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# Exterior square gamma factors for cuspidal representations of $\mathrm{GL}_n$: simple supercuspidal representations

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**Abstract**
We compute the local twisted exterior square gamma factors for simple supercuspidal representations, using which we prove a local converse theorem for simple supercuspidal representations.

**Keywords** Gamma factors · Local converse theorem

**Mathematics Subject Classification** 22E50 · 11F66

## 1 Introduction

A local conjecture of Jacquet for $\mathrm{GL}_n(F)$, where $F$ is a local non-Archimedean field, asserts that the structure of an irreducible generic representation can be determined by a family of twisted Rankin–Selberg gamma factors. This conjecture was completely settled independently by Chai [4] and Jacquet–Liu [9], using different methods:

**Theorem 1.1** (Chai [4], Jacquet–Liu [9]) *Let $\pi_1$ and $\pi_2$ be irreducible generic representations of $\mathrm{GL}_n(F)$ sharing the same central character. Suppose for any $1 \le r \le \left\lfloor \frac{n}{2} \right\rfloor$ and for any irreducible generic representation $\tau$ of $\mathrm{GL}_r(F)$,*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi).$$

*Then $\pi_1 \cong \pi_2$.*

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The bound $\lfloor \frac{n}{2} \rfloor$ for $r$ in the theorem can be shown to be sharp by constructing some pairs of generic representations. However, the sharpness of $\lfloor \frac{n}{2} \rfloor$ is not that obvious if we replace "generic" by "unitarizable supercuspidal" in the theorem. In the tame case, it is shown in [2] that $\lfloor \frac{n}{2} \rfloor$ is indeed sharp for unitarizable supercuspidal representations of $\mathrm{GL}_n(F)$ when $n$ is prime. For some certain families of supercuspidal representations, $\lfloor \frac{n}{2} \rfloor$ is no longer sharp and the $\mathrm{GL}_1(F)$ twisted Rankin–Selberg gamma factors might be enough to determine the structures of representations within these families. Such a family of supercuspidal representations can be a family of simple supercuspidal representations, see [3, Proposition 2.2] and [1, Remark 3.18], and also be a family of level zero supercuspidal representations for certain $n$, see [15, Section 4.6].

In this paper, we consider another kind of local converse theorems of Ramakrishnan using twisted exterior power gamma factors from [16].

**Conjecture 1.2** (Ramakrishnan) *Let $\pi_1$ and $\pi_2$ be irreducible unitarizable supercuspidal representations of $\mathrm{GL}_n(F)$ sharing the same central character. Suppose for any character $\chi$ of $F^*$, we have*

$$\gamma(s, \pi_1 \times \chi, \psi) = \gamma(s, \pi_2 \times \chi, \psi),$$

*and*

$$\gamma(s, \pi_1, \wedge^j \otimes \chi, \psi) = \gamma(s, \pi_2, \wedge^j \otimes \chi, \psi),$$

*for any $2 \le j \le \lfloor \frac{n}{2} \rfloor$. Then $\pi_1 \cong \pi_2$.*

We note here that the condition on sharing the same central character is redundant, since if $\gamma(s, \pi_1 \times \chi, \psi) = \gamma(s, \pi_2 \times \chi, \psi)$ for all characters $\chi$, this guarantees that the representations have the same central character as in [10, Corollary 2.7]. We leave it in the statement of the conjecture as a general requirement for a local converse problem. In fact, in the formulation of the main result Theorem 1.3, we will need to assume that the representations in consideration share the same central character.

When $j = 2$, the twisted exterior square gamma factors of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ exist due to the work of Jacquet–Shalika [8] together with Matringe [14] and Cogdell-Matringe [5], or the work of Shahidi [18] using the Langlands–Shahidi method. When $j = 3$, Ginzburg and Rallis [6] found an integral representation for the automorphic $L$-function $L(s, \pi, \wedge^3 \otimes \chi)$ attached to an irreducible cuspidal automorphic representation $\pi$ of $\mathrm{GL}_6(\mathbb{A})$ and a character $\chi$ of $\mathrm{GL}_1(\mathbb{A})$ for some adelic ring $\mathbb{A}$. In general, for $j \ge 3$, we do not have an analytic definition for $\gamma(s, \pi, \wedge^j \otimes \chi, \psi)$. Therefore, Theorem 1.2 only makes sense for $n = 4, 5$ and possibly 6 if one can prove local functional equations for the local integrals coming from [6].

