The Ramanujan Journal (2022) 58:815–834
https://doi.org/10.1007/s11139-021-00479-8

Congruences for the coefficients of the Gordon and McIntosh mock theta function $\xi(q)$

Robson da Silva[1] · **James A. Sellers**[2]

Received: 15 January 2021 / Accepted: 26 June 2021 / Published online: 26 August 2021

Abstract
Recently Gordon and McIntosh introduced the third order mock theta function $\xi(q)$ defined by

$$\xi(q) = 1 + 2\sum_{n=1}^{\infty} \frac{q^{6n^2-6n+1}}{(q;q^6)_n(q^5;q^6)_n}.$$

Our goal in this paper is to study arithmetic properties of the coefficients of this function. We present a number of such properties, including several infinite families of Ramanujan-like congruences.

Keywords Congruence · Generating function · Mock theta function

Mathematics Subject Classification 11P83 · 05A17

1 Introduction

In his last letter to Hardy in 1920, Ramanujan introduced the notion of a mock theta function. He listed 17 such functions having orders 3, 5, and 7. Since then, other mock theta functions have been found. Gordon and McIntosh [8], for example, introduced

Robson da Silva was supported by São Paulo Research Foundation (FAPESP) (Grant No. 2019/14796-8).

✉ Robson da Silva
silva.robson@unifesp.br

James A. Sellers
jsellers@d.umn.edu

1 Universidade Federal de São Paulo, Av. Cesare M. G. Lattes, 1201, São José dos Campos, SP 12247-014, Brazil

2 Department of Mathematics and Statistics, University of Minnesota Duluth, Duluth, MN 55812, USA

many additional such functions, including the following of order 3:

$$\xi(q) = 1 + 2\sum_{n=1}^{\infty} \frac{q^{6n^2-6n+1}}{(q;q^6)_n(q^5;q^6)_n}, \tag{1}$$

where we use the standard q-series notation:

$$\begin{aligned}(a;q)_0 &= 1,\\ (a;q)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad \forall n \geq 1,\\ (a;q)_\infty &= \lim_{n\to\infty}(a;q)_n, \quad |q|<1.\end{aligned}$$

Arithmetic properties of the coefficients of mock theta functions have received a great deal of attention. For instance, Zhang and Shi [15] recently proved seven congruences satisfied by the coefficients of the mock theta function $\beta(q)$ introduced by McIntosh. In a recent paper, Brietzke et al. [5] found a number of arithmetic properties satisfied by the coefficients of the mock theta function $V_0(q)$, introduced by Gordon and McIntosh [7]. Andrews et al. [2] prove a number of congruences for the partition functions $p_\omega(n)$ and $p_\nu(n)$, introduced in [1], associated with the third order mock theta functions $\omega(q)$ and $\nu(q)$, where $\omega(q)$ is defined below and

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}.$$

In a subsequent paper, Wang [14] presented some additional congruences for both $p_\omega(n)$ and $p_\nu(n)$.

This paper is devoted to exploring arithmetic properties of the coefficients $p_\xi(n)$ defined by

$$\sum_{n=0}^{\infty} p_\xi(n)q^n = \xi(q). \tag{2}$$

It is clear from (1) that $p_\xi(n)$ is even for all $n \geq 1$. In Sects. 4 and 5, we present other arithmetic properties of $p_\xi(n)$, including some infinite families of congruences.

2 Preliminaries

McIntosh [12, Theorem 3] proved a number of mock theta conjectures, including

$$\omega(q) = g_3(q,q^2) \quad \text{and} \tag{3}$$

$$\xi(q) = q^2 g_3(q^3,q^6) + \frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^2(q^6;q^6)_\infty}, \tag{4}$$

where

$$g_3(a,q)=\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n+1)/2}}{(a;q)_{n+1}(a^{-1}q;q)_{n+1}}$$

and $\omega(q)$ is the third order mock theta functions given by

$$\omega(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}.$$

It follows from (1), (3), and (4) that

$$\xi(q)=q^2\omega(q^3)+\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^2(q^6;q^6)_\infty}. \tag{5}$$

Throughout the remainder of this paper, we define

$$f_k:=(q^k;q^k)_\infty$$

in order to shorten the notation. Combining (5) and (2), we have

$$\sum_{n=0}^{\infty}p_\xi(n)q^n=q^2\omega(q^3)+\frac{f_2^4}{f_1^2 f_6}. \tag{6}$$

We recall Ramanujan's theta functions

$$f(a,b):=\sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}} \text{ for } |ab|<1,$$

$$\phi(q):=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}=\frac{f_2^5}{f_1^2f_4^2}, \quad \text{and} \tag{7}$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{f_2^2}{f_1}. \tag{8}$$

The function $\phi(q)$ satisfies many identities, including (see [3, (22.4)])

$$\phi(-q)=\frac{f_1^2}{f_2}. \tag{9}$$

In some of the proofs, we employ the classical Jacobi's identity (see [4, Theorem 1.3.9])

$$f_1^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{10}$$

We note the following identities which will be used below.

Lemma 1 *The following 2-dissection identities hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{11}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{12}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{13}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{14}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{15}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2} \tag{16}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{17}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}} \tag{18}$$

Proof By Entry 25 (i), (ii), (v), and (vi) in [3, p. 40], we have

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \tag{19}$$

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2. \tag{20}$$

Using (7) and (8) we can rewrite (19) in the form

$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_8^5}{f_4^2 f_{16}^2} + 2q\frac{f_{16}^2}{f_8},$$

from which we obtain (11) after multiplying both sides by $\frac{f_4^2}{f_2^5}$. Identity (12) can be easily deduced from (11) using the procedure described in Section 30.10 of [9].

