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Utility of integral representations for basic hypergeometric functions and orthogonal polynomials

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Dedicated to the life and mathematics of Dick Askey, 1933–2019.

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Abstract
We describe the utility of integral representations for sums of basic hypergeometric functions. In particular we use these to derive an infinite sequence of transformations for symmetrizations over certain variables which the functions possess. These integral representations were studied by Bailey, Slater, Askey, Roy, Gasper and Rahman and were also used to facilitate the computation of certain outstanding problems in the theory of basic hypergeometric orthogonal polynomials in the q-Askey scheme. We also generalize and give consequences and transformation formulas for some fundamental integrals connected to nonterminating basic hypergeometric series and the Askey–Wilson polynomials. We express a certain integral of a ratio of infinite q-shifted factorials as a symmetric sum of two basic hypergeometric series with argument q. The result is then expressed as a q-integral. Examples of integral representations applied to the derivation of generating functions for the Askey–Wilson polynomials are given and as well the computation of a missing generating function for the continuous dual q-Hahn polynomials.

Keywords Basic hypergeometric functions · Transformations · Integral representations · Basic hypergeometric orthogonal polynomials · Generating functions

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1 Introduction

The main aim of this paper is to demonstrate the utility of revisiting the application of integral representations for problems in basic hypergeometric functions and basic hypergeometric orthogonal polynomials in the q-Askey scheme. We accomplish this by proving a collection of identities which arise naturally through the utilization of this powerful method. For a detailed history of the subject of integral representations for basic hypergeometric functions, see [6] and [7, Chapter 4].

All of the results presented below are new but some rely heavily on identities which have been proven elsewhere in the literature. For instance Theorem 2.1 is essentially a restatement of [7, (4.10.5–6)], which in turn is closely connected to [20, (5.2.4) and (5.2.20)]. However, our introduction of the useful t parameter in Theorem 2.1 (see for instance Lemma 2.16, Corollary 2.17 and Corollary 2.19) is new. Also, our utilization of the powerful van de Bult–Rains notation for basic hypergeometric series with vanishing numerator or denominator parameters (see (21), (22) below) in Theorem 2.1 allows for a clear elucidation of structure which, in our opinion, is not as such in previous incarnations of this or related results in the literature. Furthermore, even though we believe that Theorem 2.4 is new in its full generality, the ideas which went into it have been used many times in the literature, such as in derivations of the Askey–Wilson integral (27), the Nassrallah–Rahman integral (28), the Rahman integral (29), the Askey–Roy integral (31) and the Gasper integral (32), as well as in fundamental results such as [7, Exercises 4.4, 4.5] which reappear in (46) and (68).

1.1 Preliminaries

We adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$, and we use the sets $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ which represent the integers, real numbers and complex numbers respectively, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and $\mathbb{C}^\dagger := \{z \in \mathbb{C}^* : |z| < 1\}$. We also adopt the following notation and conventions. Given a set $\mathbf{a} := \{a_1, \ldots, a_A\}$, for $A \in \mathbb{N}$, define $\mathbf{a}_{[k]} := \mathbf{a} \setminus \{a_k\}$, $1 \le k \le A$, $b\mathbf{a} := \{b\,a_1, b\,a_2, \ldots, b\,a_A\}$, $\mathbf{a} + b := \{a_1 + b, a_2 + b, \ldots, a_A + b\}$, where $b, a_1, \ldots, a_A \in \mathbb{C}$.

We assume that the empty sum vanishes and the empty product is unity. We will also adopt the following symmetric sum notation.

Definition 1.1 For some function $f(a_1, \ldots, a_n; \mathbf{b})$, where $\mathbf{b}$ is some set of parameters. Then

$$\mathop{\Large\mathrm{I}}^{a_1;a_2,\ldots,a_n} f(a_1, \ldots, a_n; \mathbf{b}) := f(a_1, \ldots, a_n; \mathbf{b}) + \mathrm{idem}(a_1; a_2, \ldots, a_n), \tag{1}$$

where "idem$(a_1; a_2, \ldots, a_n)$" after an expression stands for the sum of the $n-1$ expressions obtained from the preceding expression by interchanging a_1 with each a_k, $k = 2, 3, \ldots, n$.

Definition 1.2 We adopt the following conventions for succinctly writing elements of sets. To indicate sequential positive and negative elements, we write

$$\pm a := \{a, -a\}.$$

We also adopt an analogous notation

$$\mathrm{e}^{\pm i\theta} := \{\mathrm{e}^{i\theta}, \mathrm{e}^{-i\theta}\}.$$

In the same vein, consider the numbers $f_s \in \mathbb{C}$ with $s \in \mathcal{S} \subset \mathbb{N}$, with $\mathcal{S}$ finite. Then, the notation $\{f_s\}$ represents the set of all complex numbers f_s such that $s \in \mathcal{S}$. Furthermore, consider some $p \in \mathcal{S}$, then the notation $\{f_s\}_{s\neq p}$ represents the set of all complex numbers f_s such that $s \in \mathcal{S}\backslash\{p\}$.

Consider $q \in \mathbb{C}^\dagger$, $n \in \mathbb{N}_0$. Define the sets $\Omega_q^n := \{q^{-k} : k \in \mathbb{N}_0,\ 0 \leq k \leq n-1\}$, $\Omega_q := \Omega_q^\infty = \{q^{-k} : k \in \mathbb{N}_0\}$. In order to obtain our derived identities, we rely on properties of the q-shifted factorial $(a;q)_n$. It has been pointed out by the referee that Askey, partly for historical reasons and partly because he preferred descriptive names to honorifics, referred to $(a;q)_n$ as a q-shifted factorial rather than the other common nomenclature: q-Pochhammer symbol. For any $n \in \mathbb{N}_0$, $a, b, q \in \mathbb{C}$, the q-shifted factorial is defined as

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}). \tag{2}$$

One may also define

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \tag{3}$$

$$\vartheta(x;q) := (x, q/x; q)_\infty, \tag{4}$$

where $|q| < 1$, $x \neq 0$, and (4) defines the modified theta function of nome q [7, (11.2.1)]. Note that $\vartheta(q^n;q) = 0$ for all $n \in \mathbb{Z}$. Furthermore one has the following identities:

$$(a^2;q)_\infty = (\pm a, \pm q^{\frac{1}{2}}a; q)_\infty, \tag{5}$$

$$\frac{(a, q/a; q)_\infty}{(qa, 1/a; q)_\infty} = \frac{\vartheta(a;q)}{\vartheta(qa;q)} = -a, \tag{6}$$

where $a \neq 0$. Moreover, define

$$(a;q)_b := \frac{(a;q)_\infty}{(aq^b;q)_\infty}, \tag{7}$$

where $aq^b \notin \Omega_q$. We will also use the common notational product convention

$$(a_1, \ldots, a_k; q)_b := (a_1; q)_b \cdots (a_k; q)_b.$$

The following properties for the q-shifted factorial can be found in Koekoek et al. [13, (1.8.7), (1.8.10–11), (1.8.14), (1.8.19), (1.8.21–22)], namely for appropriate values of $q, a \in \mathbb{C}^*$ and $n, k \in \mathbb{N}_0$:

$$(a; q^{-1})_n = q^{-\binom{n}{2}}(-a)^n(a^{-1}; q)_n, \tag{8}$$

$$(a; q)_{n+k} = (a; q)_k(aq^k; q)_n = (a; q)_n(aq^n; q)_k, \tag{9}$$

$$(a; q)_n = q^{\binom{n}{2}}(-a)^n(q^{1-n}/a; q)_n, \tag{10}$$

$$(aq^{-n}; q)_k = q^{-nk}\frac{(q/a; q)_n}{(q^{1-k}/a; q)_n}(a; q)_k, \tag{11}$$

$$(a^2; q^2)_n = (\pm a; q)_n, \tag{12}$$

$$(a; q)_{2n} = (a, aq; q^2)_n = (\pm\sqrt{a}, \pm\sqrt{qa}; q)_n. \tag{13}$$

Observe that, by using (9) and (13), one obtains

$$(aq^n; q)_n = \frac{(\pm\sqrt{a}, \pm\sqrt{qa}; q)_n}{(a; q)_n}, \quad a \notin \Omega_q^n. \tag{14}$$

Define the Jackson q-integral as in [7, (1.11.2)]

$$\int_a^b f(u; q)\mathrm{d}_q u = (1-q)b\sum_{n=0}^{\infty} q^n f(q^n b; q) - (1-q)a\sum_{n=0}^{\infty} q^n f(q^n a; q) \tag{15}$$

$$= \frac{(1-q)ab}{a-b}\mathop{\Big[\!\!\Big[}^{a;b}\left(1-\frac{a}{b}\right)\sum_{n=0}^{\infty} q^n f(q^n a; q)$$

$$= \frac{b}{a-b}\mathop{\Big[\!\!\Big[}^{a;b}\left(1-\frac{a}{b}\right)\int_0^a f(u; q)\mathrm{d}_q u, \tag{16}$$

where we have utilized (6) to write the q-integral as a symmetric sum.

The nonterminating basic hypergeometric series, which we will often use, is defined for $s \in \mathbb{N}_0$, $r \in \mathbb{N}_0 \cup \{-1\}$, $b_j \notin \Omega_q$, $j = 1, \ldots, s$, as [13, (1.10.1)]

$${}_{r+1}\phi_s\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_s \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_k}{(q, b_1, \ldots, b_s; q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{s-r} z^k. \tag{17}$$

For $s > r$, ${}_{r+1}\phi_s$ is an entire function of z, for $s = r$ then ${}_{r+1}\phi_s$ is convergent for $|z| < 1$, and for $s < r$ the series is divergent.

Remark 1.3 Sometimes we also use generalized hypergeometric series ${}_{r+1}F_s$ which is the $q \uparrow 1$ limit of basic hypergeometric series (see for instance [13, p. 15]). For their properties, see [15, Chapter 16].

Note that we refer to a basic hypergeometric series as ℓ-balanced if $q^\ell a_1 \cdots a_{r+1} = b_1 \cdots b_s$, and balanced (Saalschützian) if $\ell = 1$ (see [1, Definition 3.3.1], [7, p. 5]). The referee has pointed out that for the very important $\ell = 1$ case, the term *balanced* was introduced by Askey, whereas the earlier term Saalschützian is due to Whipple and was used by Bailey, but lost much of its force after Askey's discovery in 1975 that Pfaff had Saalschütz's identity 93 years earlier in 1797. A basic hypergeometric series ${}_{r+1}\phi_r$ is well-poised if the parameters satisfy the relations

$$qa_1 = b_1a_2 = b_2a_3 = \cdots = b_ra_{r+1}.$$

It is very-well poised if in addition, $\{a_2, a_3\} = \pm q\sqrt{a_1}$. Define the very-well poised basic hypergeometric series ${}_{r+1}W_r$ [7, (2.1.11)]

$${}_{r+1}W_r(b; a_4, \ldots, a_{r+1}; q, z) := {}_{r+1}\phi_r\left(\begin{matrix}\pm q\sqrt{b}, b, a_4, \ldots, a_{r+1}\\ \pm\sqrt{b}, \frac{qb}{a_4}, \ldots, \frac{qb}{a_{r+1}}\end{matrix}; q, z\right), \quad (18)$$

where $\sqrt{b}, \frac{qb}{a_4}, \ldots, \frac{qb}{a_{r+1}} \notin \Omega_q$. When the very-well poised basic hypergeometric series is terminating, one has

$${}_{r+1}W_r\left(b; q^{-n}, a_5, \ldots, a_{r+1}; q, z\right) = {}_{r+1}\phi_r\left(\begin{matrix}q^{-n}, \pm q\sqrt{b}, b, a_5, \ldots, a_{r+1}\\ \pm\sqrt{b}, q^{n+1}b, \frac{qb}{a_5}, \ldots, \frac{qb}{a_{r+1}}\end{matrix}; q, z\right), \quad (19)$$

where $\sqrt{b}, \frac{qb}{a_5}, \ldots, \frac{qb}{a_{r+1}} \notin \Omega_q^n \cup \{0\}$. The Askey–Wilson polynomials are intimately connected with the terminating very-well poised ${}_8W_7$, which is given by

$${}_8W_7(b; q^{-n}, c, d, e, f; q, z) = {}_8\phi_7\left(\begin{matrix}q^{-n}, \pm q\sqrt{b}, b, c, d, e, f\\ \pm\sqrt{b}, q^{n+1}b, \frac{qb}{c}, \frac{qb}{d}, \frac{qb}{e}, \frac{qb}{f}\end{matrix}; q, z\right), \quad (20)$$

where $\sqrt{b}, \frac{qb}{c}, \frac{qb}{d}, \frac{qb}{e}, \frac{qb}{f} \notin \Omega_q^n \cup \{0\}$.