Since we have only twisted exterior square gamma factors in general, we want to know which families of supercuspidal representations of $\mathrm{GL}_n(F)$ satisfy Theorem 1.2 when $j = 2$. We show in the paper that Theorem 1.2 holds true for simple supercuspidal representations up to a sign as we will explain in the next paragraph. This result is our first step toward Theorem 1.2. We have already seen that $\mathrm{GL}_1(F)$ twists are enough to distinguish simple supercuspidal representations, see [3, Proposition 2.2] and [1, Remark 3.18]. Thus, Theorem 1.2 for simple supercuspidal representations has no context if we still require $\mathrm{GL}_1(F)$ twists. Therefore, we will drop the assumption on the $\mathrm{GL}_1(F)$ twists.

Let $\mathfrak{o}$ be the ring of integers of $F$, and $\mathfrak{p} = (\varpi)$ is the maximal prime ideal in $\mathfrak{o}$ generated by a fixed uniformizer $\varpi$. By [12], if we fix a tamely ramified central character $\omega$, i.e., $\omega$ is trivial on $1+\mathfrak{p}$, there are exactly $n(q-1)$ isomorphism classes of irreducible simple supercuspidal representations of $\mathrm{GL}_n(F)$, each of which corresponds to a pair $(t_0, \zeta)$, where $t_0 \in \mathfrak{f}^*$ is a non-zero element in the residue field $\mathfrak{f}$ of $F$ and $\zeta$ is an $n$-th root of $\omega(t^{-1}\varpi)$, where $t$ is a lift of $t_0$ to $\mathfrak{o}^*$. The main theorem of the paper is the following:

**Theorem 1.3** *Let $\pi$ and $\pi'$ be irreducible simple supercuspidal representations of $\mathrm{GL}_n(F)$ sharing the same central character $\omega$, such that $\pi$ and $\pi'$ are associated with the data $(t_0, \zeta)$ and $(t_0', \zeta')$, respectively. Assume that*

1. $\gcd(m-1, q-1) = 1$ *if* $n = 2m$,
2. *or* $\gcd(m, q-1) = 1$ *if* $n = 2m+1$.

*Suppose for every unitary tamely ramified character $\mu$ of $F^*$, we have*

$$\gamma(s, \pi, \wedge^2 \otimes \mu, \psi) = \gamma(s, \pi', \wedge^2 \otimes \mu, \psi).$$

*Then $t_0 = t_0'$ and $\zeta = \pm\zeta'$. Moreover, we have $\zeta = \zeta'$ if $n = 2m+1$ is odd.*

In the case $n = 2m$, we can only show $\zeta$ and $\zeta'$ are equal up to a sign. That is what we mean by saying that Theorem 1.2 holds true for simple supercuspidal representations up to a sign. Theorem 1.3, as far as we know, is the first result toward Theorem 1.2 of Ramakrishnan.

In Sect. 2, we recall the definitions of the twisted exterior square gamma factors following [5,8,14]. We then recall some results on simple supercuspidal representations in [12]. More importantly from [1, Section 3.3], we have explicit Whittaker functions for such simple supercuspidal representations. Using these explicit Whittaker functions, we compute in Sect. 3 the twisted exterior square gamma factors. Finally in Sect. 4, we prove our main theorem, Theorem 1.3.

## 2 Preliminaries and notation

### 2.1 Notation

Let $F$ be a non-archimedean local field. We denote by $\mathfrak{o}$ its ring of integers, by $\mathfrak{p}$ the unique prime ideal of $\mathfrak{o}$, by $\mathfrak{f} = \mathfrak{o}/\mathfrak{p}$ its residue field. Denote $q = |\mathfrak{f}|$.