By (7) and (8) we can rewrite (20) in the form

$$\frac{f_2^{10}}{f_1^4 f_4^4} = \frac{f_4^{10}}{f_2^4 f_8^4} + 4q\frac{f_8^4}{f_4^2},$$

from which we obtain (13).

Identities (14), (15), and (18) are equations (30.10.3), (30.9.9), and (30.12.3) of [9], respectively. Finally, for proofs of (16) and (17) see [13, Lemma 4]. □

The next lemma exhibits the 3-dissections of $\psi(q)$ and $1/\phi(-q)$.

Lemma 2 *We have*

$$\psi(q) = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}, \tag{21}$$

$$\frac{1}{\phi(-q)} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{22}$$

Proof Identity (21) is Eq. (14.3.3) of [9]. A proof of (22) can be seen in [10]. □

3 Dissections for $p_\xi(n)$

This section is devoted to proving the 2-, 3-, and 4-dissections of (2). We begin with the 2-dissection.

Theorem 1 *We have*

$$2\sum_{n=0}^{\infty} p_\xi(2n+1)q^{n+1} = \frac{f_6^6 f_{12}}{f_3^4 f_{24}^2} - f(q^{12}) + 4q\frac{f_2^2 f_8^2}{f_1 f_3 f_4}, \quad \textit{and} \tag{23}$$

$$\sum_{n=0}^{\infty} p_\xi(2n)q^n = q\frac{f_6^8 f_{24}^2}{f_3^4 f_{12}^5} - q^4\omega(-q^6) + \frac{f_4^5}{f_1 f_3 f_8^2}. \tag{24}$$

Proof We start with equation (4) of [2]:

$$f(q^8) + 2q\omega(q) + 2q^3\omega(-q^4) = F(q),$$

where $f(q)$ is the mock theta function

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}$$

and

$$F(q) = \frac{\phi(q)\phi(q^2)^2}{f_4^2} = \frac{f_2 f_4^6}{f_1^2 f_8^4}.$$

Thus,

$$f(q^{24}) + 2q^3\omega(q^3) + 2q^9\omega(-q^{12}) = F(q^3).$$

Using (5), it follows that

$$2\sum_{n=0}^{\infty} p_{\xi}(n)q^{n+1} = F(q^3) - f(q^{24}) - 2q^9\omega(-q^{12}) + 2q\frac{f_2^4}{f_1^2 f_6}. \tag{25}$$

By (11), we have

$$F(q^3) = \frac{f_{12}^6 f_{24}}{f_6^4 f_{48}^2} + 2q^3\frac{f_{12}^8 f_{48}^2}{f_6^4 f_{24}^5},$$

which along with (11) allows us to rewrite (25) as

$$2\sum_{n=0}^{\infty} p_{\xi}(n)q^{n+1} = \frac{f_{12}^6 f_{24}}{f_6^4 f_{48}^2} + 2q^3\frac{f_{12}^8 f_{48}^2}{f_6^4 f_{24}^5} - f(q^{24}) - 2q^9\omega(-q^{12}) + 2q\frac{f_8^5}{f_2 f_6 f_{16}^2} + 4q^2\frac{f_4^2 f_{16}^2}{f_2 f_6 f_8}.$$

Thus,

$$2\sum_{n=0}^{\infty} p_{\xi}(2n+1)q^{2n+2} = \frac{f_{12}^6 f_{24}}{f_6^4 f_{48}^2} - f(q^{24}) + 4q^2\frac{f_4^2 f_{16}^2}{f_2 f_6 f_8}, \quad \text{and} \tag{26}$$

$$\sum_{n=0}^{\infty} p_{\xi}(2n)q^{2n+1} = q^3\frac{f_{12}^8 f_{48}^2}{f_6^4 f_{24}^5} - q^9\omega(-q^{12}) + q\frac{f_8^5}{f_2 f_6 f_{16}^2}. \tag{27}$$

Dividing (27) by q and replacing q^2 by q in the resulting identity and in (26), we obtain (23) and (24). □

The next theorem exhibits the 3-dissection of (2).

Theorem 2 *We have*

$$\sum_{n=0}^{\infty} p_{\xi}(3n)q^n = \frac{f_2 f_3^4}{f_1^2 f_6^2}, \tag{28}$$

$$\sum_{n=0}^{\infty} p_{\xi}(3n+1)q^n = 2\frac{f_3 f_6}{f_1}, \quad \textit{and} \tag{29}$$

$$\sum_{n=0}^{\infty} p_{\xi}(3n+2)q^n = \omega(q) + \frac{f_6^4}{f_2 f_3^2}. \tag{30}$$

Proof In view of (8), we rewrite (6) as

$$\sum_{n=0}^{\infty} p_{\xi}(n)q^n = q^2\omega(q^3) + \frac{\psi(q)^2}{f_6}.$$

Using (21), we obtain

$$\sum_{n=0}^{\infty} p_\xi(n)q^n = q^2\omega(q^3) + \frac{f_6 f_9^4}{f_3^2 f_{18}^2} + 2q\frac{f_9 f_{18}}{f_3} + q^2\frac{f_{18}^4}{f_6 f_9^2}. \tag{31}$$

Extracting the terms of the form q^{3n+r} on both sides of (31), for $r \in \{0, 1, 2\}$, dividing both sides of the resulting identity by q^r and then replacing q^3 by q, we obtain the desired results. □

We close this section with the 4-dissection of (2).

