\right) + O\left(\frac{x^2 y L^2}{T^{\frac{23}{12}}}\right)$$
$$= \eta x^2 y + O\left(\frac{x^2 y L^2}{T^{\frac{11}{12}}}\right),$$

where the constant η is given by

$$\eta := \frac{1}{2\pi i}\int_{\frac{5}{4}-i\infty}^{\frac{5}{4}+i\infty} H_1(s_1)ds_1, \tag{5.8}$$

which is an absolutely convergent integral given by

$$\eta := \frac{1}{2\pi i} \int_{\frac{5}{4}-i\infty}^{\frac{5}{4}+i\infty} \frac{\zeta(s)\zeta(2-s)\zeta(3s)\zeta(4s)\zeta(5s)\zeta(6-3s)\zeta(8-4s)\zeta(10-5s)\kappa(s)\kappa(2-s)}{\zeta(2)\zeta(8s)\zeta(16-8s)s(2-s)} ds. \tag{5.9}$$

Now, we use the inequalities $|\zeta(s)| \le \zeta(\sigma)$ and $\left|\frac{1}{\zeta(s)}\right| \le \frac{\zeta(\sigma)}{\zeta(2\sigma)}$ for $\sigma > 1$ (see (8.4.1), (8.7.1) in [13]) to obtain

$$|\eta| \le \frac{\zeta(3)\zeta(5)\zeta(6)\zeta(10)\zeta(\frac{5}{4})\zeta(\frac{9}{4})\zeta^2(\frac{15}{4})\zeta(\frac{25}{4})\kappa\left(\frac{3}{4}\right)\kappa\left(\frac{5}{4}\right)}{\pi\zeta(2)\zeta(12)\zeta(20)} \times \int_0^\infty \frac{\left|\zeta\left(\frac{3}{4}+it\right)\right|}{\sqrt{(\frac{9}{16}+t^2)(\frac{25}{16}+t^2)}} dt.$$

Here, the integral on the right-hand side of the above is a computable constant, and that is, strictly speaking, enough for the purpose of this paper.

For I_2, we move the line of integration to $\Gamma(\alpha_1, \frac{1}{2}+\frac{1}{\log x}, T)$. The integrals over the horizontal lines are

$$\ll \frac{xyL^3}{T}\int_{\frac{1}{2}+\frac{1}{\log x}}^{\alpha_1} T^{\frac{2}{3}(1-\sigma_1)} x^{\sigma_1} d\sigma_1 \ll \frac{x^{\frac{3}{2}} y L^3}{T}\left(x^{\frac{1}{2}} + T^{\frac{1}{3}}\right)$$

and the integral over the vertical line is

$$\ll xyL^3 \int_{-T}^{T} \frac{|\zeta(\frac{1}{2}+it_1)|^2}{1+|t_1|} x^{\frac{1}{2}} dt_1 \ll x^{\frac{3}{2}} y L^5$$

by using the estimate $\int_1^T |\zeta(\frac{1}{2}+it)|^2 dt \ll T\log T$ and integration by parts. Furthermore, when moving the path of integration there is a double pole at $s_1 = 1$. Hence, using Cauchy's theorem, we have

$$
\begin{aligned}
I_2 = &\frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta(2)\zeta^2(8)} x^2 y\log x + \frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta(2)\zeta^2(8)} \\
&\times \left(2\gamma - 1 + \frac{\kappa'(1)}{\kappa(1)} + 3\frac{\zeta'(3)}{\zeta(3)} + 4\frac{\zeta'(4)}{\zeta(4)} + 5\frac{\zeta'(5)}{\zeta(5)} - 8\frac{\zeta'(8)}{\zeta(8)} - \frac{\zeta'(2)}{\zeta(2)}\right) x^2 y \\
&+ O\left(\frac{x^{\frac{3}{2}} y L^3}{T}\left(x^{\frac{1}{2}} + T^{\frac{1}{3}}\right)\right) + O(x^{\frac{3}{2}} y L^5),
\end{aligned}
$$