Let $\nu : \mathfrak{o} \to \mathfrak{f}$ be the quotient map. We continue denoting by $\nu$ the maps that $\nu$ induces on various groups, for example $\mathfrak{o}^m \to \mathfrak{f}^m$, $M_m(\mathfrak{o}) \to M_m(\mathfrak{f})$, $\mathrm{GL}_m(\mathfrak{o}) \to \mathrm{GL}_m(\mathfrak{f})$ etc.

Let $\varpi$ be a uniformizer (a generator of $\mathfrak{p}$). We denote by $|\cdot|$, the absolute value on $F$, normalized such that $|\varpi| = \frac{1}{q}$.

Let $\psi : F \to \mathbb{C}^*$ be a non-trivial additive character with conductor $\mathfrak{p}$, i.e., $\psi$ is trivial on $\mathfrak{p}$, but not on $\mathfrak{o}$.

### 2.2 The twisted Jacquet–Shalika integral

In this section, we define twisted versions of the Jacquet–Shalika integrals and discuss the functional equations that they satisfy. This will allow us to define the twisted exterior square gamma factor $\gamma\left(s, \pi, \wedge^2 \otimes \mu, \psi\right)$ for a generic representation $(\pi, V_\pi)$ of $\mathrm{GL}_n(F)$ and a unitary character $\mu : F^* \to \mathbb{C}^*$. We will need this for our local converse theorem in Sect. 4.

We denote $N_m$ the upper unipotent subgroup of $\mathrm{GL}_m(F)$, $A_m$ the diagonal subgroup of $\mathrm{GL}_m(F)$, $K_m = \mathrm{GL}_m(\mathfrak{o})$, $\mathcal{B}_m$ the upper triangular matrix subspace of $M_m(F)$, and $\mathcal{N}_m^-$ the lower triangular nilpotent matrix subspace of $M_m(F)$. We have $\mathcal{B}_m \backslash M_m(F) \cong \mathcal{N}_m^-$.

For the following pairs of groups $A \leq B$, we normalize the Haar measure on $B$ so that the compact open subgroup $A$ has measure one: $\mathfrak{o} \leq F$, $\mathfrak{o}^* \leq F^*$, $K_m = \mathrm{GL}_m(\mathfrak{o}) \leq \mathrm{GL}_m(F)$, $\mathcal{N}_m^-(\mathfrak{o}) \leq \mathcal{N}_m^-(F)$.

Recall the Iwasawa decomposition: $\mathrm{GL}_m(F) = N_m A_m K_m$. It follows from this decomposition that for an integrable function $f : N_m \backslash \mathrm{GL}_m(F) \to \mathbb{C}$, we have

$$\int_{N_m \backslash \mathrm{GL}_m(F)} f(g)\, d^\times g = \int_{K_m} \int_{A_m} f(ak)\, \delta_{B_m}^{-1}(a)\, d^\times a d^\times k,$$

where $B_m \leq \mathrm{GL}_m(F)$ is the Borel subgroup, and for $a = \mathrm{diag}(a_1, \dots, a_m)$, $\delta_{B_m}^{-1}(a) = \prod_{1 \leq i < j \leq m} \left|\frac{a_j}{a_i}\right|$ is the Haar measure module character.

Let $(\pi, V_\pi)$ be an irreducible generic representation of $\mathrm{GL}_n(F)$ and denote its Whittaker model with respect to $\psi$ by $\mathcal{W}(\pi, \psi)$. Let $\mu : F^* \to \mathbb{C}^*$ be a unitary character. We now define the twisted Jacquet–Shalika integrals and their duals. These initially should be thought as formal integrals. We discuss their convergence domains later and explain how to interpret them for arbitrary $s \in \mathbb{C}$.

We have a map $\mathcal{W}(\pi, \psi) \to \mathcal{W}\left(\widetilde{\pi}, \psi^{-1}\right)$, denoted by $W \mapsto \widetilde{W}$, where $\widetilde{\pi}$ is the contragredient representation, and $\widetilde{W}$ is given by $\widetilde{W}(g) = W(w_n g^\iota)$, where $w_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ and $g^\iota = {}^t g^{-1}$.