Theorem 3 *We have*

$$\sum_{n=0}^{\infty} p_\xi(4n)q^n = 4q^2\frac{f_{12}^6}{f_3^2 f_6^3} - q^2\omega(-q^3) + \frac{f_2^4 f_6^5}{f_1^2 f_3^4 f_{12}^2}, \tag{32}$$

$$\sum_{n=0}^{\infty} p_\xi(4n+1)q^n = 2q\frac{f_6^3 f_{12}^2}{f_3^4} + 2\frac{f_4^4 f_6^5}{f_2^2 f_3^4 f_{12}^2}, \tag{33}$$

$$\sum_{n=0}^{\infty} p_\xi(4n+2)q^n = \frac{f_6^9}{f_3^6 f_{12}^2} + \frac{f_2^{10} f_{12}^2}{f_1^4 f_3^2 f_4^4 f_6}, \text{ and} \tag{34}$$

$$2\sum_{n=0}^{\infty} p_\xi(4n+3)q^{n+1} = \frac{f_6^{15}}{f_3^8 f_{12}^6} - f(q^6) + 4q\frac{f_2^4 f_{12}^2}{f_1^2 f_3^2 f_6}. \tag{35}$$

Proof In order to prove (32), we use (13) and (18) to obtain the even part of (24), which is given by

$$\sum_{n=0}^{\infty} p_\xi(4n)q^{2n} = 4q^4\frac{f_{24}^6}{f_6^2 f_{12}^3} - q^4\omega(-q^6) + \frac{f_4^4 f_{12}^5}{f_2^2 f_6^4 f_{24}^2}.$$

Replacing q^2 by q we obtain (32).

Using (13) and (18) we can extract the odd part of (23):

$$2\sum_{n=0}^{\infty} p_\xi(4n+1)q^{2n+1} = 4q^3\frac{f_{12}^3 f_{24}^2}{f_6^4} + 4q\frac{f_8^4 f_{12}^5}{f_4^2 f_6^4 f_{24}^2}.$$

After simplifications we arrive at (33).

Next, extracting the odd part of (24) with the help of (13) and (18) yields

$$\sum_{n=0}^{\infty} p_\xi(4n+2)q^{2n+1} = q\frac{f_{12}^9}{f_6^6 f_{24}^2} + q\frac{f_4^{10} f_{24}^2}{f_2^4 f_6^2 f_8^4 f_{12}},$$

which, after simplifications, gives us (34).

In order to obtain (35), we use (13) and (18) in (23) to extract its even part:

$$2\sum_{n=0}^{\infty} p_\xi(4n+3)q^{2n+2} = \frac{f_{12}^{15}}{f_6^8 f_{24}^6} - f(q^{12}) + 4q^2 \frac{f_4^4 f_{24}^2}{f_2^2 f_6^2 f_{12}}.$$

Replacing q^2 by q in this identity, we obtain (35). □

4 Arithmetic properties of $p_\xi(n)$

Our first observation provides a characterization of $p_\xi(3n) \pmod 4$.

Theorem 4 *For all $n \geq 0$, we have*

$$p_\xi(3n) \equiv \begin{cases} 1 \pmod 4 & \textit{if } n = 0, \\ 2 \pmod 4 & \textit{if } n \textit{ is a square}, \\ 0 \pmod 4 & \textit{otherwise}. \end{cases}$$

Proof By (28), using (9) and the fact that $f_k^4 \equiv f_{2k}^2 \pmod 4$ for all $k \geq 1$, it follows that

$$\sum_{n=0}^{\infty} p_\xi(3n)q^n = \frac{f_2 f_3^4}{f_1^2 f_6^2} \equiv \frac{f_2}{f_1^2} = \frac{f_1^2 f_2}{f_1^4} \equiv \frac{f_1^2}{f_2} = \phi(-q) \pmod 4.$$

By (7), we obtain

$$\sum_{n=0}^{\infty} p_\xi(3n)q^n \equiv \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \equiv 1 + 2\sum_{n=1}^{\infty} q^{n^2} \pmod 4,$$

which completes the proof. □

Theorem 4 yields an infinite family of Ramanujan-like congruences modulo 4.

Corollary 1 *For all primes $p \geq 3$ and all $n \geq 0$, we have*

$$p_\xi(3(pn+r)) \equiv 0 \pmod 4,$$

if r is a quadratic nonresidue modulo p.

Proof If $pn + r = k^2$, then $r \equiv k^2 \pmod p$, which contradicts the fact that r is a quadratic nonresidue modulo p. □

Since $\gcd(3, p) = 1$, among the $p-1$ residues modulo p, we have $\frac{p-1}{2}$ residues r for which r is a quadratic nonresidue modulo p. Thus, for instance, the above corollary yields the following congruences:

$$\begin{aligned} p_\xi(9n+6) &\equiv 0 \pmod 4, \\ p_\xi(15n+k) &\equiv 0 \pmod 4, \ \text{for } k \in \{6, 9\}, \\ p_\xi(21n+k) &\equiv 0 \pmod 4, \ \text{for } k \in \{9, 15, 18\}, \\ p_\xi(33n+k) &\equiv 0 \pmod 4, \ \text{for } k \in \{6, 18, 21, 24, 30\}. \end{aligned}$$

Theorem 5 *For all $n \geq 0$, we have*

$$p_\xi(3n+1) \equiv \begin{cases} 2 \pmod 4 & \text{if } 3n+1 \text{ is a square}, \\ 0 \pmod 4 & \text{otherwise}. \end{cases}$$

Proof From Theorem 2,

$$\sum_{n=0}^{\infty} p_\xi(3n+1)q^n = 2\frac{f_3 f_6}{f_1}. \tag{36}$$

So we only need to consider the parity of

$$\frac{f_3 f_6}{f_1}.$$

Note that

$$\frac{f_3 f_6}{f_1} \equiv \frac{f_3^3}{f_1} = \sum_{n=0}^{\infty} a_3(n)q^n \pmod 2,$$

where $a_3(n)$ is the number of 3-core partitions of n (see [11, Theorem 1]). Thanks to [6, Theorem 7], we know that

$$a_3(n) \equiv \begin{cases} 1 \pmod 2 & \text{if } 3n+1 \text{ is a square}, \\ 0 \pmod 2 & \text{otherwise}. \end{cases}$$

This completes the proof. □

Theorem 5 yields an infinite family of congruences modulo 4.