where γ is the Euler constant. Combining these results we have

$$
\begin{aligned}
S_{2,1}^{(3)}(x,y) = &\frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta(2)\zeta^2(8)} x^2 y\log x + \eta x^2 y \\
&+ \frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta(2)\zeta^2(8)} \left(2\gamma - 1 + \frac{\kappa'(1)}{\kappa(1)} + 3\frac{\zeta'(3)}{\zeta(3)} + 4\frac{\zeta'(4)}{\zeta(4)}\right. \\
&\left. + 5\frac{\zeta'(5)}{\zeta(5)} - 8\frac{\zeta'(8)}{\zeta(8)} - \frac{\zeta'(2)}{\zeta(2)}\right) x^2 y + O\left(x^2 y L^5 \cdot x^{-\frac{1}{3}}\right).
\end{aligned}
\tag{5.10}
$$

Here, we substituted $T = x$ into the error term of $S_{2,1}(x,y)$.

5.2 Estimation of $S_{2,4}^{(3)}(x, y)$

Explicitly we have

$$
\begin{aligned}
S_{2,4}^{(3)}(x,y) = &\frac{y^3}{(2\pi i)^2}\int_{\alpha_1 - iT}^{\alpha_1 + iT}\int_{\alpha_2 - iT}^{\alpha_2 + iT} \frac{\zeta(3-s_1-s_2)\zeta(2-s_1)\zeta(2-s_2)}{\zeta(4-s_1-s_2)(3-s_1-s_2)s_1 s_2} \\
&\times \frac{\zeta(3s_1)\zeta(4s_1)\zeta(5s_1)\kappa(s_1)\zeta(3s_2)\zeta(4s_2)\zeta(5s_2)\kappa(s_2)}{\zeta(8s_1)\zeta(8s_2)} \left(\frac{x}{y}\right)^{s_1+s_2} ds_2\, ds_1.
\end{aligned}
$$

For this purpose, we move the line of integral with respect to s_2 to contour $\Gamma(\alpha_2, \beta, T)$, where $\beta = \frac{5}{2} - \alpha_1 = \frac{3}{2} - \frac{1}{\log x}$. We denote the integrals over the horizontal line segments by $J_{4,1}$ and $J_{4,3}$, and the integral over the vertical line segment by $J_{4,2}$, respectively. There are no poles when we deform the path of integral over s_2. The contribution from the horizontal lines are

$$J_{4,1},\, J_{4,3} \ll xy^2 \left(\frac{x}{y}\right)^{\frac{1}{\log x}} \int_{-T}^{T} \frac{\left|\zeta\left(1-\frac{1}{\log x}-it_1\right)\right|}{1+|t_1|} dt_1$$
$$\times \int_{\alpha_2}^{\beta} \frac{\left|\zeta\left(2-\frac{1}{\log x}-\sigma_2-i(t_1+T)\right)\zeta\left(2-\sigma_2-iT\right)\right|}{(1+|t_1+T|)T} \left(\frac{x}{y}\right)^{\sigma_2} d\sigma_2.$$

The inner integral is estimated as

$$\ll \frac{L^3}{T(1+|t_1+T|)} \left(\frac{x}{y}\right) \left(1+T^{\frac{1}{3}}\left(\frac{x}{y}\right)^{\frac{1}{2}}\right),$$

where we have used the assumption $y \ll x^M$. Hence, we have

$$J_{4,1},\, J_{4,3} \ll \frac{x^2yL^3}{T}\left(1+T^{\frac{1}{3}}\left(\frac{x}{y}\right)^{\frac{1}{2}}\right) \int_{-T}^{T} \frac{\left|\zeta\left(1-\frac{1}{\log x}-it_1\right)\right|}{(1+|t_1|)(1+|t_1|+T)} dt_1$$
$$\ll \frac{x^2yL^4}{T^2}\left(1+T^{\frac{1}{3}}\left(\frac{x}{y}\right)^{\frac{1}{2}}\right).$$