Denote by $\mathcal{S}(F^m)$ the space of Schwartz functions $\phi : F^m \to \mathbb{C}$ that is the space of locally constant functions with compact support.

Suppose $n = 2m$. We define for $s \in \mathbb{C}$, $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F^m)$

$$J(s, W, \phi, \mu, \psi) = \int_{N_m \backslash \mathrm{GL}_m(F)} \int_{\mathcal{B}_m \backslash M_m(F)} W\left(\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right) \psi(-\mathrm{tr} X) \cdot |\det g|^s \mu(\det g)\, \phi(\varepsilon g)\, dX d^\times g,$$

where $\varepsilon = \varepsilon_m = \begin{pmatrix} 0 \dots 0 \ 1 \end{pmatrix} \in F^m$, and $\sigma_{2m}$ is the column permutation matrix corresponding to the permutation

$$\begin{pmatrix} 1 \ 2 \dots & m & | \ m+1 \ m+2 \dots 2m \\ 1 \ 3 \dots & 2m-1 & | \quad 2 \qquad 4 \quad \dots 2m \end{pmatrix},$$

i.e.,

$$\sigma_{2m} = \begin{pmatrix} e_1 \ e_3 \ \dots \ e_{2m-1} \ e_2 \ e_4 \ \dots \ e_{2m} \end{pmatrix},$$

where $e_i$ is the $i$-th standard column vector, for $1 \le i \le 2m$. In this case, we define the dual Jacquet–Shalika as

$$\tilde{J}(s, W, \phi, \mu, \psi) = J\left(1-s, \widetilde{\pi}\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\widetilde{W}, \mathcal{F}_\psi \phi, \mu^{-1}, \psi^{-1}\right),$$

where $J$ on the right-hand side is the Jacquet–Shalika integral for $\widetilde{\pi}$, and

$$\mathcal{F}_\psi \phi(y) = q^{\frac{m}{2}} \int_{F^m} \phi(x)\, \psi(\langle x, y \rangle)\, dx$$

is the Fourier transform, normalized such that it is self-dual (here $\langle \cdot, \cdot \rangle$ is the standard bilinear form on $F^m$).

Suppose $n = 2m+1$. We define for $s \in \mathbb{C}$, $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F^m)$

$$\begin{aligned} J(s, W, \phi, \mu, \psi) = & \int_{N_m \backslash \mathrm{GL}_m(F)} \int_{M_{1\times m}(F)} \int_{\mathcal{B}_m \backslash M_m(F)} \\ & \cdot W\left(\sigma_{2m+1}\begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1 \end{pmatrix}\begin{pmatrix} I_m & & \\ & I_m & \\ & Z & 1 \end{pmatrix}\right) \\ & \cdot \psi(-\mathrm{tr} X)\, |\det g|^{s-1}\, \mu(\det g)\, \phi(Z)\, dX dZ d^\times g, \end{aligned}$$

where $\sigma_{2m+1}$ is the column permutation matrix corresponding to the permutation

$$\begin{pmatrix} 1 \ 2 \dots m & | \ m+1 \ m+2 \dots 2m \ | \ 2m+1 \\ 1 \ 3 \dots 2m-1 & | \ 2 \qquad 4 \qquad \dots 2m \ | \ 2m+1 \end{pmatrix},$$

i.e.,

$$\sigma_{2m+1} = \begin{pmatrix} e_1 \ e_3 \ \dots \ e_{2m-1} \ e_2 \ e_4 \ \dots \ e_{2m} \ e_{2m+1} \end{pmatrix}.$$

In this case, we define the dual Jacquet–Shalika as follows:

$$\tilde{J}(s, W, \phi, \mu, \psi) = J\left(1-s, \widetilde{\pi}\begin{pmatrix} & I_m & \\ I_m & & \\ & & 1 \end{pmatrix}\widetilde{W}, \mathcal{F}_\psi \phi, \mu^{-1}, \psi^{-1}\right).$$

In both cases, we denote $J(s, W, \phi, \psi) = J(s, W, \phi, 1, \psi)$ and $\tilde{J}(s, W, \phi, \psi) = \tilde{J}(s, W, \phi, 1, \psi)$, where 1 denotes the trivial character $F^* \to \mathbb{C}^*$.