Corollary 2 *For all primes $p > 3$ and all $n \geq 0$, we have*

$$p_\xi(3(pn+r)+1) \equiv 0 \pmod 4,$$

if $3r+1$ is a quadratic nonresidue modulo p.

Proof If $3(pn+r)+1=k^2$, then $3r+1 \equiv k^2 \pmod{p}$, which would be a contradiction with $3r+1$ being a quadratic nonresidue modulo p. □

For example, the following congruences hold for all $n \geq 0$:

$$\begin{aligned} p_\xi(15n+k) &\equiv 0 \pmod{4} \text{ for } k \in \{7, 13\}, \\ p_\xi(21n+k) &\equiv 0 \pmod{4} \text{ for } k \in \{10, 13, 19\}, \\ p_\xi(33n+k) &\equiv 0 \pmod{4} \text{ for } k \in \{7, 10, 13, 19, 28\}. \end{aligned}$$

We next turn our attention to the arithmetic progression $4n+2$ to yield an additional infinite family of congruences.

Theorem 6 *For all $n \geq 0$, we have*

$$p_\xi(4n+2) \equiv \begin{cases} 2 \pmod{4} & \text{if } n = 6k(3k \pm 1), \\ 0 \pmod{4} & \text{otherwise}. \end{cases}$$

Proof From (34), we obtain

$$\sum_{n=0}^{\infty} p_\xi(4n+2)q^n \equiv \frac{f_6^7}{f_3^2 f_{12}^2} + \frac{f_{12}^2}{f_3^2 f_6} \equiv 2\frac{f_6^3}{f_3^2} \equiv 2f_6^2 \equiv 2f_{12} \pmod{4}. \tag{37}$$

Using Euler's identity (see [9, Eq. (1.6.1)])

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}, \tag{38}$$

we obtain

$$\sum_{n=0}^{\infty} p_\xi(4n+2)q^n \equiv 2 \sum_{n=-\infty}^{\infty} (-1)^n q^{6n(3n-1)} \pmod{4},$$

which concludes the proof. □

Theorem 6 yields an infinite family of congruences modulo 4.

Corollary 3 *Let $p > 3$ be a prime and r an integer such that $2r+1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$,*

$$p_\xi(4(pn+r)+2) \equiv 0 \pmod{4}.$$

Proof If $pn+r = 6k(3k \pm 1)$, then $r \equiv 18k^2 \pm 6k \pmod{p}$. Thus, $2r+1 \equiv (6k \pm 1)^2 \pmod{p}$, which contradicts the fact that $2r+1$ is a quadratic nonresidue modulo p. □

Thanks to Corollary 3, the following example congruences hold for all $n \geq 0$:

$$\begin{aligned} p_\xi(20n+j) &\equiv 0 \pmod 4 \quad \text{for} \quad j \in \{6, 14\}, \\ p_\xi(28n+j) &\equiv 0 \pmod 4 \quad \text{for} \quad j \in \{6, 10, 26\}, \\ p_\xi(44n+j) &\equiv 0 \pmod 4 \quad \text{for} \quad j \in \{14, 26, 34, 38, 42\}, \\ p_\xi(52n+j) &\equiv 0 \pmod 4 \quad \text{for} \quad j \in \{10, 14, 22, 30, 38, 42\}. \end{aligned}$$

We now provide a mod 8 characterization for $p_\xi(3n)$.

Theorem 7 *For all $n \geq 0$, we have*

$$p_\xi(3n) \equiv \begin{cases} 1 \pmod 8 & \textit{if } n = 0, \\ 6(-1)^k \pmod 8 & \textit{if } n = k^2, \\ 4 \pmod 8 & \textit{if } n = 2k^2, n = 3k^2, \textit{ or } n = 6k^2, \\ 0 \pmod 8 & \textit{otherwise}. \end{cases}$$

Proof By (28), using (7) and (9), we have

$$\begin{aligned} \sum_{n=0}^{\infty} p_\xi(3n)q^n &= \frac{f_1^6 f_2 f_3^4}{f_1^8 f_6^2} \equiv \left(\frac{f_1^2}{f_2}\right)^3 \left(\frac{f_3^2}{f_6}\right)^2 \equiv \phi(-q)^3 \phi(-q^3)^2 \\ &\equiv \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}\right)^3 \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2}\right)^2 \\ &\equiv \left(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\right)^3 \left(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{3n^2}\right)^2 \pmod 8, \end{aligned}$$

which yields

$$\begin{aligned} \sum_{n=0}^{\infty} p_\xi(3n)q^n \equiv 1 + 6\sum_{n=1}^{\infty}(-1)^n q^{n^2} + 4\left(\sum_{n=1}^{\infty}(-1)^n q^{n^2}\right)^2 \\ + 4\sum_{n=1}^{\infty}(-1)^n q^{3n^2} + 4\left(\sum_{n=1}^{\infty}(-1)^n q^{3n^2}\right)^2 \pmod 8. \end{aligned}$$