For the integral on the vertical line we find that

$$J_{4,2} \ll y^3 \int_{-T}^{T}\int_{-T}^{T} \frac{\left|\zeta(\frac{1}{2}-i(t_1+t_2))\zeta(1-\frac{1}{\log x}-it_1)\zeta(\frac{1}{2}+\frac{1}{\log x}-it_2)\right|}{(1+|t_1+t_2|)(1+|t_1|)(1+|t_2|)} \left(\frac{x}{y}\right)^{\frac{5}{2}} dt_1\, dt_2$$
$$\ll y^3 \left(\frac{x}{y}\right)^{\frac{5}{2}} \int_{-2T}^{2T} \frac{\left|\zeta\left(\frac{1}{2}-iu\right)\right|}{1+|u|} \int_{-T}^{T} \frac{\left|\zeta\left(\frac{1}{2}+\frac{1}{\log x}-it_2\right)\right|}{(1+|t_2|)(1+|u-t_2|)} dt_2\, du$$
$$\ll x^2y\left(\frac{x}{y}\right)^{\frac{1}{2}} L^3.$$

Hence, we take $T = x$ to get

$$S_{2,4}^{(3)}(x, y) \ll x^2y\left(\frac{x}{y}\right)^{\frac{1}{2}} L^3. \tag{5.11}$$

5.3 Estimation of $S_{2,3}^{(3)}(x, y)$

It is given explicitly by

$$S_{2,3}^{(3)}(x,y) = \frac{y^2}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-iT}^{\alpha_2+iT}\frac{\zeta(2-s_2)\zeta(1+s_1-s_2)\zeta(s_1)}{\zeta(2+s_1-s_2)(2-s_2)}$$
$$\times\frac{\zeta(3s_1)\zeta(4s_1)\zeta(5s_1)\kappa(s_1)\zeta(3s_2)\zeta(4s_2)\zeta(5s_2)\kappa(s_2)}{\zeta(8s_1)\zeta(8s_2)}\frac{x^{s_1+s_2}y^{-s_2}}{s_1s_2}ds_2\,ds_1.$$

We move the path of integration with respect to s_2 to $\Gamma(\alpha_2, \frac{3}{2}, T)$. We denote the integrals over the horizontal line segments by $J_{3,1}$ and $J_{3,3}$, and the integral over the vertical line segment by $J_{3,2}$, respectively. Note that there exist no poles with this deformation. The contribution from the horizontal lines are

$$\begin{aligned}J_{3,1},\, J_{3,3} &\ll \frac{y^2xL}{T^2}\int_{-T}^{T}\frac{|\zeta(\alpha_1+it_1)|}{1+|t_1|}\\ &\quad\times\int_{\alpha_2}^{\frac{3}{2}}|\zeta(2-\sigma_2-iT)\zeta(1+\alpha_1-\sigma_2+i(t_1-T))|\left(\frac{x}{y}\right)^{\sigma_2}d\sigma_2\,dt_1\\ &\ll \frac{y^2xL^2}{T^2}\int_{-T}^{T}\frac{|\zeta(\alpha_1+it_1)|}{1+|t_1|}\\ &\quad\times\int_{\alpha_2}^{\frac{3}{2}}T^{\frac{1}{3}(-1+\sigma_2)}(1+|t_1-T|)^{\frac{1}{3}(-1+\sigma_2)}\left(\frac{x}{y}\right)^{\sigma_2}d\sigma_2\,dt_1\\ &\ll yx^2L^3\left(T^{-2}+T^{-\frac{5}{3}}\left(\frac{x}{y}\right)^{\frac{1}{2}}\right).\end{aligned}$$

On the other hand, the contribution from the vertical lines is

$$\begin{aligned}J_{3,2} &\ll y^2x\int_{-T}^{T}\frac{|\zeta(\alpha_1+it_1)|}{1+|t_1|}\\ &\quad\times\int_{-T}^{T}\frac{|\zeta(\frac{1}{2}-it_2)\zeta(\frac{1}{2}+\frac{1}{\log x}+i(t_1-t_2))|}{(1+|t_2|)^2}\left(\frac{x}{y}\right)^{\frac{3}{2}}dt_2\,dt_1\\ &\ll y^2x\left(\frac{x}{y}\right)^{\frac{3}{2}}L.\end{aligned}$$

Hence, we take $T = x$ into the above to obtain

$$S_{2,3}^{(3)}(x,y) \ll x^2yL\left(\frac{x}{y}\right)^{\frac{1}{2}}. \tag{5.12}$$