The definitions of the twisted Jacquet–Shalika integrals are motivated from [5,8,14]. We now list properties of the twisted Jacquet–Shalika integrals, some of which are only proven for the (untwisted) Jacquet–Shalika integrals in the literature.

From now and on suppose $n = 2m$ or $n = 2m + 1$.

**Theorem 2.1** ([8, Section 7, Proposition 1, Section 9, Proposition 3]) *There exists $r_{\pi,\wedge^2} \in \mathbb{R}$, such that for every $s \in \mathbb{C}$ with $\operatorname{Re}(s) > r_{\pi,\wedge^2}$, $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(F^m)$, the integral $J(s, W, \phi, \mu, \psi)$ converges absolutely.*

Similarly, there exists a half left plane $\operatorname{Re}(s) < r_{\widetilde{\pi},\wedge^2}$ (where $r_{\widetilde{\pi},\wedge^2} = 1 - r_{\pi,\wedge^2}$), in which the dual twisted Jacquet–Shalika integrals $\tilde{J}(s, W, \phi, \mu, \psi)$ converge absolutely for every $W, \phi$.

**Theorem 2.2** ([11, Theorem 2.3], [5, Lemma 3.1]) *For fixed $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F^m)$. The map $s \mapsto J(s, W, \phi, \mu, \psi)$ for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > r_{\pi,\wedge^2}$ results in an element of $\mathbb{C}\left(q^{-s}\right)$ that is a rational function in the variable $q^{-s}$ and, therefore, has a meromorphic continuation to the entire plane, which we continue to denote as $J(s, W, \phi, \mu, \psi)$. Similarly, we continue to denote the meromorphic continuation of $\tilde{J}(s, W, \phi, \mu, \psi)$ by the same symbol. Furthermore, denote*

$$I_{\pi,\psi,\mu} = \operatorname{span}_{\mathbb{C}} \left\{ J(s, W, \phi, \mu, \psi) \mid W \in \mathcal{W}(\pi, \psi), \phi \in \mathcal{S}\left(F^m\right) \right\},$$

*then there exists a unique element $p(Z) \in \mathbb{C}[Z]$, such that $p(0) = 1$ and $I_{\pi,\psi,\mu} = \frac{1}{p(q^{-s})}\mathbb{C}\left[q^{-s}, q^s\right]$. $p(Z)$ does not depend on $\psi$, and we denote $L\left(s, \pi, \wedge^2 \otimes \mu\right) = \frac{1}{p(q^{-s})}$.*

**Proposition 2.3** ([19, Proposition 3.2])

1. *For $n = 2m$,*

$$\tilde{J}(s, W, \phi, \mu, \psi) = \int_{N_m \backslash \mathrm{GL}_m(F)} \int_{\mathcal{B}_m \backslash M_m(F)} W\left(\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right) \cdot \psi(-\operatorname{tr} X) \cdot |\det g|^{s-1} \mu(\det g) \mathcal{F}_\psi \phi\left(\varepsilon_1 g^\iota\right) dX d^\times g,$$

*where $\varepsilon_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$.*

2. *For $n = 2m + 1$,*

$$\begin{aligned} &\tilde{J}(s, W, \phi, \mu, \psi) \\ &= \int_{N_m \backslash \mathrm{GL}_m(F)} \int_{M_{1\times m}(F)} \int_{\mathcal{B}_m \backslash M_m(F)} \\ &\cdot W\left(\begin{pmatrix} & 1 \\ I_{2m} & \end{pmatrix} \sigma_{2m+1} \begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix} \begin{pmatrix} g & & \\ & g & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_m & & -{}^tZ \\ & I_m & \\ & & 1 \end{pmatrix}\right) \\ &\cdot \psi(-\operatorname{tr} X) |\det g|^s \mu(\det g) \mathcal{F}_\psi \phi(Z)\, dX dZ d^\times g. \end{aligned}$$