Since

$$\left(\sum_{n=1}^{\infty}(-1)^n q^{n^2}\right)^2 \equiv \sum_{n=1}^{\infty} q^{2n^2} \pmod 2,$$

we have

$$\left(\sum_{n=1}^{\infty}(-1)^n q^{3n^2}\right)^2 \equiv \sum_{n=1}^{\infty} q^{6n^2} \pmod{2}.$$

Therefore

$$\sum_{n=0}^{\infty} p_{\xi}(3n)q^n \equiv 1 + 6\sum_{n=1}^{\infty}(-1)^n q^{n^2} + 4\sum_{n=1}^{\infty} q^{2n^2} + 4\sum_{n=1}^{\infty}(-1)^n q^{3n^2} + 4\sum_{n=1}^{\infty} q^{6n^2} \pmod{8},$$

which completes the proof. □

As with the prior results, Theorem 7 provides an effective way to yield an infinite family of congruences modulo 8.

Corollary 4 *Let p be a prime such that $p \equiv \pm 1 \pmod{24}$. Then*

$$p_{\xi}(3(pn+r)) \equiv 0 \pmod{8},$$

if r is a quadratic nonresidue modulo p.

Proof Since $p \equiv \pm 1 \pmod{8}$ and $p \equiv \pm 1 \pmod{12}$, it follows that 2 and 3 are quadratic residues modulo p. Thus, r, $2r$, $3r$, and $6r$ are quadratic nonresidues modulo p. Indeed, according to the properties of Legendre's symbol, for $j \in \{1, 2, 3, 6\}$, we have

$$\left(\frac{jr}{p}\right) = \left(\frac{j}{p}\right)\left(\frac{r}{p}\right) = \left(\frac{r}{p}\right) = -1.$$

It follows that we cannot have $3(pn+r) = jk^2$, for some $k \in \mathbb{N}$ and $j \in \{1, 2, 3, 6\}$. In fact, $3(pn+r) = jk^2$ would imply $3(pn+r) \equiv 3r \equiv jk^2 \pmod{p}$. However, for $j = 1, 2, 3, 6$, this would imply that $3r$, $6r$, r, or $2r$, respectively, is a quadratic residue modulo p, which would be a contradiction since 2, 3, and 6 are quadratic residues modulo p. The result follows from Theorem 7. □

As an example, we note that, for $p = 23$ and all $n \geq 0$, we have

$$p_{\xi}(69n+k) \equiv 0 \pmod{8} \text{ for } k \in \{15, 21, 30, 33, 42, 45, 51, 57, 60, 63, 66\}.$$

Theorem 8 *For all $n \geq 0$, we have*

$$p_{\xi}(12n+4) \equiv p_{\xi}(3n+1) \pmod{8}.$$

Proof Initially we use (14) to extract the odd part on both sides of (29). The resulting identity is

$$\sum_{n=0}^{\infty} p_{\xi}(6n+4)q^n = 2\frac{f_3^2 f_4^2 f_{24}}{f_1^2 f_8 f_{12}}. \tag{39}$$

Using (15) in (39), we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(12n+4)q^n = 2\frac{f_2^6 f_3 f_6}{f_1^5 f_4^2} = 2\frac{f_1^3 f_2^6 f_3 f_6}{f_1^8 f_4^2} \equiv 2\frac{f_3 f_6}{f_1} \pmod{8}.$$

The result follows using (29). □

Now we present complete characterizations of $p_{\xi}(48n+4)$ and $p_{\xi}(12n+1)$ modulo 8.

Theorem 9 *For all* $n \geq 0$, *we have*

$$p_{\xi}(48n+4) \equiv p_{\xi}(12n+1) \equiv \begin{cases} 2(-1)^k \pmod{8} & \text{if } n = k(3k \pm 1), \\ 0 \pmod{8} & \text{otherwise.} \end{cases}$$

Proof The first congruence follows directly from Theorem 8. Replacing (14) in (29), we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(3n+1)q^n = 2\frac{f_4 f_6^2 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + 2q\frac{f_6^2 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}.$$

Extracting the terms of the form q^{2n}, we have

$$\sum_{n=0}^{\infty} p_{\xi}(6n+1)q^{2n} = 2\frac{f_4 f_6^2 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}},$$

which, after replacing q^2 by q, yields

$$\sum_{n=0}^{\infty} p_{\xi}(6n+1)q^n = 2\frac{f_2 f_3^2 f_8 f_{12}^2}{f_1^2 f_4 f_6 f_{24}}. \tag{40}$$

Now we use (15) to obtain

$$\sum_{n=0}^{\infty} p_{\xi}(12n+1)q^n = 2\frac{f_2^3 f_6^4}{f_1^4 f_{12}^2}$$
$$\equiv 2f_2 \equiv 2\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)} \pmod{8} \quad \text{(by (38))},$$

which completes the proof. □

Theorem 9 also provides an effective way to yield an infinite family of congruences modulo 8.

Corollary 5 *For all primes $p > 3$ and all $n \geq 0$, we have*

$$p_\xi(48(pn+r)+4) \equiv p_\xi(12(pn+r)+1) \equiv 0 \pmod 8$$

if $12r+1$ is a quadratic nonresidue modulo p.