In the sequel, we will use the following notation ${}_{r+1}\phi_s^m$, $m \in \mathbb{Z}$ (originally due to van de Bult & Rains [21, p. 4]), for basic hypergeometric series with zero parameter entries. Consider $p \in \mathbb{N}_0$. Then define

$$
{}_{r+1}\phi_s^{-p}\left(\begin{matrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_s \end{matrix};q,z\right) := {}_{r+p+1}\phi_s\left(\begin{matrix} a_1,a_2,\ldots,a_{r+1},\overbrace{0,\ldots,0}^{p} \\ b_1,b_2,\ldots,b_s \end{matrix};q,z\right), \tag{21}
$$

$$
{}_{r+1}\phi_s^{p}\left(\begin{matrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_s \end{matrix};q,z\right) := {}_{r+1}\phi_{s+p}\left(\begin{matrix} a_1,a_2,\ldots,a_{r+1} \\ b_1,b_2,\ldots,b_s,\underbrace{0,\ldots,0}_{p} \end{matrix};q,z\right), \tag{22}
$$

where $b_1,\ldots,b_s \notin \Omega_q \cup \{0\}$, and ${}_{r+1}\phi_s^0 := {}_{r+1}\phi_s$. The nonterminating basic hypergeometric series ${}_{r+1}\phi_s^m(\mathbf{a};\mathbf{b};q,z)$, $\mathbf{a} := \{a_1,\ldots,a_{r+1}\}$, $\mathbf{b} := \{b_1,\ldots,b_s\}$, is well-defined for $s-r+m \geq 0$. In particular ${}_{r+1}\phi_s^m$ is an entire function of z for $s-r+m > 0$, convergent for $|z| < 1$ for $s-r+m = 0$ and divergent if $s-r+m < 0$. Note that we will move interchangeably between the van de Bult and Rains notation and the alternative notation with vanishing numerator and denominator parameters which are used on the right-hand sides of (21) and (22).

1.2 The Askey–Wilson polynomials and related fundamental integrals

Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$. For the Askey–Wilson polynomials $p_n(x;\mathbf{a}|q)$, which are symmetric in four free parameters, we will switch interchangeably with the notation $\mathbf{a} := \{a_1,a_2,a_3,a_4\}$ and $\mathfrak{a} := \{a,b,c,d\}$, $\mathbf{a} = \mathfrak{a}$, $a,b,c,d \in \mathbb{C}^*$, and similarly for the continuous dual q-Hahn polynomials which are symmetric in three free parameters. Define $a_{12} := a_1a_2$, $a_{13} := a_1a_3$, $a_{23} := a_2a_3$, $a_{123} := a_1a_2a_3$, $a_{1234} := a_1a_2a_3a_4$, etc.

The Askey–Wilson polynomials can be defined in terms of the terminating basic hypergeometric series [13, (14.1.1)]

$$
p_n(x;\mathfrak{a}|q) := a^{-n}(ab,ac,ad;q)_n\, {}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}abcd,a\mathrm{e}^{\pm i\theta} \\ ab,ac,ad \end{matrix};q,q\right), \tag{23}
$$

where $x = \cos\theta$. The Askey–Wilson polynomials are orthogonal on $(-1,1)$ with respect to the weight function

$$
w_q(\cos\theta;\mathbf{a}) := \frac{(\mathrm{e}^{\pm 2i\theta};q)_\infty}{(\mathbf{a}\mathrm{e}^{\pm i\theta};q)_\infty} = \frac{(\pm\mathrm{e}^{\pm i\theta},\pm q^{\frac12}\mathrm{e}^{\pm i\theta};q)_\infty}{(\mathbf{a}\mathrm{e}^{\pm i\theta};q)_\infty}, \tag{24}
$$

where the second equality is due to (5). The orthogonality relation for Askey–Wilson polynomials is [13, (14.1.2)]

$$
\int_0^\pi p_m(x;\mathbf{a}|q)p_n(x;\mathbf{a}|q)w_q(x;\mathbf{a})\,\mathrm{d}\theta = h_n(\mathbf{a};q)\delta_{m,n}, \tag{25}
$$

where

$$h_n(\mathbf{a};q) := \frac{2\pi(q^{n-1}a_{1234};q)_n(q^{2n}a_{1234};q)_\infty}{(q^{n+1},q^na_{12},q^na_{13},q^na_{14},q^na_{23},q^na_{24},q^na_{34};q)_\infty}. \tag{26}$$

We will also rely on several important generalized q-beta integrals. The first is the Askey–Wilson integral [7, (6.1.1)] (the integral over the full domain of the Askey–Wilson weight (24))

$$\int_0^\pi \frac{(\mathrm{e}^{\pm 2i\theta};q)_\infty}{(\mathbf{a}\mathrm{e}^{\pm i\theta};q)_\infty}\,\mathrm{d}\theta := \frac{2\pi(a_{1234};q)_\infty}{(q,a_{12},\ldots,a_{34};q)_\infty}, \tag{27}$$

where $\max(|a_1|,\ldots,|a_4|) < 1$. Note the Askey–Wilson norm for $n = 0$ is equal to the evaluation of the Askey–Wilson integral (27). The second is the Nassrallah–Rahman integral (in symmetrical form) [7, (6.3.9)] which generalizes the Askey–Wilson integral, namely [17, (3.1)]. Let $\mathbf{a} := \{a_1,a_2,a_3,a_4,a_5\}$. Then

$$\int_0^\pi \frac{(\mathrm{e}^{\pm 2i\theta},\lambda\mathrm{e}^{\pm i\theta};q)_\infty}{(\mathbf{a}\mathrm{e}^{\pm i\theta};q)_\infty}\,\mathrm{d}\theta = \frac{2\pi(\lambda\mathbf{a},\lambda^{-1}a_{12345};q)_\infty}{(q,a_{12},\ldots,a_{45},\lambda^2;q)_\infty}\,{}_8W_7\left(\frac{\lambda^2}{q};\frac{\lambda}{\mathbf{a}};q,\frac{a_{12345}}{\lambda}\right), \tag{28}$$

where $\max(|a_1|, ..., |a_5|) < 1$ and $|a_{12345}| < |\lambda|$. The Nassrallah–Rahman integral (28) becomes the Askey–Wilson integral (27) for $\lambda = a_5$.

The third is the Rahman integral [7, Exercise 6.7] which generalizes the Nassrallah–Rahman integral. Let $\mathbf{a} := \{a_1,a_2,a_3,a_4,a_5,a_6\}$. Then

$$\begin{aligned}\int_0^\pi \frac{(\mathrm{e}^{\pm 2i\theta},\lambda\mathrm{e}^{\pm i\theta},\mu\mathrm{e}^{\pm i\theta};q)_\infty}{(\mathbf{a}\mathrm{e}^{\pm i\theta};q)_\infty}\,\mathrm{d}\theta &= \frac{2\pi}{(q,a_{12},\ldots,a_{56};q)_\infty}\\ &\quad\times\left(\frac{(\lambda\mathbf{a},\frac{\mu}{\mathbf{a}};q)_\infty}{(\lambda^2,\mu/\lambda;q)_\infty}\,{}_{10}W_9\left(\frac{\lambda^2}{q};\frac{\lambda\mu}{q},\frac{\lambda}{\mathbf{a}};q,q\right)\right.\\ &\quad\left.+\frac{(\mu\mathbf{a},\frac{\lambda}{\mathbf{a}};q)_\infty}{(\mu^2,\lambda/\mu;q)_\infty}\,{}_{10}W_9\left(\frac{\mu^2}{q};\frac{\lambda\mu}{q},\frac{\mu}{\mathbf{a}};q,q\right)\right)\end{aligned} \tag{29}$$

$$= \frac{2\pi}{(q,a_{12},\ldots,a_{56};q)_\infty}\overset{\lambda;\mu}{\Big[\!\!\Big]}\frac{(\lambda\mathbf{a},\frac{\mu}{\mathbf{a}};q)_\infty}{(\lambda^2,\mu/\lambda;q)_\infty}\,{}_{10}W_9\left(\frac{\lambda^2}{q};\frac{\lambda\mu}{q},\frac{\lambda}{\mathbf{a}};q,q\right), \tag{30}$$

where $\lambda\mu = a_{123456}$ and $\max(|a_1|,\ldots,|a_6|) < 1$. Given that $\lambda \neq 0$, then if $\mu = a_6 \to 0$, then the Rahman integral (29) becomes the Nassrallah–Rahman integral (28). See [18] for some other interesting limits of the Rahman integral (29).

Some other important integrals related to basic hypergeometric functions are the q-beta integrals. The first one we mention is due to Askey–Roy [2, (2.8)]

$$\int_{-\pi}^{\pi} \frac{\left((fc, \frac{q}{f}d)\frac{\sigma}{z}, (\frac{f}{d}, \frac{q}{fc})\frac{z}{\sigma}; q\right)_\infty}{\left((c,d)\frac{\sigma}{z}, (a,b)\frac{z}{\sigma}; q\right)_\infty} \mathrm{d}\psi = 2\pi \frac{\vartheta(f, f\frac{c}{d}; q)(abcd; q)_\infty}{(q, ac, ad, bc, bd; q)_\infty}, \tag{31}$$

where $z = \mathrm{e}^{i\psi}$, $\max(|q|, |a/\sigma|, |b/\sigma|, |\sigma c|, |\sigma d|) < 1$ and $cdf \neq 0$. The second important q-beta integral is due to Gasper [6, (1.8)], namely

$$\int_{-\pi}^{\pi} \frac{((fc, \frac{q}{f}d)\frac{\sigma}{z}, (\frac{f}{d}, \frac{q}{fc}, abcde)\frac{z}{\sigma}; q)_\infty}{((c,d)\frac{\sigma}{z}, (a,b,e)\frac{z}{\sigma}; q)_\infty} \mathrm{d}\psi$$
$$= 2\pi \frac{\vartheta(f, f\frac{c}{d}; q)(abcd, bcde, acde; q)_\infty}{(q, ac, ad, bc, bd, ce, de; q)_\infty}, \tag{32}$$

where $z = \mathrm{e}^{i\psi}$, $\max(|q|, |a/\sigma|, |b/\sigma|, |\sigma c|, |\sigma d|, |e/\sigma|) < 1$ and $cdf \neq 0$. This integral extends the Askey–Roy integral (31) and reduces to it when e is set to 0. Note that (32) reduces to an expression equivalent to [6, (1.8)] by taking $\sigma \to 1$ and $f \mapsto f/c$.

Remark 1.4 Note that it is the special choice of numerator parameter behavior in the Askey–Roy and Gasper integrals which allows one to obtain these closed-form infinite product representations. We will return to this in Theorem 2.4.

2 Integral representations for basic hypergeometric functions

Here we present a result which follows by contour integration of products and quotients of q-gamma functions multiplied by integer powers of a complex exponential. Much of the derivations presented here follow the pioneering work of Bailey [4, Chapter 8], his student Slater [20, Chapters 5 and 7] and especially from such works of Askey and Roy [2], Nassrallah and Rahman [14], Rahman [16], Gasper [6], and Gasper and Rahman who carefully reviewed early preliminary results as well as deriving fundamental extensions in [7, Chapters 4 and 6]. The following theorem is a straightforward generalization of Corollary 2.4 in [10], and essentially a restatement of [7, (4.10.5–6)] using the van de Bult–Rains notation (21), (22) for basic hypergeometric series with vanishing numerator or denominator parameters.