**Theorem 2.4** ([14, Theorem 4.1] [5, Theorem 3.1]) *There exists a non-zero element* $\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)\in\mathbb{C}\left(q^{-s}\right)$*, such that for every* $W\in\mathcal{W}(\pi,\psi)$*,* $\phi\in\mathcal{S}(F^m)$*, we have*

$$\tilde{J}(s,W,\phi,\mu,\psi)=\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)J(s,W,\phi,\mu,\psi).$$

*Furthermore,*

$$\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=\epsilon\left(s,\pi,\wedge^2\otimes\mu,\psi\right)\frac{L\left(1-s,\widetilde{\pi},\wedge^2\otimes\mu^{-1}\right)}{L\left(s,\pi,\wedge^2\otimes\mu\right)},$$

*where* $\epsilon\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=c\cdot q^{-ks}$*, for* $k\in\mathbb{Z}$ *and* $c\in\mathbb{C}^*$*.*

The proof of the functional equation is very similar to the proofs of the referred theorems and requires only slight modifications.

As before, we denote $L\left(s,\pi,\wedge^2\right)=L\left(s,\pi,\wedge^2\otimes 1\right)$, $\gamma\left(s,\pi,\wedge^2,\psi\right)=\gamma\left(s,\pi,\wedge^2\otimes 1,\psi\right)$, and $\epsilon\left(s,\pi,\wedge^2,\psi\right)=\epsilon\left(s,\pi,\wedge^2\otimes 1,\psi\right)$.

**Theorem 2.5** *Suppose that* $(\pi,V_\pi)$ *is an irreducible supercuspidal representation of* $\mathrm{GL}_n(F)$

1. *If* $n=2m+1$*, then* $L\left(s,\pi,\wedge^2\otimes\mu\right)=1$*.*
2. *If* $n=2m$*, then* $L\left(s,\pi,\wedge^2\otimes\mu\right)=\frac{1}{p\left(q^{-s}\right)}$*, where* $p(Z)\in\mathbb{C}[Z]$ *is a polynomial, such that* $p(0)=1$ *and* $p(Z)\mid 1-\omega_\pi(\varpi)\mu(\varpi)^m Z^m$*.*

The proof of this statement is very similar to the proof of [11, Theorem 3.6]. Its proof uses a slight modification of [11, Proposition 3.4] for the twisted Jacquet–Shalika integral. See also [8, Section 8, Theorem 1], [8, Section 9, Theorem 2] for the analogous global statements.

**Lemma 2.6** *Let* $n=2m$*. Suppose that* $\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=c\cdot q^{-ks}\frac{p_1\left(q^{-s}\right)}{p_2\left(q^{-(1-s)}\right)}$*, where* $c\in\mathbb{C}^*$*,* $k\in\mathbb{Z}$*,* $p_1,p_2\in\mathbb{C}[Z]$*, such that* $p_1(0)=p_2(0)=1$ *and* $p_1(Z)$ *and* $p_2\left(q^{-1}Z^{-1}\right)$ *do not have any mutual roots. Then* $L\left(s,\pi,\wedge^2\otimes\mu\right)=\frac{1}{p_1\left(q^{-s}\right)}$*,* $L\left(s,\widetilde{\pi},\wedge^2\otimes\mu^{-1}\right)=\frac{1}{p_2\left(q^{-s}\right)}$*,* $\epsilon\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=c\cdot q^{-ks}$*.*

***Proof*** Write $L\left(s,\pi,\wedge^2\otimes\mu\right)=\frac{1}{p_\pi\left(q^{-s}\right)}$, $L\left(s,\widetilde{\pi},\wedge^2\otimes\mu^{-1}\right)=\frac{1}{p_{\widetilde{\pi}}\left(q^{-s}\right)}$, and $\epsilon\left(s,\pi,\wedge^2,\psi\right)=c_\pi\cdot q^{-k_\pi\cdot s}$, where $c_\pi\in\mathbb{C}^*$, $k_\pi\in\mathbb{Z}$ and $p_\pi,p_{\widetilde{\pi}}\in\mathbb{C}[Z]$ satisfy $p_\pi(0)=p_{\widetilde{\pi}}(0)=1$.