Proof Let $p > 3$ be a prime and $12r+1$ a quadratic nonresidue modulo p. If $pn+r = k(3k\pm 1)$, then $r \equiv 3k^2 \pm k \pmod p$, which implies that $12r+1 \equiv (6k\pm 1)^2 \pmod p$, a contradiction. The result follows from Theorem 9. □

5 Additional congruences

In this section, we prove several additional Ramanujan-like congruences that are not included in the results of the previous section.

Theorem 10 *For all $n \geq 0$, we have*

$$p_\xi(24n+19) \equiv 0 \pmod 3, \tag{41}$$
$$p_\xi(27n+18) \equiv 0 \pmod 3, \text{ and} \tag{42}$$
$$p_\xi(72n+51) \equiv 0 \pmod 3. \tag{43}$$

Proof Using (15) we can now 2-dissect (40) to obtain

$$\sum_{n=0}^{\infty} p_\xi(6n+1)q^n = 2\frac{f_4^3 f_{12}^4}{f_2^4 f_{24}^2} + 4q\frac{f_6 f_8^2 f_{12}}{f_2^3},$$

from which we have

$$\sum_{n=0}^{\infty} p_\xi(12n+7)q^{2n+1} = 4q\frac{f_6 f_8^2 f_{12}}{f_2^3}.$$

Now, dividing both sides of the above expression by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} p_\xi(12n+7)q^n = 4\frac{f_3 f_4^2 f_6}{f_1^3}. \tag{44}$$

Using (17) we rewrite (44) as

$$\sum_{n=0}^{\infty} p_\xi(12n+7)q^n = 4\frac{f_4^8 f_6^4}{f_2^9 f_{12}^2} + 12q\frac{f_4^4 f_6^2 f_{12}^2}{f_2^7}.$$

Taking the odd parts on both sides of the last equation, we are left with

$$\sum_{n=0}^{\infty} p_\xi(24n+19)q^n = 12\frac{f_2^4 f_3^2 f_6^2}{f_1^7},$$

which proves (41).

In order to prove (42), we use (22) to extract the terms of the form q^{3n} of (28). The resulting identity is

$$\sum_{n=0}^{\infty} p_\xi(9n)q^{3n} = \frac{f_6^2 f_9^6}{f_3^4 f_{18}^3},$$

which, after replacing q^3 by q and using (8), yields

$$\sum_{n=0}^{\infty} p_\xi(9n)q^n = \frac{f_2^2 f_3^6}{f_1^4 f_6^3} \equiv \frac{f_2^2 f_3^5}{f_1 f_6^3} = \psi(q)\frac{f_3^5}{f_6^3} \pmod 3.$$

By (8), we have

$$\sum_{n=0}^{\infty} p_\xi(9n)q^n \equiv \frac{f_3^5}{f_6^3}\sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod 3.$$

Since $n(n+1)/2 \not\equiv 2 \pmod 3$ for all $n \geq 0$, all terms of the form q^{3n+2} in the last expression have coefficients congruent to $0 \pmod 3$, which proves (42).

We now prove (43). Replacing (22) in (28) and extracting the terms of the form q^{3n+2}, we obtain

$$\sum_{n=0}^{\infty} p_\xi(9n+6)q^{3n+2} = 4q^2\frac{f_{18}^3}{f_3^2}. \tag{45}$$

Dividing both sides of (45) by q^2 and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} p_\xi(9n+6)q^n = 4\frac{f_6^3}{f_1^2}. \tag{46}$$

Now we use (11) to extract the odd part of (46) and obtain

$$\sum_{n=0}^{\infty} p_\xi(18n+15)q^n = 8\frac{f_2^2 f_3^3 f_8^2}{f_1^5 f_4}.$$

Since $f_1^3 \equiv f_3 \pmod 3$, we have

$$\sum_{n=0}^{\infty} p_\xi(18n+15)q^n \equiv 2\frac{f_2^2 f_3^2 f_8^2}{f_1^2 f_4} \pmod 3.$$

Using (15) we obtain

$$\sum_{n=0}^{\infty} p_\xi(36n+15)q^n \equiv 2\frac{f_2^3 f_3 f_4 f_6^2}{f_1^3 f_{12}} \pmod 3.$$

Since the odd part of (17) is divisible by 3, then the coefficients of the terms of the form q^{2n+1} in $\sum_{n=0}^{\infty} p_\xi(36n+15)q^n$ are congruent to 0 modulo 3. This completes the proof of (43). □

We now prove a pair of unexpected congruences modulo 5 satisfied by $p_\xi(n)$.

Theorem 11 *For all $n \geq 0$, we have*

$$p_\xi(45n+33) \equiv 0 \pmod 5, \tag{47}$$
$$p_\xi(45n+42) \equiv 0 \pmod 5. \tag{48}$$

Proof By (46), we have

$$\sum_{n=0}^{\infty} p_\xi(9n+6)q^n = 4\frac{f_6^3}{f_1^2} = 4\frac{f_1^3 f_6^3}{f_1^5} \equiv 4\frac{f_1^3 f_6^3}{f_5} \pmod 5.$$

Thanks to Jacobi's identity (10) we know

$$f_1^3 f_6^3 = \sum_{j,k=0}^{\infty} (-1)^{j+k}(2j+1)(2k+1)q^{3j(j+1)+k(k+1)/2}.$$

Note that, for all integers j and k, $3j(j+1)$ and $k(k+1)/2$ are congruent to either 0, 1 or 3 modulo 5. The only way to obtain $3j(j+1)+k(k+1)/2 = 5n+3$ is the following:

- $3j(j+1) \equiv 0 \pmod 5$ and $k(k+1)/2 \equiv 3 \pmod 5$, or
- $3j(j+1) \equiv 3 \pmod 5$ and $k(k+1)/2 \equiv 0 \pmod 5$.