Theorem 2.1 *Let* $q \in \mathbb{C}^\dagger$, $m \in \mathbb{Z}$, $t \in \mathbb{C}^*$, $\sigma \in (0, \infty)$, $\mathbf{a} := \{a_1, \ldots, a_A\}$, $\mathbf{b} := \{b_1, \ldots, b_B\}$, $\mathbf{c} := \{c_1, \ldots, c_C\}$, $\mathbf{d} := \{d_1, \ldots, d_D\}$ *be sets of complex numbers with cardinality* $A, B, C, D \in \mathbb{N}_0$ *(not all zero) respectively with* $|c_k| < \sigma/|t|$, $|d_l| < 1/\sigma$, *for any* $a_i, b_j, c_k, d_l \in \mathbb{C}$ *elements of* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, *and* $z = \mathrm{e}^{i\psi}$. *Define*

$$G_{m,t} := G_{m,t}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}; \sigma, q) := \frac{(q;q)_\infty}{2\pi} \left(\frac{\sqrt{t}}{\sigma}\right)^m \int_{-\pi}^{\pi} \frac{(\mathbf{b}\frac{\sigma}{z}, t\mathbf{a}\frac{z}{\sigma}; q)_\infty}{(\mathbf{d}\frac{\sigma}{z}, t\mathbf{c}\frac{z}{\sigma}; q)_\infty} \mathrm{e}^{im\psi} \mathrm{d}\psi, \tag{33}$$

such that the integral exists. Then

$$G_{m,t}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}; \sigma, q) = G_{-m,t}(\mathbf{b}, \mathbf{a}, \mathbf{d}, \mathbf{c}; \sigma, q), \tag{34}$$

if $|c_k|, |d_l| < \min\{1/\sigma, \sigma/|t|\}$. *Furthermore, let* $td_lc_k \notin \Omega_q$. *If* $D \geq B$, $d_l/d_{l'} \notin \Omega_q$, $l \neq l'$, *then*

$$G_{m,t} = t^{\frac{m}{2}} \sum_{k=1}^{D} \frac{(td_k\mathbf{a}, \mathbf{b}/d_k; q)_\infty d_k^m}{(td_k\mathbf{c}, \mathbf{d}_{[k]}/d_k; q)_\infty} \\ \times {}_{B+C}\phi_{A+D-1}^{C-A} \left(\begin{matrix} td_k\mathbf{c}, qd_k/\mathbf{b} \\ td_k\mathbf{a}, qd_k/\mathbf{d}_{[k]} \end{matrix} ; q, q^m (qd_k)^{D-B} \frac{b_1 \cdots b_B}{d_1 \cdots d_D} \right), \quad (35)$$

and/or if $C \geq A$, $c_k/c_{k'} \notin \Omega_q$, $k \neq k'$, *then*

$$G_{m,t} = \frac{1}{t^{\frac{m}{2}}} \sum_{k=1}^{C} \frac{(tc_k\mathbf{b}, \mathbf{a}/c_k; q)_\infty c_k^{-m}}{(tc_k\mathbf{d}, \mathbf{c}_{[k]}/c_k; q)_\infty} \\ \times {}_{A+D}\phi_{B+C-1}^{D-B} \left(\begin{matrix} tc_k\mathbf{d}, qc_k/\mathbf{a} \\ tc_k\mathbf{b}, qc_k/\mathbf{c}_{[k]} \end{matrix} ; q, q^{-m} (qc_k)^{C-A} \frac{a_1 \cdots a_A}{c_1 \cdots c_C} \right), \quad (36)$$

where the nonterminating basic hypergeometric series in (35) (*resp.* (36)) *is entire if* $D > B$ (*resp.* $C > A$), *convergent for* $|q^m b_1 \cdots b_B| < |d_1 \cdots d_D|$ *if* $D = B$ *(resp.* $|q^{-m} a_1 \cdots a_A| < |c_1 \cdots c_C|$ *if* $C = A$*), and divergent otherwise.*

Proof We obtain the integral expression for $G_{m,t}$ (33) by starting with [7, (4.9.3)] and replacing the sets of parameters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ with $\mathbf{a}t/\sigma$, $\mathbf{b}\sigma$, $\mathbf{c}t/\sigma$, $\mathbf{d}\sigma$. The relation (34) follows by replacing ψ with $-\psi$ in (33). To produce (35) and (36), in Gasper & Rahman [7, (4.10.5–6)], make the above parameter replacements and use the van de Bult–Rains notation (21), (22). Note that (35), (36) reduce to [7, (4.10.5–6)] when $t = \sigma = 1$. As mentioned in [7, §4.9], these integrals were used in Slater [20, Chapter 5] with $m = 0, 1$. $\square$

Remark 2.2 Note that in the case where the arguments of the basic hypergeometric functions in (35), (36) are greater than unity, the integral representations for $G_{m,t}$, when convergent, may provide an analytic continuation for these basic hypergeometric functions. For the integrals in (33) with respect to the variable of integration ψ the line of integration from $-\pi$ to π in the ψ-plane would have to be replaced by a suitably deformed line in the ψ-plane separating the sequences of poles that have infinitely many poles in the upper half ψ-plane from those with infinitely many poles in the lower half ψ-plane.

Remark 2.3 Observe that in the case where (33) can be written as (35) or (36) (e.g., $m \in \mathbb{Z}$) then $G_{m,t}$ does not depend on σ.

2.1 Argument q applications of Theorem 2.1

Note that when identifying integral representations for basic hypergeometric series, Theorem 2.1 is extremely useful. However, in applications even though one may use it to identify the parameters of a symmetric sum of basic hypergeometric functions, the restriction on the argument is often problematic. On the other hand, with the special choice of parameters given in the following corollaries which leads to an argument q, the ability to tie parameters to specific basic hypergeometric functions is greatly enhanced. Here we present some generalized results which gives the symmetric sum of two terms each containing a basic hypergeometric function with argument q.

Theorem 2.4 *Let $q \in \mathbb{C}^\dagger$, $\mathbf{a} := \{a_1, \ldots, a_A\}$, $\mathbf{c} := \{c_1, \ldots, c_C\}$, be sets of complex numbers with cardinality $A, C \in \mathbb{N}_0$ (not both zero) respectively, $\mathbf{d} := \{d_1, d_2\}$, $c_k d_l \notin \Omega_q$, $z = \mathrm{e}^{i\psi}$, $\sigma \in (0, \infty)$, $d_1, d_2 \in \mathbb{C}^*$, such that $|c_k| < \sigma$, $|d_1|, |d_2| < 1/\sigma$, for any $c_k \in \mathbf{c}$. Define*

$$H(\mathbf{a}, \mathbf{c}, \mathbf{d}; q) := \mathop{\Big[\!\!\Big]}^{d_1;d_2} \frac{(d_1\mathbf{a}; q)_\infty}{\left(\frac{d_2}{d_1}, d_1\mathbf{c}; q\right)_\infty} \, {}_C\phi_{A+1}^{C-A-2}\left(\begin{matrix} d_1\mathbf{c} \\ d_1\mathbf{a}, qd_1/d_2 \end{matrix}; q, q\right) \tag{37}$$

$$= \frac{(d_1\mathbf{a}; q)_\infty}{\left(\frac{d_2}{d_1}, d_1\mathbf{c}; q\right)_\infty} \, {}_C\phi_{A+1}^{C-A-2}\left(\begin{matrix} d_1\mathbf{c} \\ d_1\mathbf{a}, qd_1/d_2 \end{matrix}; q, q\right) + \frac{(d_2\mathbf{a}; q)_\infty}{\left(\frac{d_1}{d_2}, d_2\mathbf{c}; q\right)_\infty} \, {}_C\phi_{A+1}^{C-A-2}\left(\begin{matrix} d_2\mathbf{c} \\ d_2\mathbf{a}, qd_2/d_1 \end{matrix}; q, q\right), \tag{38}$$

where $d_l/d_{l'} \notin \Omega_q$, $l \neq l'$, and if $C \geq A + 2$,

$$J(\mathbf{a}, \mathbf{c}, \mathbf{d}; f, q) := \sum_{k=1}^{C} \frac{\vartheta(fc_kd_1, \frac{f}{c_kd_2}; q)(\mathbf{a}/c_k; q)_\infty}{(c_k\mathbf{d}, \mathbf{c}_{[k]}/c_k; q)_\infty} \times {}_{A+2}\phi_{C-1}\left(\begin{matrix} c_k\mathbf{d}, qc_k/\mathbf{a} \\ qc_k/\mathbf{c}_{[k]} \end{matrix}; q, \frac{q(qc_k)^{C-A-2}a_1\cdots a_A}{d_1d_2c_1\cdots c_C}\right), \tag{39}$$

where $c_k/c_{k'} \notin \Omega_q$, $k \neq k'$, and ${}_{A+2}\phi_{C-1}$ is convergent for $C = A + 2$ if $|qa_1\cdots a_A| < |d_1d_2c_1\cdots c_C|$, and is an entire function if $C > A + 2$. Then

$$\int_{-\pi}^{\pi} \frac{((fd_1, \frac{q}{f}d_2)\frac{\sigma}{z}, (\frac{f}{d_2}, \frac{q}{fd_1}, \mathbf{a})\frac{z}{\sigma}; q)_\infty}{((d_1, d_2)\frac{\sigma}{z}, \mathbf{c}\frac{z}{\sigma}; q)_\infty} \mathrm{d}\psi = \frac{2\pi\vartheta(f, f\frac{d_1}{d_2}; q)}{(q; q)_\infty} H(\mathbf{a}, \mathbf{c}, \mathbf{d}; q) \tag{40}$$

$$= \frac{2\pi}{(q; q)_\infty} J(\mathbf{a}, \mathbf{c}, \mathbf{d}; f, q), \quad (C \geq A + 2), \tag{41}$$

and none of the arguments of the modified theta functions are equal to some q^m, $m \in \mathbb{Z}$.

Proof Starting with (33), (35) with $m = 0$, $t = 1$, and substituting the parameters as in the integrand of (40), two of the numerator parameters cancel with two of the denominator parameters and the argument of the basic hypergeometric function reduces to q. Noting that the nonterminating basic hypergeometric series are either of the form

${}_C\phi_{C-1}$ or ${}_{A+2}\phi_{A+1}$, depending on whether $C-A-2$ is negative or positive respectively, the series is convergent for $|q|<1$, and this produces the right-hand side of (40). One produces (41) by starting with (33), (36) with $m=0$, $t=1$ and using the convergence properties of the nonterminating basic hypergeometric series described in Theorem 2.1. This completes the proof. $\square$

Corollary 2.5 *Let $q\in\mathbb{C}^\dagger$, $\mathbf{b}:=\{b_1,\ldots,b_B\}$, $\mathbf{d}:=\{d_1,\ldots,d_D\}$, be sets of complex numbers with cardinality $B,D\in\mathbb{N}_0$ (not both zero) respectively, $\mathbf{c}:=\{c_1,c_2\}$, $z=\mathrm{e}^{i\psi}$, $\sigma\in(0,\infty)$, $c_1,c_2\in\mathbb{C}^*$, $f\in\mathbb{C}^*\setminus\{1\}$, such that $|d_l|<1/\sigma$, $|c_1|,|c_2|<\sigma$, for any $d_l\in\mathbf{d}$. Then*

$$\int_{-\pi}^{\pi}\frac{((\frac{f}{c_2},\frac{q}{fc_1},\mathbf{b})\frac{\sigma}{z},(fc_1,\frac{q}{f}c_2)\frac{z}{\sigma};q)_\infty}{(\mathbf{d}\frac{\sigma}{z},(c_1,c_2)\frac{z}{\sigma};q)_\infty}\mathrm{d}\psi=\frac{2\pi\vartheta(f,f\frac{c_1}{c_2};q)}{(q;q)_\infty}H(\mathbf{b},\mathbf{d},\mathbf{c};q) \quad (42)$$

$$=\frac{2\pi}{(q;q)_\infty}J(\mathbf{b},\mathbf{d},\mathbf{c};f,q),\quad (D\ge B+2), \quad (43)$$

and ${}_{B+2}\phi_{D-1}$ in J is convergent for $D=B+2$ if $|qb_1\cdots b_B|<|c_1c_2d_1\cdots d_D|$, and is an entire function if $D>B+2$, and none of the arguments of the modified theta functions are equal to some q^m, $m\in\mathbb{Z}$.