Then by Theorem 2.4, we have the equality

$$\begin{aligned}\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)&=c\cdot\left(q^{-s}\right)^k\frac{p_1\left(q^{-s}\right)}{p_2\left(q^{-1}\left(q^{-s}\right)^{-1}\right)}\\&=c_\pi\cdot\left(q^{-s}\right)^{k_\pi}\frac{p_\pi\left(q^{-s}\right)}{p_{\widetilde{\pi}}\left(q^{-1}\left(q^{-s}\right)^{-1}\right)},\end{aligned}$$

which implies that

$$cZ^{k}p_{1}\left(Z\right)p_{\tilde{\pi}}\left(q^{-1}Z^{-1}\right)=c_{\pi}Z^{k_{\pi}}p_{\pi}\left(Z\right)p_{2}\left(q^{-1}Z^{-1}\right),$$

as elements of the polynomial ring $\mathbb{C}\left[Z,Z^{-1}\right]$.

By Theorem 2.5, $p_{\pi}\left(Z\right) \mid 1-\omega_{\pi}\left(\varpi\right)\mu\left(\varpi\right)^{m}Z^{m}$ and $p_{\tilde{\pi}}\left(Z\right) \mid 1-\omega_{\pi}^{-1}\left(\varpi\right)\mu\left(\varpi\right)^{-m}Z^{m}$. Therefore, we get that $p_{\pi}\left(Z\right)$ and $p_{\tilde{\pi}}\left(q^{-1}Z^{-1}\right)$ have no mutual roots. Note that they also do not have zero or infinity as a root. Therefore, we conclude that every root of $p_{\pi}$ (including multiplicity) is a root of $p_1$, which implies that $p_1\left(Z\right)=h_1\left(Z\right)p_{\pi}\left(Z\right)$, where $h_1\in\mathbb{C}\left[Z\right]$, with $h_1\left(0\right)=1$. Similarly, we get that $p_2\left(Z\right)=h_2\left(Z\right)p_{\tilde{\pi}}\left(Z\right)$, where $h_2\in\mathbb{C}\left[Z\right]$, with $h_2\left(0\right)=1$. Hence, we get that $cZ^{k}h_1\left(Z\right)=c_{\pi}Z^{k_{\pi}}h_2\left(q^{-1}Z^{-1}\right)$. Since $p_1\left(Z\right)$ and $p_2\left(q^{-1}Z^{-1}\right)$ do not have any mutual roots, and since both do not have zero or infinity as a root, we get that $h_1\left(Z\right),h_2\left(Z\right)$ are constants. Therefore, $h_1=h_2=1$, and the result follows. □

### 2.3 Simple supercuspidal representations

Let $n$ be a positive integer.

Let $\omega : F^{*}\to\mathbb{C}^{*}$ be a multiplicative character such that $\omega\restriction_{1+\mathfrak{p}}=1$.

Let $I_n^{+}=\nu^{-1}\left(N_n\left(\mathfrak{f}\right)\right)$ be the pro-unipotent radical of the standard Iwahori subgroup of $\mathrm{GL}_n\left(F\right)$, where $N_n\left(\mathfrak{f}\right)$ is the upper unipotent subgroup of $\mathrm{GL}_n\left(\mathfrak{f}\right)$. Denote $H_n=F^{*}I_n^{+}$.

Let $t_0\in\mathfrak{o}^{*}/1+\mathfrak{p}\cong\mathfrak{f}^{*}$. Let $t\in\mathfrak{o}^{*}$ be a lift of $t_0$, i.e., $\nu\left(t\right)=t_0$. We define an affine generic character $\chi : H_n\to\mathbb{C}^{*}$ by

$$\chi\left(zk\right)=\omega\left(z\right)\psi\left(\sum_{i=1}^{n-1}a_i+ta_n\right),$$

where $z\in F^{*}$, and

$$k=\begin{pmatrix} x_1 & a_1 & * & \cdots & * \\ * & x_2 & a_2 & \cdots & * \\ \vdots & & \ddots & \ddots & \vdots \\ * & * & \cdots & x_{n-1} & a_{n-1} \\ \varpi a_n & * & \cdots & * & x_n \end{pmatrix}\in I_n^{+}.$$

Note that $\chi$ does not depend on the choice $t$, because the conductor of $\psi$ is $\mathfrak{p}$.