Thus, $j \equiv 2 \pmod 5$ or $k \equiv 2 \pmod 5$ in all possible cases, and this means

$$(2j+1)(2k+1) \equiv 0 \pmod 5.$$

Therefore, for all $n \geq 0$, $p_\xi(45n+33) = p_\xi(9(5n+3)+6) \equiv 0 \pmod 5$, which is (47).

In order to complete the proof of (48), we want to see when

$$3j(j+1) + k(k+1)/2 = 5n+4.$$

Four possible cases arise:

- $k \equiv 1 \pmod 5$ and $j \equiv 2 \pmod 5$,
- $k \equiv 3 \pmod 5$ and $j \equiv 2 \pmod 5$,
- $j \equiv 1 \pmod 5$ and $k \equiv 2 \pmod 5$, or
- $j \equiv 3 \pmod 5$ and $k \equiv 2 \pmod 5$.

In all four cases above, either $j \equiv 2 \pmod 5$ or $k \equiv 2 \pmod 5$. So

$$(2j+1)(2k+1) \equiv 0 \pmod 5$$

in all these cases. Therefore,

$$p_\xi(45n+42) = p_\xi(9(5n+4)+6) \equiv 0 \pmod 5,$$

which completes the proof of (48). □

Next, we prove three congruences modulo 8 which are not covered by the above results.

Theorem 12 *For all $n \geq 0$, we have*

$$p_\xi(16n+14) \equiv 0 \pmod 8, \tag{49}$$
$$p_\xi(24n+13) \equiv 0 \pmod 8, \tag{50}$$
$$p_\xi(24n+22) \equiv 0 \pmod 8. \tag{51}$$

Proof Initially we prove (49). From (34) and (7) we have

$$\sum_{n=0}^{\infty} p_\xi(4n+2)q^n \equiv \frac{f_3^2 f_6^5}{f_{12}^2} + \frac{f_{12}^2}{f_3^2 f_6}\phi(q)^2 \pmod 8.$$

Now we can use (11), (12), and (20) to extract the terms involving q^{2n+1} from both sides of the previous congruence:

$$\sum_{n=0}^{\infty} p_\xi(8n+6)q^{2n+1} \equiv -2q^3 \frac{f_6^6 f_{48}^2}{f_{12}^2 f_{24}} + 2q^3 \frac{f_4^{10} f_{12}^4 f_{48}^2}{f_2^4 f_6^6 f_8^4 f_{24}} + 4q \frac{f_8^4 f_{12}^2 f_{24}^5}{f_4^2 f_6^6 f_{48}^2} \pmod 8.$$

After dividing both sides by q and then replacing q^2 by q, we are left with

$$\begin{aligned}\sum_{n=0}^{\infty} p_{\xi}(8n+6)q^n &\equiv -2q\frac{f_3^6 f_{24}^2}{f_6^2 f_{12}} + 2q\frac{f_2^{10} f_6^4 f_{24}^2}{f_1^4 f_3^6 f_4^4 f_{12}} + 4\frac{f_4^4 f_6^2 f_{12}^5}{f_2^2 f_3^6 f_{24}^2} \\ &\equiv -2q\frac{f_3^6 f_{24}^2}{f_6^2 f_{12}} + 2q\frac{f_3^6 f_6^4 f_{24}^2}{f_3^{12} f_{12}} + 4\frac{f_4^4 f_{12}^5}{f_2^2 f_6 f_{24}^2} \\ &\equiv 4\frac{f_4^3 f_{12}}{f_6} \pmod{8},\end{aligned}$$

whose odd part is congruent to 0 modulo 8, which implies (49).

In order to prove (50), we use (15) to obtain the even part of identity (40), which is

$$\sum_{n=0}^{\infty} p_{\xi}(12n+1)q^n = 2\frac{f_2^3 f_6^4}{f_1^4 f_{12}^2}.$$

Now, employing (13), we obtain the odd part of the last identity, which is

$$\sum_{n=0}^{\infty} p_{\xi}(24n+13)q^n = 8\frac{f_2^2 f_3^4 f_4^4}{f_1^7 f_6^2},$$

which implies (50).

Now we prove (51). We employ (15) in (39) to obtain

$$\sum_{n=0}^{\infty} p_{\xi}(12n+10)q^n = 4\frac{f_2^3 f_3^2 f_{12}^2}{f_1^4 f_6^2}. \tag{52}$$

By (12) and (13), we rewrite (52) in the form

$$\sum_{n=0}^{\infty} p_{\xi}(12n+10)q^n = 4\frac{f_2^3 f_{12}^2}{f_6^2}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)\left(\frac{f_6 f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_6 f_{48}^2}{f_{24}}\right),$$

from which we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(24n+22)q^{2n+1} = 4\frac{f_2^3 f_{12}^2}{f_6^2}\left(-2q^3\frac{f_4^{14} f_6 f_{48}^2}{f_2^{14} f_8^4 f_{24}} + 4q\frac{f_4^2 f_6 f_8^4 f_{24}^5}{f_2^{10} f_{12}^2 f_{48}^2}\right).$$

Dividing both sides by q and replacing q^2 by q, we are left with

$$\sum_{n=0}^{\infty} p_{\xi}(24n+22)q^n = -8q\frac{f_2^{14} f_6^2 f_{24}^2}{f_1^{11} f_3 f_4^4 f_{12}} + 16\frac{f_2^2 f_4^4 f_{12}^5}{f_1^7 f_3 f_{24}^2},$$

which implies (51). □

We close this section by proving a congruence modulo 9.