Proof As in the proof of Theorem 2.4, start with Theorem 2.1 with $m=0$, $t=1$. Use (33), (36), and substitute the parameters as in the integrand of (42). Noting that the nonterminating basic hypergeometric series are either of the form ${}_D\phi_{D-1}$ or ${}_{B+2}\phi_{B+1}$, depending on whether $D-B-2$ is negative or positive respectively, the series is convergent for $|q|<1$, and this produces the right-hand side of (42). One produces (43) by starting with (42) with (33), (35) and using the convergence properties of nonterminating basic hypergeometric series. This completes the proof. $\square$

Theorem 2.6 *Let H, $\mathbf{a}$, $\mathbf{c}$, $\mathbf{d}$, q be defined as in Theorem 2.4 and $s\in\mathbb{C}^*$, $d_2{d_1}^{-1}\neq q^m$, $m\in\mathbb{Z}$. Then*

$$H(\mathbf{a},\mathbf{c},\mathbf{d};q)=\frac{\sqrt{\frac{d_2}{d_1}}}{(1-q)s(q;q)_\infty\vartheta(\frac{d_2}{d_1};q)}$$
$$\times\int_{s\sqrt{\frac{d_2}{d_1}}}^{s\sqrt{\frac{d_1}{d_2}}}\frac{((q\sqrt{d_1/d_2},q\sqrt{d_2/d_1},\mathbf{a}\sqrt{d_1d_2})\frac{u}{s};q)_\infty}{(\mathbf{c}\sqrt{d_1d_2}\frac{u}{s};q)_\infty}\mathrm{d}_qu, \quad (44)$$

which is symmetric in $\{d_1,d_2\}$, as in (16).

Proof Start with the q-integral on the right-hand side of (44) using the definition (15). Replacing the q-shifted factorials using (7) identifies the argument q basic hypergeometric series in question. Then identifying common factors using (6) and comparing with (38) derives (44). The symmetry in $\{d_1,d_2\}$ is clear from (38). This completes the proof. $\square$

Note that from Theorems 2.4 and 2.6, we arrive at an interesting relation of a definite integral with a q-integral.

Corollary 2.7 *Let* $\mathbf{a}$, $\mathbf{c}$, $\mathbf{d}$, q, s, f, σ, $z=\mathrm{e}^{i\psi}$ *be defined as in Theorems* 2.4 *and* 2.6. *Then*

$$\int_{-\pi}^{\pi}\frac{((fd_1,\frac{q}{f}d_2)\frac{\sigma}{z},(\frac{f}{d_2},\frac{q}{fd_1},\mathbf{a})\frac{z}{\sigma};q)_\infty}{((d_1,d_2)\frac{\sigma}{z},\mathbf{c}\frac{z}{\sigma};q)_\infty}\mathrm{d}\psi=\frac{2\pi\sqrt{\frac{d_2}{d_1}}\vartheta(f,f\frac{d_1}{d_2};q)}{(1-q)s(q,q;q)_\infty\vartheta(\frac{d_2}{d_1};q)}$$
$$\times\int_{s\sqrt{\frac{d_2}{d_1}}}^{s\sqrt{\frac{d_1}{d_2}}}\frac{((q\sqrt{d_1/d_2},q\sqrt{d_2/d_1},\mathbf{a}\sqrt{d_1d_2})\frac{u}{s};q)_\infty}{(\mathbf{c}\sqrt{d_1d_2}\frac{u}{s};q)_\infty}\mathrm{d}_qu. \tag{45}$$

Proof Comparing Theorem 2.4 with Theorem 2.6 completes the proof. □

A useful consequence of this formula is given in [7, Exercise 4.4], which is an application of (40) with $C=4$, $A=2$. It takes advantage of Bailey's transformation of a very-well-poised ${}_8W_7$ [15, (17.9.16)] and is given as follows:

$$\int_{-\pi}^{\pi}\frac{((cf,\frac{qd}{f})\frac{\sigma}{z},(\frac{f}{d},\frac{q}{fc},k,\frac{abcdgh}{k})\frac{z}{\sigma};q)_\infty}{((c,d)\frac{\sigma}{z},(a,b,g,h)\frac{z}{\sigma};q)_\infty}\mathrm{d}\psi$$
$$=\frac{2\pi\vartheta(f,f\frac{c}{d};q)(kc,kd,acdg,bcdg,cdgh,\frac{abcdh}{k};q)_\infty}{(q,ac,ad,bc,bd,cg,dg,ch,dh,kcdg;q)_\infty}$$
$$\times{}_8W_7\left(\frac{kcdg}{q};cg,dg,\frac{k}{a},\frac{k}{b},\frac{k}{h};q,\frac{abcdh}{k}\right), \tag{46}$$

where $z=\mathrm{e}^{i\psi}$, $\max(|a|,|b|,|c|,|d|,|g|,|h|)<1$, and $|abcdh|<|k|$. Note that if $h=k$ and $g\mapsto e$ then (46) reduces to Gasper's integral (32).

Using Theorem 2.1 one can derive the following generalization of Rahman's integral (29) which does not include the constraint $\lambda\mu=a_{123456}$.

Theorem 2.8 *Let* $\mathbf{a}:=\{a_1,\ldots,a_6\}$, $a_1,\ldots,a_6,\lambda,\mu\in\mathbb{C}^*$, $q\in\mathbb{C}^\dagger$. *Then*

$$\int_{-\pi}^{\pi}\frac{(\mathrm{e}^{\pm2i\psi},\lambda\mathrm{e}^{\pm i\psi},\mu\mathrm{e}^{\pm i\psi};q)_\infty}{(\mathbf{a}\mathrm{e}^{\pm i\psi};q)_\infty}\mathrm{d}\psi=\frac{2\pi}{(q;q)_\infty}\sideset{}{^{a_1;a_2,\ldots,a_6}}{\left[\!\!\left[\right.\right.}\frac{\left(a_1^{-2},a_1\lambda,a_1\mu,\frac{\lambda}{a_1},\frac{\mu}{a_1};q\right)_\infty}{\left(a_{12},\ldots,a_{16},\frac{a_2}{a_1},\ldots,\frac{a_6}{a_1};q\right)_\infty}$$
$$\times{}_{10}W_9\left(a_1^2;a_{12},\ldots,a_{16},\frac{qa_1}{\lambda},\frac{qa_1}{\mu};q,\frac{q\lambda\mu}{a_{123456}}\right), \tag{47}$$

where $|q\lambda\mu|<|a_{123456}|<1$ *and* $\max(|a_1|,\ldots,|a_6|)<1$.

Proof Starting with the left-hand side of (47) and applying Theorem 2.1, noting $qa_1^2/(\pm q^{\frac12}a_1)=\pm q^{\frac12}a_1$, completes the proof. □

If we set $\lambda\mu=a_{123456}$ then (47) specializes to Rahman's integral (29). This results in the following transformation law for the symmetrized sum of six ${}_{10}W_9$'s with argument q being equal to the symmetrized sum of two ${}_{10}W_9$'s with argument q.

Corollary 2.9 *Let* $\mathbf{a} := \{a_1, \ldots, a_6\}$, $a_1, \ldots, a_6, \lambda, \mu \in \mathbb{C}^*$, $q \in \mathbb{C}^\dagger$. *Then*

$$\overset{a_1;a_2,\ldots,a_6}{\mathbb{I}} \frac{\left(a_1^{-2}, a_1\lambda, a_1\mu, \frac{\lambda}{a_1}, \frac{\mu}{a_1}; q\right)_\infty}{\left(a_{12}, \ldots, a_{16}, \frac{a_2}{a_1}, \ldots, \frac{a_6}{a_1}; q\right)_\infty} {}_{10}W_9\left(a_1^2; a_{12}, \ldots, a_{16}, \frac{qa_1}{\lambda}, \frac{qa_1}{\mu}; q, q\right)$$
$$= 2 \overset{\lambda;\mu}{\mathbb{I}} \frac{\left(\lambda\mathbf{a}, \frac{\mu}{\mathbf{a}}; q\right)_\infty}{(a_{12}, \ldots, a_{56}; q)_\infty \left(\lambda^2, \frac{\mu}{\lambda}; q\right)_\infty} {}_{10}W_9\left(\frac{\lambda^2}{q}; \frac{\lambda}{\mathbf{a}}, \frac{a_{123456}}{q}; q, q\right). \tag{48}$$

Proof Comparing (47) to (29) and noting that the integrand is an even function of ψ, and that $qa_1^2/(\pm q^{\frac{1}{2}}a_1) = \pm q^{\frac{1}{2}}a_1$, completes the proof. □

Using Theorem 2.1 we can find an alternative expression for the Nassrallah–Rahman integral (28) as a symmetrized sum of five ${}_8W_7$'s.

Theorem 2.10 *Let* $\mathbf{a} := \{a_1, \ldots, a_5\}$, $a_1, \ldots, a_5, \lambda, \mu \in \mathbb{C}^*$, $q \in \mathbb{C}^\dagger$. *Then*

$$\int_{-\pi}^{\pi} \frac{(\mathrm{e}^{\pm 2i\psi}, \lambda\mathrm{e}^{\pm i\psi}; q)_\infty}{(\mathbf{a}\mathrm{e}^{\pm i\psi}; q)_\infty} \mathrm{d}\psi = \frac{2\pi}{(q;q)_\infty} \overset{a_1;a_2,\ldots,a_5}{\mathbb{I}} \frac{\left(a_1^{-2}, a_1\lambda, \frac{\lambda}{a_1}; q\right)_\infty}{\left(a_{12}, \ldots, a_{15}, \frac{a_2}{a_1}, \ldots, \frac{a_5}{a_1}; q\right)_\infty}$$
$$\times {}_8W_7\left(a_1^2; a_{12}, \ldots, a_{15}, \frac{qa_1}{\lambda}; q, \frac{q\lambda}{a_{12345}}\right), \tag{49}$$

where $|q\lambda| < |a_{12345}| < 1$ *and* $\max(|a_1|, \ldots, |a_5|) < 1$.

Proof Starting with the left-hand side of (49) and applying Theorem 2.1 completes the proof. □

By comparing the above expression for the Nassrallah–Rahman integral to (49), one can obtain the following transformation of a symmetrized sum of five ${}_8W_7$'s is equal to a symmetric ${}_8W_7$.

Corollary 2.11 *Let* $\mathbf{a} := \{a_1, \ldots, a_5\}$, $a_1, \ldots, a_5, \lambda \in \mathbb{C}^*$, $q \in \mathbb{C}^\dagger$. *Then*

$$\overset{a_1;a_2,\ldots,a_5}{\mathbb{I}} \frac{\left(\lambda a_1, a_1^{-2}, \frac{\lambda}{a_1}; q\right)_\infty}{\left(a_{12}, \ldots, a_{15}, \frac{a_2}{a_1}, \ldots, \frac{a_5}{a_1}; q\right)_\infty} {}_8W_7\left(a_1^2; a_{12}, \ldots, a_{15}, \frac{qa_1}{\lambda}; q, \frac{q\lambda}{a_{12345}}\right)$$
$$= \frac{2\left(\lambda\mathbf{a}, \frac{a_{12345}}{\lambda}; q\right)_\infty}{(a_{12}, \ldots, a_{45}, \lambda^2; q)_\infty} {}_8W_7\left(\frac{\lambda^2}{q}; \frac{\lambda}{\mathbf{a}}; q, \frac{a_{12345}}{\lambda}\right), \tag{50}$$

where $|q| < \left|\frac{a_{12345}}{\lambda}\right| < 1$.

Proof Comparing (49) to (28) and noting that the integrand is an even function of ψ completes the proof. □

By taking $\lambda = a_5$ one reduces the Nassrallah–Rahman integrals (28), (49) to the Askey–Wilson integral (27) (the ${}_8W_7$ becomes unity). Comparing these limit expressions produces the following nonterminating summation formula which relates a symmetric sum of four ${}_6W_5$'s to an infinite product that we now give.