5.4 Evaluation of $S_{2,2}^{(3)}(x, y)$

The explicit form of $S_{2,2}^{(3)}(x, y)$ is given by

$$S_{2,2}^{(3)}(x,y)=\frac{y^2}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-iT}^{\alpha_2+iT}\frac{\zeta(2-s_1)\zeta(1-s_1+s_2)\zeta(s_2)}{\zeta(2-s_1+s_2)(2-s_1)}$$
$$\times\frac{\zeta(3s_1)\zeta(4s_1)\zeta(5s_1)\kappa(s_1)\zeta(3s_2)\zeta(4s_2)\zeta(5s_2)\kappa(s_2)}{\zeta(8s_1)\zeta(8s_2)}\frac{x^{s_1+s_2}y^{-s_1}}{s_1s_2}ds_2\,ds_1. \tag{5.13}$$

This time we firstly move the line of the integration over s_1 to $\Gamma(\alpha_1,\frac{3}{2},T)$. The estimates over the horizontal lines and the vertical line are the same as that of $S_{2,3}^{(3)}(x,y)$, but there is a simple pole at $s_1=s_2$ inside this contour. The residue of the integrand of (5.13) at this pole is

$$-\frac{\zeta(2-s_2)\zeta(s_2)\zeta^2(3s_2)\zeta^2(4s_2)\zeta^2(5s_2)\kappa^2(s_2)}{\zeta(2)\zeta^2(8s_2)(2-s_2)s_2^2}x^{2s_2}y^{-s_2}.$$

Hence, we have

$$S_{2,2}^{(3)}(x,y)=\frac{x^2y}{2\pi i}\int_{\alpha_2-iT}^{\alpha_2+iT}\frac{\zeta(2-s_2)\zeta(s_2)\zeta^2(3s_2)\zeta^2(4s_2)\zeta^2(5s_2)\kappa^2(s_2)}{\zeta(2)\zeta^2(8s_2)(2-s_2)s_2^2}\left(\frac{y}{x^2}\right)^{1-s_2}ds_2$$
$$+O\left(x^2yL\left\{x^{-1}+\left(\frac{x}{y}\right)^{\frac{1}{2}}\right\}\right)$$

by taking $T=x$. We move the line of integration to $\Gamma(\alpha_2,\frac{1}{2}+\frac{2}{\log x},T)$. By the same method as before, the integrals over the horizontal lines are estimated as

$$\ll\frac{x^2y}{T^3}\left(L^4\left(\frac{y}{x^2}\right)^{-\frac{2}{\log x}}+L^2T^{\frac{1}{2}}\left(\frac{y}{x^2}\right)^{\frac{1}{2}}\right)\ll\frac{x^2yL^4}{T^3}\left(1+T^{\frac{1}{2}}\left(\frac{y}{x^2}\right)^{\frac{1}{2}}\right)$$

and the vertical lines are estimated as

$$\ll x^2y\left(\frac{y}{x^2}\right)^{\frac{1}{2}}L^2.$$

Furthermore, there is a contribution from the pole $s_2=1$ of order 2, hence $S_{2,2}^{(3)}(x,y)$ has the form

$$S_{2,2}^{(3)}(x,y)=\frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta(2)\zeta^2(8)}x^2y\log\frac{x^2}{y}+\frac{\zeta^2(3)\zeta^2(4)\zeta^2(5)\kappa^2(1)}{\zeta(2)\zeta^2(8)}$$
$$\times\left(6\frac{\zeta'(3)}{\zeta(3)}+8\frac{\zeta'(4)}{\zeta(4)}+10\frac{\zeta'(5)}{\zeta(5)}-16\frac{\zeta'(8)}{\zeta(8)}-1+2\frac{\kappa'(1)}{\kappa(1)}\right)$$
$$+O\left(x^2y\left(L^5x^{-\frac{1}{3}}+L^6y^{-\frac{1}{2}}+L^3\left(\frac{x}{y}\right)^{\frac{1}{2}}+L^2\left(\frac{y}{x^2}\right)^{\frac{1}{2}}\right)\right) \tag{5.14}$$

by taking $T=x$.

5.5 Asymptotic formula of (1.12)

Now, we substitute (5.5), (5.10), (5.11), (5.12) and (5.14) into (5.4) to obtain the assertion of Theorem 3. □

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