Theorem 13 *For all $n \geq 0$, we have*

$$p_{\xi}(96n+76) \equiv 0 \pmod 9. \tag{53}$$

Proof We use (21) to extract the terms of the form q^{3n+1} from (32). The resulting identity is

$$\sum_{n=0}^{\infty} p_{\xi}(12n+4)q^{3n+1} = 2q\frac{f_6^6 f_9 f_{18}}{f_3^5 f_{12}^2},$$

which, after dividing by q and replacing q^3 by q, yields

$$\sum_{n=0}^{\infty} p_{\xi}(12n+4)q^{n} = 2\frac{f_2^6 f_3 f_6}{f_1^5 f_4^2} = 2\frac{f_2^6 f_6}{f_4^2}\frac{f_3}{f_1}\frac{1}{f_1^4}.$$

Using (13) and (14), we extract the even part on both sides of the above identity to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_{\xi}(24n+4)q^{n} &= 2\frac{f_2^{13} f_3^2 f_8 f_{12}^2}{f_1^{10} f_4^5 f_6 f_{24}} + 8q\frac{f_3^2 f_4^6 f_{24}}{f_1^6 f_8 f_{12}} \\ &\equiv 2\frac{f_2^{13} f_8 f_{12}^2}{f_4^5 f_6 f_{24}}\frac{1}{f_1 f_3} + 8q\frac{f_4^6 f_{24}}{f_8 f_{12}}\frac{f_1^3}{f_3} \pmod 9.\end{aligned}$$

Now we employ (18) and (16) to extract the odd part on both sides of the last congruence:

$$\sum_{n=0}^{\infty} p_{\xi}(48n+28)q^{n} \equiv 2\frac{f_1^9 f_6 f_{12}}{f_3^3 f_4} + 8\frac{f_2^9 f_{12}}{f_4 f_6^2} \equiv \frac{f_6 f_{12}}{f_4} \pmod 9,$$

which implies (53). □

6 Concluding remarks

Computational evidence indicates that $p_{\xi}(n)$ satisfies many other congruences. The interested reader may wish to consider the following two conjectures.

Conjecture 1

$$\sum_{n=0}^{\infty} p_{\xi}(8n+3)q^{n} \equiv 2\sum_{n=0}^{\infty} q^{3n(n+1)/2} \pmod 3$$

Conjecture 2

$$\sum_{n=0}^{\infty} p_\xi(32n+12)q^n \equiv 6\sum_{n=0}^{\infty} q^{3n(n+1)/2} \pmod 9)$$

Clearly, once proven, Conjectures 1 and 2 would immediately lead to infinite families of Ramanujan-like congruences. Morever, Conjecture 2 would immediately imply Theorem 13 since $96n+76 = 32(3n+2)+12$ while the right-hand side of Conjecture 2 is clearly a function of q^3. The same argument would imply that, for all $n \geq 0$,

$$p_\xi(96n+44) \equiv 0 \pmod 9).$$

since $96n + 44 = 32(3n + 1) + 12$.

Acknowledgements The authors thank the anonymous referee for carefully reading the manuscript and his/her helpful comments and suggestions.

References

1. Andrews, G.E., Dixit, A., Yee, A.J.: Partitions associated with the Ramanujan/Watson mock theta functions $\omega(q)$ and $\phi(q)$. Res. Number Theory **1**, 1–19 (2015)
2. Andrews, G.E., Passary, D., Sellers, J.A., Yee, A.J.: Congruences related to the Ramanujan/Watson mock theta functions $\omega(q)$ and $\nu(q)$. Ramanujan J. **43**, 347–357 (2017)
3. Berndt, B.C.: Ramanujan's Notebooks. Part III, Springer, New York (1991)
4. Berndt, B.C.: Number Theory in the Spirit of Ramanujan. American Mathematical Society, Providence (2006)
5. Brietzke, E.H.M., da Silva, R., Sellers, J.A.: Congruences related to an eighth order mock theta function of Gordon and McIntosh. J. Math. Anal. Appl. **479**, 62–89 (2019)
6. da Silva, R., Sellers, J.A.: Parity Considerations for the Mex-Related Partition Functions of Andrews and Newman, J. Integer Seq. **23**, Article 20.5.7 (2020)
7. Gordon, B., McIntosh, R.J.: Some eight order mock theta functions. J. Lond. Math. Soc. **62**, 321–335 (2000)
8. Gordon, B., McIntosh, R.J.: Modular transformations of Ramanujan's fifth and seventh order mock theta functions. Ramanujan J. **7**, 193–222 (2003)
9. Hirschhorn, M.D.: The Power of q, a Personal Journey. Developments in Mathematics, vol. 49. Springer, Cham (2017)
10. Hirschhorn, M.D., Sellers, J.A.: Arithmetic relations for overpartitions. J. Combin. Math. Combin. Comput. **53**, 65–73 (2005)
11. Hirschhorn, M.D., Sellers, J.A.: Elementary proofs of various facts about 3-cores. Bull. Aust. Math. Soc. **79**, 507–512 (2009)
12. McIntosh, R.J.: New mock theta conjectures Part I. Ramanujan J. **46**, 593–604 (2018)
13. Naika, M.S.M., Gireesh, D.S.: Congruences for 3-regular partitions with designated summands. Integers **16**, A25 (2016)
14. Wang, L.: New congruences for partitions related to mock theta functions. J. Number Theory **175**, 51–65 (2017)
15. Zhang, W., Shi, J.: Congruences for the coefficients of the mock theta function $\beta(q)$. Ramanujan J. **49**, 257–267 (2019)