Corollary 2.12 *Let* $a_1, \ldots, a_4 \in \mathbb{C}^*$, $q \in \mathbb{C}^\dagger$ *and none of the arguments of the modified theta functions are equal to some* q^m, $m \in \mathbb{Z}$, *and none of infinite* q*-shifted factorials vanish. Then*

$$\sum_{a_1;a_2,a_3,a_4} \frac{(a_1^{-2};q)_\infty}{\left(a_{12},a_{13},a_{14},\frac{a_2}{a_1},\frac{a_3}{a_1},\frac{a_4}{a_1};q\right)_\infty} {}_6W_5\left(a_1^2;a_{12},a_{13},a_{14};q,\frac{q}{a_{1234}}\right)$$
$$= \frac{(a_{1234};q)_\infty}{(a_{12},a_{13},a_{14},a_{23},a_{24},a_{34};q)_\infty} \sum_{a_1;a_2,a_3,a_4} \frac{\vartheta(a_1^{-2},a_{23},a_{24},a_{34};q)}{\vartheta(a_{1234},\frac{a_2}{a_1},\frac{a_3}{a_1},\frac{a_4}{a_1};q)}$$
$$= \frac{2(a_{1234};q)_\infty}{(a_{12},a_{13},a_{14},a_{23},a_{24},a_{34};q)_\infty}, \tag{51}$$

where $|q| < |a_{1234}| < 1$.

Proof Setting $\lambda = a_5$ in (49), and comparing with (27) completes the proof. □

Remark 2.13 Note that for the ${}_8W_7(a;b,c,d,e,f;q,z)$'s which appear in this subsection, instead of the argument being $q^2a^2/(bcdef)$ it is $-q^2a^2/(bcdef)$. Compare with Bailey's transformation of a very-well poised ${}_8W_7$ [15, (17.9.16)]. So these ${}_8W_7$'s cannot be written as a sum of two balanced ${}_4\phi_3$'s.

Some other applications of Theorem 2.4 arise when one encounters a sum of two basic hypergeometric functions with argument q. In this case, you are almost certainly guaranteed to be able to find a corresponding integral representation. Below we present some examples of this. First we present two integral representations of a nonterminating ${}_2\phi_1$. Note that other integral representations for the arbitrary ${}_2\phi_1$ have been presented such as Watson's contour integral (see [7, (4.2.2)] for more details)

$${}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,z\right) = -\frac{1}{2i}\frac{(a,b;q)_\infty}{(q,c;q)_\infty}\int_{-i\infty}^{i\infty}\frac{((q,c)q^s;q)_\infty}{((a,b)q^s;q)_\infty}\frac{(-z)^s}{\sin(\pi s)}\mathrm{d}s, \tag{52}$$

where $\pm i\infty := \pm\lim_{x\uparrow\infty} ix$, where $x \in (0,\infty)$.

Corollary 2.14 *Let $a, b, c, z \in \mathbb{C}^*$, such that $|z| < 1$, $q \in \mathbb{C}^\dagger$, $\tau \in (0,1)$, $w = e^{i\eta}$. Then*

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) \\
&= \frac{(q, a, \frac{c}{b}, \frac{abz}{c}; q)_\infty}{2\pi \vartheta(f, f\frac{c}{bz}; q)(c; q)_\infty} \int_{-\pi}^{\pi} \frac{((f\sqrt{\frac{c}{bz}}, \frac{q}{f}\sqrt{\frac{bz}{c}}, \sqrt{bcz})\frac{\tau}{w}, (f\sqrt{\frac{c}{bz}}, \frac{q}{f}\sqrt{\frac{bz}{c}})\frac{w}{\tau}; q)_\infty}{((\sqrt{\frac{cz}{b}}, \sqrt{\frac{bz}{c}}a)\frac{\tau}{w}, (\sqrt{\frac{bz}{c}}, \sqrt{\frac{c}{bz}})\frac{w}{\tau}; q)_\infty} \mathrm{d}\eta
\end{aligned} \tag{53}
$$

$$
= \frac{(q, a, b, \frac{c}{a}, \frac{c}{b}, \frac{abz}{c}; q)_\infty}{2\pi \vartheta(f, f\frac{ab}{c}; q)(c; q)_\infty} \int_{-\pi}^{\pi} \frac{((f\sqrt{\frac{ab}{c}}, \frac{q}{f}\sqrt{\frac{c}{ab}})\frac{\tau}{w}, (f\sqrt{\frac{ab}{c}}, \frac{q}{f}\sqrt{\frac{c}{ab}})\frac{w}{\tau}; q)_\infty}{((\sqrt{\frac{ab}{c}}, \sqrt{\frac{c}{ab}})\frac{\tau}{w}, (\sqrt{\frac{ac}{b}}, \sqrt{\frac{bc}{a}}, \sqrt{\frac{ab}{c}}z)\frac{w}{\tau}; q)_\infty} \mathrm{d}\eta, \tag{54}
$$

and none of the arguments of the modified theta functions are equal to some q^m, $m \in \mathbb{Z}$.

Proof For (53) start with cf. [15, (17.9.3)]

$$
\begin{aligned}
\frac{(c; q)_\infty}{(a, \frac{c}{b}, \frac{abz}{c}; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) &= \frac{(c; q)_\infty}{(a, \frac{c}{b}, \frac{bz}{c}; q)_\infty} {}_2\phi_2^{-1}\left(\begin{matrix} a, \frac{c}{b} \\ c, \frac{qc}{bz} \end{matrix}; q, q\right) \\
&\quad + \frac{(bz; q)_\infty}{(z, \frac{c}{bz}, \frac{abz}{c}; q)_\infty} {}_2\phi_2^{-1}\left(\begin{matrix} z, \frac{abz}{c} \\ bz, \frac{qbz}{c} \end{matrix}; q, q\right),
\end{aligned} \tag{55}
$$

then apply (40) with

$$
\mathbf{a} := \{\sqrt{bcz}\}, \;\; \mathbf{c} := \left\{\sqrt{\frac{bz}{c}}a, \sqrt{\frac{cz}{b}}\right\}, \;\; \mathbf{d} := \left\{\sqrt{\frac{c}{bz}}, \sqrt{\frac{bz}{c}}\right\}, \tag{56}
$$

and therefore $C - A - 2 = -1$. For (54) start with cf. [15, (17.9.3_5)]

$$
\begin{aligned}
\frac{(c; q)_\infty}{(a, b, \frac{c}{a}, \frac{c}{b}, \frac{abz}{c}; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) &= \frac{1}{(a, \frac{c}{b}, \frac{bz}{c}; q)_\infty} {}_3\phi_1^1\left(\begin{matrix} a, b, \frac{abz}{c} \\ \frac{qab}{c} \end{matrix}; q, q\right) \\
&+ \frac{1}{(z, c, \frac{c}{a}, \frac{c}{b}; q)_\infty} {}_3\phi_1^1\left(\begin{matrix} \frac{c}{a}, \frac{c}{b}, z \\ \frac{qc}{ab} \end{matrix}; q, q\right),
\end{aligned} \tag{57}
$$

then apply (40) with

$$
\mathbf{a} := \emptyset, \;\; \mathbf{c} := \left\{\sqrt{\frac{ac}{b}}, \sqrt{\frac{bc}{a}}, \sqrt{\frac{ab}{c}}\right\}, \;\; \mathbf{d} := \left\{\sqrt{\frac{ab}{c}}, \sqrt{\frac{c}{ab}}\right\}, \tag{58}
$$

and therefore $C - A - 2 = 1$. This completes the proof. □

Another example where an integral representation for a nonterminating basic hypergeometric function may be found is for a well-poised ${}_3\phi_2$.

Corollary 2.15 *Let $a, b, c, x \in \mathbb{C}^*$, such that $|qax| < |bc|$, $q \in \mathbb{C}^\dagger$, $\tau \in (0,1)$, $w = e^{i\eta}$, $f, fx \neq q^m$, $m \in \mathbb{Z}$. Then*

$$
{}_3\phi_2\left(\begin{matrix} a, b, c \\ \frac{qa}{b}, \frac{qa}{c} \end{matrix}; q, \frac{qax}{bc}\right) = \frac{(q, a, \frac{qa}{bc}; q)_\infty}{2\pi\vartheta(f, fx; q)(\frac{qa}{b}, \frac{qa}{c}; q)_\infty}
\times \int_{-\pi}^{\pi} \frac{((f\sqrt{x}, \frac{q}{f\sqrt{x}})\frac{\tau}{w}, (f\sqrt{x}, \frac{q}{f\sqrt{x}}, \frac{qa\sqrt{x}}{b}, \frac{qa\sqrt{x}}{c}, ax^{\frac{3}{2}})\frac{w}{\tau}; q)_\infty}{((\sqrt{x}, \frac{1}{\sqrt{x}})\frac{\tau}{w}, (\pm\sqrt{ax}, \pm\sqrt{qax}, \frac{qa\sqrt{x}}{bc})\frac{w}{\tau}; q)_\infty} \mathrm{d}\eta. \tag{59}
$$

Proof Start with cf. [7, (III.35)], then

$$
\begin{aligned}
&\frac{(\frac{qa}{b}, \frac{qa}{c}; q)_\infty}{(a, \frac{qa}{bc}; q)_\infty} {}_3\phi_2\left(\begin{matrix} a, b, c \\ \frac{qa}{b}, \frac{qa}{c} \end{matrix}; q, \frac{qax}{bc}\right) \\
&= \frac{(\frac{qax}{b}, \frac{qax}{c}; q)_\infty}{(1/x, \frac{qax}{bc}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \pm x\sqrt{a}, \pm x\sqrt{qa}, \frac{qax}{bc} \\ qx, \frac{qax}{b}, \frac{qax}{c}, ax^2 \end{matrix}; q, q\right) \\
&\quad + \frac{(\frac{qa}{b}, \frac{qa}{c}, ax; q)_\infty}{(a, \frac{qa}{bc}, x; q)_\infty} {}_5\phi_4\left(\begin{matrix} \pm\sqrt{a}, \pm\sqrt{qa}, \frac{qa}{bc} \\ q/x, \frac{qa}{b}, \frac{qa}{c}, ax \end{matrix}; q, q\right).
\end{aligned} \tag{60}
$$

Applying (40) with

$$
\begin{aligned}
&\mathbf{a} := \left\{\frac{qa\sqrt{x}}{b}, \frac{qa\sqrt{x}}{c}, ax^{\frac{3}{2}}\right\}, \ \mathbf{c} := \left\{\pm\sqrt{ax}, \pm\sqrt{qax}, \frac{qa\sqrt{x}}{bc}\right\}, \\
&\mathbf{d} := \left\{\sqrt{x}, \frac{1}{\sqrt{x}}\right\},
\end{aligned} \tag{61}
$$

and therefore $C - A - 2 = 0$, completes the proof. □

2.2 Unbalanced symmetrization transformations for basic hypergeometric functions and some of their specializations and limits

A direct consequence of Theorem 2.4 is the following integral.

Lemma 2.16 *Let $\mathbf{a} := \{a_1, \ldots, a_A\}$, $\mathbf{b} := \{b_1, \ldots, b_B\}$, $\mathbf{c} := \{c_1, \ldots, c_C\}$, $\mathbf{d} := \{d_1, \ldots, d_D\}$ be sets of complex numbers with cardinality $A, B, C, D \in \mathbb{N}_0$ (not all zero) respectively, $z = \mathrm{e}^{i\psi}$, $w = \mathrm{e}^{i\eta}$. Let $\sigma, \tau \in (0, \infty)$, $t \in \mathbb{C}^*$, so that $|b_j| < |d_l| < \min\{\tau/|t|, 1/\sigma\}$, $|d_l a_i| < 1/|t|$, and $|a_i| < |c_k| < \min\{\sigma/|t|, 1/\tau\}$ for any a_i, b_j, c_k, d_l elements of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ respectively. Then*

$$
\int_{-\pi}^{\pi} \frac{(\mathbf{b}\frac{\sigma}{z}, t\,\mathbf{a}\frac{z}{\sigma}; q)_\infty}{(\mathbf{d}\frac{\sigma}{z}, t\,\mathbf{c}\frac{z}{\sigma}; q)_\infty} \mathrm{d}\psi = \int_{-\pi}^{\pi} \frac{(\mathbf{a}\frac{\tau}{w}, t\,\mathbf{b}\frac{w}{\tau}; q)_\infty}{(\mathbf{c}\frac{\tau}{w}, t\,\mathbf{d}\frac{w}{\tau}; q)_\infty} \mathrm{d}\eta = \frac{2\pi}{(q;q)_\infty} G_{0,t}. \tag{62}
$$

Proof Setting $m = 0$ in (33), it is straightforward to check the identity by comparing the first summation expression of (33) to the second summation expression of (33). Hence the result holds. □

Therefore taking into account the definition of $G_{m,t}$ (see expression (33)) the following identity holds.

Corollary 2.17 *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, *and the other variables be as defined as in Lemma 2.16. Then one has the following (in general non-balanced) transformation of symmetrization over variables for basic hypergeometric functions:*

$$\sum_{k=1}^{D} \frac{(td_k\mathbf{a}, d_k^{-1}\mathbf{b}; q)_\infty}{(td_k\mathbf{c}, d_k^{-1}\mathbf{d}_{[k]}; q)_\infty} {}_{B+C}\phi_{A+D-1}^{A-C}\left(\begin{matrix} td_k\mathbf{c}, qd_k\mathbf{b}^{-1} \\ td_k\mathbf{a}, qd_k\mathbf{d}_{[k]}^{-1} \end{matrix}; q, \frac{b_1\cdots b_B}{d_1\cdots d_D}\right)$$
$$= \sum_{k=1}^{C} \frac{(tb_k\mathbf{c}, b_k^{-1}\mathbf{a}; q)_\infty}{(tc_k\mathbf{d}, c_k^{-1}\mathbf{c}_{[k]}; q)_\infty} {}_{A+D}\phi_{B+C-1}^{B-D}\left(\begin{matrix} tc_k\mathbf{d}, qc_k\mathbf{a}^{-1} \\ tc_k\mathbf{b}, qc_k\mathbf{c}_{[k]}^{-1} \end{matrix}; q, \frac{a_1\cdots a_A}{c_1\cdots c_C}\right). \tag{63}$$

Proof The identity follows by using Theorem 2.1 with $m = 0$. □

Now we treat the $t = 1$ case which has an extra degree of symmetry that can be exploited.

Lemma 2.18 *Let* $\mathbf{a} := \{a_1, \ldots, a_A\}$, $\mathbf{b} := \{b_1, \ldots, b_B\}$, $\mathbf{c} := \{c_1, \ldots, c_C\}$, $\mathbf{d} := \{d_1, \ldots, d_D\}$ *be sets of complex numbers with cardinality* $A, B, C, D \in \mathbb{N}_0$ *(not all zero) respectively,* $z = \mathrm{e}^{i\psi}$, $w = \mathrm{e}^{i\eta}$. *Let* $\sigma, \tau \in (0, \infty)$. *so that* $|b_j| < |d_l| < \min\{\tau, 1/\sigma\}$, $|d_l a_i| < 1$, *and* $|a_i| < |c_k| < \min\{\sigma, 1/\tau\}$ *for any* a_i, b_j, c_k, d_l *elements of* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *respectively. Then*

$$\int_{-\pi}^{\pi} \frac{(\mathbf{b}\frac{\sigma}{z}, \mathbf{a}\frac{z}{\sigma}; q)_\infty}{(\mathbf{d}\frac{\sigma}{z}, \mathbf{c}\frac{z}{\sigma}; q)_\infty} \mathrm{d}\psi = \int_{-\pi}^{\pi} \frac{(\mathbf{a}\frac{\tau}{w}, \mathbf{b}\frac{w}{\tau}; q)_\infty}{(\mathbf{c}\frac{\tau}{w}, \mathbf{d}\frac{w}{\tau}; q)_\infty} \mathrm{d}\eta. \tag{64}$$

The $A = B = C = D = 2$ case of Corollary 2.17 is quite interesting. It is only one example of an infinite sequence of such results with arbitrary values of $A, B, C, D \in \mathbb{N}$ in Corollary 2.17—it relates the sum of two ${}_4\phi_3$'s to a different sum of two ${}_4\phi_3$'s and provides a generalization of Corollary 2.4 in Ismail and Stanton [9] (see also Ismail [10, Corollary 15.8.3]).

Corollary 2.19 *Let* $t, a, b, c, d, e, f, g, h \in \mathbb{C}^*$, $|ab| < |ef|$, $|cd| < |gh|$. *Then*

$$\coprod^{e;f} \frac{(etc, etd, \frac{a}{e}, \frac{b}{e}; q)_\infty}{(etg, eth, \frac{f}{e}; q)_\infty} {}_4\phi_3\left(\begin{matrix} etg, eth, \frac{qe}{a}, \frac{qe}{b} \\ etc, etd, \frac{qe}{f} \end{matrix}; q, \frac{ab}{ef}\right)$$
$$= \coprod^{g;h} \frac{(gta, gtb, \frac{c}{g}, \frac{d}{g}; q)_\infty}{(gte, gtf, \frac{h}{g}; q)_\infty} {}_4\phi_3\left(\begin{matrix} gte, gtf, \frac{qg}{c}, \frac{qg}{d} \\ gta, gtb, \frac{qg}{h} \end{matrix}; q, \frac{cd}{gh}\right). \tag{65}$$

The $t=1$ case is interesting.

Corollary 2.20 *Let $a, b, c, d, e, f, g, h \in \mathbb{C}^*$, $|ab| < |ef|$, $|cd| < |gh|$. Then*

$$\int^{e;f} \frac{(ec, ed, \frac{a}{e}, \frac{b}{e}; q)_\infty}{(eg, eh, \frac{f}{e}; q)_\infty} {}_4\phi_3\left(\begin{matrix} eg, eh, \frac{qe}{a}, \frac{qe}{b} \\ ec, ed, \frac{qe}{f} \end{matrix}; q, \frac{ab}{ef}\right)$$

$$= \int^{g;h} \frac{(ga, gb, \frac{c}{g}, \frac{d}{g}; q)_\infty}{(ge, gf, \frac{h}{g}; q)_\infty} {}_4\phi_3\left(\begin{matrix} ge, gf, \frac{qg}{c}, \frac{qg}{d} \\ ga, gb, \frac{qg}{h} \end{matrix}; q, \frac{cd}{gh}\right). \tag{66}$$

This exploits the trick adopted in [7, Exercise 4.4] which converts those basic hypergeometric functions with specific argument to those with argument q and reduces the number of numerator parameters and denominator parameters by two. By mapping

$$(e, f, a, b, c, d) \mapsto (e, f, \kappa e, qf/\kappa, q/(\kappa e), \kappa/f),$$

one converts the left-hand side of Corollary 2.20 to that with an argument q and reduces the ${}_4\phi_3$'s to ${}_2\phi_1$'s. Furthermore, by mapping $(g, h) \mapsto (1/(\mu e), \mu/f)$, this converts the right-hand side of the above Corollary to that with an argument q. The resulting relation can be easily verified using the q-Gauss sum [7, (II.8)].

3 Generating functions and integral representations

One powerful application of integral representations for basic hypergeometric functions is the determination of generating functions for basic hypergeometric orthogonal polynomials in the q-Askey scheme.

3.1 The Askey–Wilson polynomials

In this section we study integral representations for the Askey–Wilson polynomials and some useful applications of these.

3.1.1 Integral representations of the Askey–Wilson polynomials

A key formula which allows for this is given in [7, Exercise 4.5] that is equivalent to the following.

Theorem 3.1 *Let* $a, b, c, d, f \in \mathbb{C}^*$, $\sigma \in (0, 1)$, $\max(|a|, |b|, |c|, |d|) < 1$, $q \in \mathbb{C}^\dagger$, $x = \cos\theta \in [-1, 1]$, $z = e^{i\psi}$, $f, f e^{2i\theta} \neq q^m$, $m \in \mathbb{Z}$. *Then*

$$p_n(x; \mathfrak{a}|q) = \frac{(q, a e^{\pm i\theta}, b e^{\pm i\theta}, c e^{\pm i\theta}; q)_\infty (ab, ac, bc; q)_n}{2\pi \vartheta(f, f e^{2i\theta}; q)(ab, ac, bc; q)_\infty} D_n(x; \mathfrak{a}, f, \sigma|q), \tag{67}$$

where

$$D_n(x; \mathfrak{a}, f, \sigma|q)$$
$$= \int_{-\pi}^{\pi} \frac{((f e^{i\theta}, \frac{q}{f} e^{-i\theta})\frac{\sigma}{z}, (f e^{i\theta}, \frac{q}{f} e^{-i\theta}, abc)\frac{z}{\sigma}; q)_\infty}{(e^{\pm i\theta}\frac{\sigma}{z}, (a, b, c)\frac{z}{\sigma}; q)_\infty} \frac{(d\frac{\sigma}{z}; q)_n}{(abc\frac{z}{\sigma}; q)_n} \left(\frac{z}{\sigma}\right)^n d\psi \tag{68}$$

$$= \int_{-\pi}^{\pi} \frac{((f abc\, e^{i\theta}, \frac{q}{f} abc\, e^{-i\theta})\frac{\sigma}{z}, (f \frac{1}{abc} e^{i\theta}, \frac{q}{f}\frac{1}{abc} e^{-i\theta}, 1)\frac{z}{\sigma}; q)_\infty}{(abc\, e^{\pm i\theta}\frac{\sigma}{z}, (\frac{1}{ab}, \frac{1}{ac}, \frac{1}{bc})\frac{z}{\sigma}; q)_\infty}$$
$$\times \frac{(abcd\frac{\sigma}{z}; q)_n}{(\frac{z}{\sigma}; q)_n} \left(\frac{1}{abc}\frac{z}{\sigma}\right)^n d\psi. \tag{69}$$

Proof The integral representation (68) is Exercise 4.5 in [7]. The integral representation (69) is derived as follows. Start with [5, (30)] then apply [7, (III.23)]. This produces the following nonterminating representation of the Askey–Wilson polynomials:

$$ {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, a e^{\pm i\theta} \\ ab, ac, ad \end{matrix}; q, q\right) = \frac{(a^2cd, cd; q)_n (\frac{qa}{b}, \frac{q}{ab}, acd\, e^{\pm i\theta}; q)_\infty}{(acd\, e^{\pm i\theta}; q)_n (\frac{q}{b} e^{\pm i\theta}, a^2cd, cd; q)_\infty}$$
$$\times {}_8W_7\left(q^{n-1}a^2cd; q^n ac, q^n ad, q^{n-1}abcd, a e^{\pm i\theta}; q, \frac{q^{1-n}}{ab}\right), \tag{70}$$

where $|q^{1-n}| < |ab|$. Using this nonterminating representation, comparing it with (46), and simplifying completes the proof. □

3.1.2 Generating functions for the Askey–Wilson polynomials

Many researchers have investigated series and q-integral (15) representations for Askey–Wilson polynomials. On the other hand, it seems that regular integral representations for the Askey–Wilson polynomials have been largely disregarded. In the following we will demonstrate how these representations for the Askey–Wilson polynomials allow for simple and straightforward evaluations of some of their fundamental properties, particularly their generating functions. We begin with the Rahman generating function for the Askey–Wilson polynomials, which is the q-analogue of the following generating function for the Wilson polynomials [12, (6.2)].

For Wilson polynomials there is the following generating function. Let $a, b, c, d, t \in \mathbb{C}^*$, $\max(|a|, |b|, |c|, |d|, |t|) < 1$, $x = \cos\theta \in [-1, 1]$, $4|t| < |1-t|^2$,

$$\sum_{n=0}^{\infty} \frac{(a+b+c+d-1)_n}{n!(a+b, a+c, a+d)_n} W_n(x^2; \mathfrak{a}) t^n = (1-t)^{1-a-b-c-d} \times {}_4F_3\left(\begin{matrix} \frac{1}{2}(a+b+c+d-1), \frac{1}{2}(a+b+c+d), a \pm ix \\ a+b, a+c, a+d \end{matrix}; \frac{-4t}{(1-t)^2}\right). \tag{71}$$

Rahman computed a q-analogue of (71) in [19, (4.9)] by using a q-integral representation of Askey–Wilson polynomials. We will prove the same generating function using the above integral representation (68).

Theorem 3.2 (Rahman [19]) *Let* $k, p \in \{1, 2, 3, 4\}$, $\mathbf{a} := \{a_1, a_2, a_3, a_4\}$, $t, a_k \in \mathbb{C}^*$, $x = \cos\theta \in [-1, 1]$, $q \in \mathbb{C}^\dagger$, $|ta_p| < 1$. *Then*

$$\sum_{n=0}^{\infty} \frac{t^n\,(q^{-1}a_{1234}; q)_n\, p_n(x; \mathbf{a}|q)}{(q, \{a_p a_s\}_{s\neq p}; q)_n} = \frac{(ta_{1234}(qa_p)^{-1}; q)_\infty}{(ta_p^{-1}; q)_\infty}\, {}_6\phi_5\left(\begin{matrix} \pm(q^{-1}a_{1234})^{\frac{1}{2}}, \pm(a_{1234})^{\frac{1}{2}}, a_p e^{\pm i\theta} \\ \{a_p a_s\}_{s\neq p}, ta_{1234}(qa_p)^{-1}, qa_p t^{-1} \end{matrix}; q, q\right) + \frac{(\{ta_s\}_{s\neq p}, q^{-1}a_{1234}, a_p e^{\pm i\theta}; q)_\infty}{(\{a_p a_s\}_{s\neq p}, a_p t^{-1}, te^{\pm i\theta}; q)_\infty}\, {}_6\phi_5\left(\begin{matrix} \pm ta_p^{-1}(q^{-1}a_{1234})^{\frac{1}{2}}, \pm ta_p^{-1}(a_{1234})^{\frac{1}{2}}, te^{\pm i\theta} \\ \{ta_s\}_{s\neq p}, q^{-1}a_{1234}(ta_p^{-1})^2, qta_p^{-1} \end{matrix}; q, q\right). \tag{72}$$

Proof Start with the left-hand side of (72) and insert (68). This produces

$$\sum_{n=0}^{\infty} \frac{(abcd/q; q)_n t^n p_n(x; \mathfrak{a}|q)}{(q, ab, ac, ad; q)_n} = \frac{(q, ae^{\pm i\theta}, be^{\pm i\theta}, ce^{\pm i\theta}; q)_\infty}{2\pi\vartheta(f, fe^{2i\theta}; q)(ab, ac, bc; q)_\infty} \times \int_{-\pi}^{\pi} \frac{((fe^{i\theta}, \frac{q}{f}e^{-i\theta})\frac{\sigma}{z}, (fe^{i\theta}, \frac{q}{f}e^{-i\theta}, abc)\frac{z}{\sigma}; q)_\infty}{(e^{\pm i\theta}\frac{\sigma}{z}, (a, b, c)\frac{z}{\sigma}; q)_\infty}\, {}_3\phi_2\left(\begin{matrix} d\frac{\sigma}{z}, abcd/q, bc \\ abc\frac{z}{\sigma}, ad \end{matrix}; q, \frac{tz}{\sigma}\right) \mathrm{d}\psi. \tag{73}$$

The ${}_3\phi_2$ can be written as a sum of two ${}_5\phi_4(q, q)$ using [7, (3.4.1)], since it is well-poised. Comparing this sum using (40) (see also Corollary 2.15) produces

$${}_3\phi_2\left(\begin{matrix} d\frac{\sigma}{z}, abcd/q, bc \\ abc\frac{z}{\sigma}, ad \end{matrix}; q, \frac{tz}{\sigma}\right) = \frac{(q, abcd/q, \frac{az}{\sigma}; q)_\infty}{2\pi\vartheta(h, h\frac{a}{t}; q)(abc\frac{z}{\sigma}, ad; q)_\infty} \times \int_{-\pi}^{\pi} \frac{((h\sqrt{\frac{a}{t}}, \frac{q}{h}\sqrt{\frac{t}{a}})\frac{\tau}{w}, (h\sqrt{\frac{a}{t}}, \frac{q}{h}\sqrt{\frac{t}{a}}, d\sqrt{ta}, \frac{bcdt^{3/2}}{q\sqrt{a}}, bc\sqrt{ta}\frac{z}{\sigma})\frac{w}{\tau}; q)_\infty}{((\sqrt{\frac{a}{t}}, \sqrt{\frac{t}{a}})\frac{\tau}{w}, (\pm\sqrt{\frac{tbcd}{q}}, \pm\sqrt{tbcd}, \sqrt{ta}\frac{z}{\sigma})\frac{w}{\tau}; q)_\infty} \mathrm{d}\eta, \tag{74}$$

where $w = e^{i\eta}$. Inserting the above integral representation, rearranging the integrals and evaluating the outer integral using Gasper's integral (32) completes the proof. □

We can also derive an integral representation for a product of two ${}_2\phi_1$'s by using the other generating function which is known for Askey–Wilson polynomials [13, (14.1.15)]. Integral representations for products of basic hypergeometric functions is an interesting direction of research.

Theorem 3.3 *Let $a, b, c, d, t, f \in \mathbb{C}^*$, $q \in \mathbb{C}^\dagger$, $\sigma \in (0,1)$, $|t| < \sigma$, $z = \mathrm{e}^{i\psi}$, $f, f\mathrm{e}^{2i\theta} \neq q^m$, $m \in \mathbb{Z}$. Then*

$$\int_{-\pi}^{\pi} \frac{((f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta})\frac{\sigma}{z}, (f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta}, abc)\frac{z}{\sigma}; q)_\infty}{(\mathrm{e}^{\pm i\theta}\frac{\sigma}{z}, (a,b,c)\frac{z}{\sigma}; q)_\infty}\, {}_3\phi_2\left(\begin{matrix} d\frac{\sigma}{z}, ab, ac \\ abc\frac{z}{\sigma}, ad \end{matrix}; q, \frac{tz}{\sigma}\right) \mathrm{d}\psi$$
$$= 2\pi \frac{\vartheta(f, f\mathrm{e}^{2i\theta}; q)(ab, ac, bc; q)_\infty}{(q, a\mathrm{e}^{\pm i\theta}, b\mathrm{e}^{\pm i\theta}, c\mathrm{e}^{\pm i\theta}; q)_\infty}\, {}_2\phi_1\left(\begin{matrix} a\mathrm{e}^{i\theta}, d\mathrm{e}^{i\theta} \\ ad \end{matrix}; q, t\mathrm{e}^{-i\theta}\right) {}_2\phi_1\left(\begin{matrix} b\mathrm{e}^{-i\theta}, c\mathrm{e}^{-i\theta} \\ bc \end{matrix}; q, t\mathrm{e}^{i\theta}\right). \tag{75}$$

Proof Start with the generating function for the Askey–Wilson polynomials [11, (1.9)], [13, (14.1.15)],

$$\sum_{n=0}^{\infty} \frac{t^n p_n(x; \mathfrak{a}|q)}{(q, ad, bc; q)_n} = {}_2\phi_1\left(\begin{matrix} a\mathrm{e}^{i\theta}, d\mathrm{e}^{i\theta} \\ ad \end{matrix}; q, t\mathrm{e}^{-i\theta}\right) {}_2\phi_1\left(\begin{matrix} b\mathrm{e}^{-i\theta}, c\mathrm{e}^{-i\theta} \\ bc \end{matrix}; q, t\mathrm{e}^{i\theta}\right). \tag{76}$$

Inserting the integral representation (68) into the left-hand side of (76) produces the following integral representation for (76), namely:

$$\sum_{n=0}^{\infty} \frac{t^n p_n(x; \mathfrak{a}|q)}{(q, ad, bc; q)_n} = \frac{(q, a\mathrm{e}^{\pm i\theta}, b\mathrm{e}^{\pm i\theta}, c\mathrm{e}^{\pm i\theta}; q)_\infty}{2\pi\vartheta(f, f\mathrm{e}^{2i\theta}; q)(ab, ac, bc; q)_\infty}$$
$$\times \int_{-\pi}^{\pi} \frac{((f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta})\frac{\sigma}{z}, (f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta}, abc)\frac{z}{\sigma}; q)_\infty}{(\mathrm{e}^{\pm i\theta}\frac{\sigma}{z}, (a,b,c)\frac{z}{\sigma}; q)_\infty}\, {}_3\phi_2\left(\begin{matrix} d\frac{\sigma}{z}, ab, ac \\ abc\frac{z}{\sigma}, ad \end{matrix}; q, \frac{tz}{\sigma}\right) \mathrm{d}\psi. \tag{77}$$

Comparing (76) with (77) completes the proof. □

3.2 Continuous dual q-Hahn polynomials

If you let $a_4 \to 0$ $(d \to 0)$ in the Askey–Wilson polynomials you obtain the three parameter symmetric continuous dual q-Hahn polynomials [13, Section 14.3]. In this case the Askey–Wilson polynomials $p_n(x; \mathbf{a}|q)$ with $\mathbf{a} = \{a_1, a_2, a_3, a_4\}$ reduce to the continuous dual q-Hahn polynomials $p_n(x; \mathbf{a}|q)$ with $\mathbf{a} := \{a_1, a_2, a_3\}$, $\mathfrak{a} := \{a, b, c\}$, $\mathfrak{a} = \mathbf{a}$.

3.2.1 Integral representations for the continuous dual q-Hahn polynomials

One can obtain several integral representations for the continuous dual q-Hahn polynomials by starting with (68).

Corollary 3.4 *Let $a, b, c, f \in \mathbb{C}^*$, $\max(|a|, |b|, |c|) < 1$, $\sigma \in (0, 1)$, $q \in \mathbb{C}^\dagger$, $x = \cos\theta \in [-1, 1]$, $z = \mathrm{e}^{i\psi}$, $f, f\mathrm{e}^{2i\theta} \neq q^m$, $m \in \mathbb{Z}$. Then*

$$p_n(x; \mathfrak{a}|q) = \frac{(q, a\mathrm{e}^{\pm i\theta}, b\mathrm{e}^{\pm i\theta}; q)_\infty (ab; q)_n}{2\pi\vartheta(f, f\mathrm{e}^{2i\theta}; q)(ab; q)_\infty} E_n(\mathfrak{a}; f, \sigma|q) \tag{78}$$

$$= \frac{(q, a\mathrm{e}^{\pm i\theta}, b\mathrm{e}^{\pm i\theta}, c\mathrm{e}^{\pm i\theta}; q)_\infty (ab, ac, bc; q)_n}{2\pi\vartheta(f, f\mathrm{e}^{2i\theta}; q)(ab, ac, bc; q)_\infty} F_n(\mathfrak{a}; f, \sigma|q), \tag{79}$$

where

$$E_n(\mathfrak{a}; f, \sigma|q) = \int_{-\pi}^{\pi} \frac{((f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta})\frac{\sigma}{z}, (f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta}, abc)\frac{z}{\sigma}; q)_\infty}{(\mathrm{e}^{\pm i\theta}\frac{\sigma}{z}, (a, b)\frac{z}{\sigma}; q)_\infty} \times \left(c\frac{\sigma}{z}; q\right)_n \left(\frac{z}{\sigma}\right)^n \mathrm{d}\psi, \tag{80}$$

$$F_n(\mathfrak{a}; f, \sigma|q) = \int_{-\pi}^{\pi} \frac{((f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta})\frac{\sigma}{z}, (f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta}, abc)\frac{z}{\sigma}; q)_\infty}{(\mathrm{e}^{\pm i\theta}\frac{\sigma}{z}, (a, b, c)\frac{z}{\sigma}; q)_\infty (abc\frac{z}{\sigma}; q)_n} \left(\frac{z}{\sigma}\right)^n \mathrm{d}\psi \tag{81}$$

$$= \int_{-\pi}^{\pi} \frac{((fabc\,\mathrm{e}^{i\theta}, \frac{q}{f}abc\,\mathrm{e}^{-i\theta})\frac{\sigma}{z}, (f\frac{1}{abc}\mathrm{e}^{i\theta}, \frac{q}{f}\frac{1}{abc}\mathrm{e}^{-i\theta}, 1)\frac{z}{\sigma}; q)_\infty}{(abc\,\mathrm{e}^{\pm i\theta}\frac{\sigma}{z}, (\frac{1}{ab}, \frac{1}{ac}, \frac{1}{bc})\frac{z}{\sigma}; q)_\infty (\frac{z}{\sigma}; q)_n} \times \left(\frac{1}{abc}\frac{z}{\sigma}\right)^n \mathrm{d}\psi. \tag{82}$$

Proof Starting with (68), letting $d \mapsto 0$ produces (81), and taking $c \mapsto 0$ followed by $d \mapsto c$ produces (80). Starting with (69), letting $d \mapsto 0$ produces (82). This completes the proof. □

Remark 3.5 The continuous dual q-Hahn polynomials are symmetric in the three parameters a, b, and c. The symmetry in the parameters is evident in the integral representation (82).

Note that starting with (69) and taking either a, b or $c \mapsto 0$ does not yield a finite result.

3.2.2 Generating functions for the continuous dual q-Hahn polynomials

There are several generating functions known for the continuous dual q-Hahn polynomials. Some of them follow by taking the $a_4 \to 0$ limit for generating functions of the Askey–Wilson polynomials. One such example which hasn't appeared frequently in the literature is the $a_4 \to 0$ limit of the Rahman generating function (72). This is given as follows.

Corollary 3.6 *Let* $k, p \in \{1, 2, 3\}$, $\mathbf{a} := \{a_1, a_2, a_3\}$, $a_k, t \in \mathbb{C}^*$, $x = \cos\theta \in [-1, 1]$, $q \in \mathbb{C}^\dagger$, $|t| < 1$. *Then*

$$\sum_{n=0}^{\infty} \frac{p_n(x; \mathbf{a}|q)}{(q, \{a_p a_s\}_{s\neq p}; q)_n} t^n = \frac{1}{(ta_p^{-1}; q)_\infty} {}_4\phi_3\left(\begin{matrix} a_p \mathrm{e}^{\pm i\theta}, 0, 0 \\ \{a_p a_s\}_{s\neq p}, q a_p t^{-1} \end{matrix}; q, q\right) + \frac{(\{ta_s\}_{s\neq p}, a_p \mathrm{e}^{\pm i\theta}; q)_\infty}{(\{a_p a_s\}_{s\neq p}, a_p t^{-1}, t\mathrm{e}^{\pm i\theta}; q)_\infty} {}_4\phi_3\left(\begin{matrix} t\mathrm{e}^{\pm i\theta}, 0, 0 \\ \{ta_s\}_{s\neq p}, qta_p^{-1} \end{matrix}; q, q\right). \tag{83}$$

A non-standard generating function for continuous dual q-Hahn polynomials was presented in [3, (3.5)]. We will show how integral representations for continuous dual q-Hahn polynomials lead to an easy proof of this formula.

Theorem 3.7 (Atakishiyeva and Atakishiyev [3]) *Let* $a, b, c \in \mathbb{C}^*$, $q \in \mathbb{C}^\dagger$, $t \in \mathbb{C}$, *such that* $|t| < 1$. *Then*

$$\sum_{n=0}^{\infty} \frac{t^n p_n(x; \mathfrak{a}|q)}{(q, tabc; q)_n} = \frac{(ta, tb, tc; q)_\infty}{(tabc, t\mathrm{e}^{\pm i\theta}; q)_\infty}. \tag{84}$$

Proof Starting with the left-hand side of (84) and inserting (81), one obtains

$$\sum_{n=0}^{\infty} \frac{t^n p_n(x; \mathfrak{a}|q)}{(q, tabc; q)_n} = \frac{(q, a\mathrm{e}^{\pm i\theta}, b\mathrm{e}^{\pm i\theta}, c\mathrm{e}^{\pm i\theta}; q)_\infty}{\vartheta(f, f\mathrm{e}^{2i\theta}; q)(ab, ac, bc; q)_\infty} \times \int_{-\pi}^{\pi} \frac{((f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta})\frac{\sigma}{z}, (f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta}, abc)\frac{z}{\sigma}; q)_\infty}{(\mathrm{e}^{\pm i\theta}\frac{\sigma}{z}, (a, b, c)\frac{z}{\sigma}; q)_\infty} \times {}_3\phi_2\left(\begin{matrix} ab, ac, bc \\ abc\frac{z}{\sigma}, tabc \end{matrix}; q, \frac{tz}{\sigma}\right) \mathrm{d}\psi, \tag{85}$$

where $z = \mathrm{e}^{i\psi}$. Using [7, (III.34)], we can write the ${}_3\phi_2$ in the integrand as a sum of two ${}_3\phi_2$'s with argument q. Then using (40) we can express it as an integral representation, namely

$$ {}_3\phi_2\left(\begin{matrix} ab, ac, bc \\ abc\frac{z}{\sigma}, tabc \end{matrix}; q, \frac{tz}{\sigma}\right) = \frac{(q, tb, tc, ab, ac, \frac{az}{\sigma}; q)_\infty}{2\pi\vartheta(f, f\frac{a}{t}; q)(tabc, abc\frac{z}{\sigma}; q)_\infty} \int_{-\pi}^{\pi} \frac{((f\sqrt{\frac{a}{t}}, \frac{q}{f}\sqrt{\frac{t}{a}})\frac{\tau}{w}, (f\sqrt{\frac{a}{t}}, \frac{q}{f}\sqrt{\frac{t}{a}}, \sqrt{ta}\, bc\frac{z}{\sigma})\frac{w}{\tau}; q)_\infty}{((\sqrt{\frac{a}{t}}, \sqrt{\frac{t}{a}})\frac{\tau}{w}, (\sqrt{ta}\, b, \sqrt{ta}\, c, \sqrt{ta}\,\frac{z}{\sigma})\frac{w}{\tau}; q)_\infty} \mathrm{d}\eta, \tag{86}$$

where $w = \mathrm{e}^{i\eta}$, and none of the arguments of the modified theta functions are equal to some q^m, $m \in \mathbb{Z}$. Inserting the integral representation (86) into the right-hand side of (85) and using Gasper's integral (32) completes the proof. □

For continuous dual Hahn polynomials [13, Section 9.3] $S_n(x^2;\mathfrak{a})$, there is a generating function which until recently there has been no known q-analogue for. This is the generating function [13, (9.3.16)]

$$\sum_{n=0}^{\infty}\frac{(\gamma)_n S_n(x^2;\mathfrak{a})}{n!(a+b,a+c)_n}t^n=(1-t)^{-\gamma}\,{}_3F_2\left(\begin{matrix}\gamma, a\pm ix\\ a+b, a+c\end{matrix};\frac{t}{t-1}\right), \tag{87}$$

where γ is a free parameter, $|t|<1$, $|t|<|1-t|$. Using the integral representation method we may readily compute the following q-analogue which we present now.

Theorem 3.8 *Let $q\in\mathbb{C}^{\dagger}$, $\gamma\in\mathbb{C}$, $t,a,b,c\in\mathbb{C}^*$, $|t|<1$. Then one has the following generating function for continuous dual q-Hahn polynomials:*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\gamma;q)_n p_n(x;\mathfrak{a}|q)}{(q,ab,ac;q)_n}t^n&=\frac{(a\mathrm{e}^{\pm i\theta};q)_\infty}{(ab,ac;q)_\infty}\\&\quad\times\left(\frac{(ab,ac,\gamma t/a;q)_\infty}{(a\mathrm{e}^{\pm i\theta},t/a;q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\gamma, a\mathrm{e}^{\pm i\theta},0\\ ab,ac,qa/t\end{matrix};q,q\right)\right.\\&\qquad\left.+\frac{(tb,tc,\gamma;q)_\infty}{(t\mathrm{e}^{\pm i\theta},a/t;q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\gamma t/a, t\mathrm{e}^{\pm i\theta},0\\ tb,tc,qt/a\end{matrix};q,q\right)\right).\end{aligned} \tag{88}$$

Proof Start with the left-hand side of (88) and insert (82). This produces

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\gamma;q)_n t^n p_n(x;\mathfrak{a}|q)}{(q,ab,ac;q)_n}&=\frac{(q,a\mathrm{e}^{\pm i\theta},b\mathrm{e}^{\pm i\theta},c\mathrm{e}^{\pm i\theta};q)_\infty}{2\pi\vartheta(f,f\mathrm{e}^{2i\theta};q)(ab,ac,bc;q)_\infty}\\&\times\int_{-\pi}^{\pi}\frac{((fabc\,\mathrm{e}^{i\theta},\frac{q}{f}abc\,\mathrm{e}^{-i\theta})\frac{\sigma}{z},(f\frac{1}{abc}\mathrm{e}^{i\theta},\frac{q}{f}\frac{1}{abc}\mathrm{e}^{-i\theta},1)\frac{z}{\sigma};q)_\infty}{(abc\,\mathrm{e}^{\pm i\theta}\frac{\sigma}{z},(\frac{1}{ab},\frac{1}{ac},\frac{1}{bc})\frac{z}{\sigma};q)_\infty}\\&\quad\times{}_2\phi_1\left(\begin{matrix}\gamma, bc\\ \frac{z}{\sigma}\end{matrix};q,\frac{tz}{abc\sigma}\right)\mathrm{d}\psi.\end{aligned} \tag{89}$$

The ${}_2\phi_1$ can be written either as a sum of two ${}_2\phi_2^{-1}$'s [15, (17.9.3)] or as a sum of two ${}_3\phi_1^{1}$'s [15, (17.9.3_5)]. We use (53) which corresponds to the expansion of a ${}_2\phi_1$ with a ${}_3\phi_2$ with one vanishing numerator parameter. Comparing this sum using (40) produces

$$\begin{aligned}{}_2\phi_1\left(\begin{matrix}\gamma, bc\\ \frac{z}{\sigma}\end{matrix};q,\frac{tz}{abc\sigma}\right)&=\frac{(q,\gamma,\gamma\frac{t}{a},\frac{z}{bc\sigma};q)_\infty}{2\pi\vartheta(h,h\frac{a}{t};q)(\frac{z}{\sigma};q)_\infty}\\&\times\int_{-\pi}^{\pi}\frac{((h\sqrt{\frac{a}{t}},\frac{q}{h}\sqrt{\frac{t}{a}})\frac{\tau}{w},(h\sqrt{\frac{a}{t}},\frac{q}{h}\sqrt{\frac{t}{a}},\sqrt{\frac{t}{a}\frac{z}{\sigma}})\frac{w}{\tau};q)_\infty}{((\sqrt{\frac{a}{t}},\sqrt{\frac{t}{a}})\frac{\tau}{w},(\gamma\sqrt{\frac{t}{a}},\sqrt{\frac{t}{a}\frac{z}{bc\sigma}})\frac{w}{\tau};q)_\infty}\mathrm{d}\eta,\end{aligned} \tag{90}$$

where $w = e^{i\eta}$, and none of the arguments of the modified theta functions are equal to some q^m, $m \in \mathbb{Z}$. Inserting the above integral representation, rearranging the integrals and evaluating the outer integral using Gasper's integral (32) completes the proof. □

If one lets $\gamma \to 0$ in (88) then one obtains (83). Furthermore, if you let $a_3 \to 0$ you produce the following well-known generating function for Al-Salam–Chihara polynomials [13, (14.8.16)]:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\gamma;q)_n t^n p_n(x;\mathfrak{a}|q)}{(q,ab;q)_n} &= \frac{(ae^{\pm i\theta};q)_\infty}{(ab;q)_\infty} \\ &\quad\times \left(\frac{(ab,\gamma t/a;q)_\infty}{(ae^{\pm i\theta},t/a;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} \gamma, ae^{\pm i\theta} \\ ab, qa/t \end{matrix}; q, q \right) \right. \\ &\qquad \left. + \frac{(tb,\gamma;q)_\infty}{(te^{\pm i\theta},a/t;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} \gamma t/a, te^{\pm i\theta} \\ tb, qt/a \end{matrix}; q, q \right) \right) \\ &= \frac{(\gamma te^{i\theta};q)_\infty}{(te^{i\theta};q)_\infty}\, {}_3\phi_2\left(\begin{matrix} \gamma, ae^{i\theta}, be^{i\theta} \\ ab, \gamma te^{i\theta} \end{matrix}; q, te^{-i\theta} \right), \end{aligned} \tag{91}$$

where the second equality is obtained by using the nonterminating basic hypergeometric series transformation [7, (III.34)].

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