Let $\zeta\in\mathbb{C}$ be an $n$th root of $\omega\left(t^{-1}\varpi\right)$.

Denote $g_n=\begin{pmatrix} & I_{n-1} \\ t^{-1}\varpi & \end{pmatrix}$, $H_n'=\langle g_n\rangle H_n$. We define a character $\chi_{\zeta} : H_n'\to\mathbb{C}^{*}$ by $\chi_{\zeta}\left(g_n^{j}h\right)=\zeta^{j}\chi\left(h\right)$, for $j\in\mathbb{Z}$ and $h\in H_n$.

**Theorem 2.7** ([12, Section 4.3]) *The representation* $\sigma_{\chi}^{\zeta}=\mathrm{ind}_{H_n'}^{\mathrm{GL}_n\left(F\right)}\left(\chi_{\zeta}\right)$ *is an irreducible supercuspidal representation of* $\mathrm{GL}_n\left(F\right)$.

A representation $\sigma_\chi^\zeta$ such as in Theorem 2.7 is called a **simple supercuspidal representation**. We say that $\pi = \sigma_\chi^\zeta$ is a simple supercuspidal representation with central character $\omega$, associated with the data $(t_0, \zeta)$. Simple supercuspidal representations were first constructed by Gross and Reeder in [7] for groups that are simply connected, almost simple and split over the non-Archimedean field $F$.

By the proof of [12, Corollary 5.3], there exist exactly $n\,(q-1)$ equivalence classes of simple supercuspidal representations with a given central character, each of which corresponds to a pair $(t_0, \zeta)$.

By [1, Section 3.3], we have that if $W : \mathrm{GL}_n(F) \to \mathbb{C}$ is the function supported on $N_n H_n'$ (where $N_n$ is the upper triangular unipotent subgroup of $\mathrm{GL}_n(F)$), defined by

$$W\left(uh'\right) = \psi(u)\,\chi_\zeta\left(h'\right), \quad u \in N_n, h' \in H_n', \tag{2.1}$$

then $W \in \mathcal{W}\left(\sigma_\chi^\zeta, \psi\right)$ is a Whittaker function.

## 3 Computation of the twisted exterior square factors

In this section, we compute the twisted exterior square factors of a simple supercuspidal representation. Throughout this section, let $t_0 \in \mathfrak{o}^*/1+\mathfrak{p} \cong \mathfrak{f}^*, t \in \mathfrak{o}^*, \omega : F^* \to \mathbb{C}^*$, $\zeta \in \mathbb{C}^*$ be as in Sect. 2.3. We denote $\pi = \sigma_\chi^\zeta$. Our goal is to compute the twisted exterior square factors of $\pi$.

### 3.1 Preliminary lemmas

In order to compute the twisted exterior factors of $\pi$, we will use the function $\pi\left(\sigma_{2m}^{-1}\right) W$, where $W$ is the Whittaker function from Sect. 2.3. Before beginning our computation, we need some lemmas regarding the support of the integrand of the twisted Jacquet–Shalika integral $J\left(s, \pi\left(\sigma_{2m}^{-1}\right) W, \phi, \mu, \psi\right)$.

Denote for $1 \le l \le m$, $d_l = \begin{pmatrix} I_{m-l} & \\ & t^{-1}\varpi I_l \end{pmatrix}$, $w_l = \begin{pmatrix} & I_{m-l} \\ I_l & \end{pmatrix}$, and denote by $\tau_l$ the permutation defined by the columns of $w_l$, i.e.,

$$w_l = \left(e_{\tau_l(1)} \ldots e_{\tau_l(m)}\right).$$

**Lemma 3.1** *Suppose that $g \in \mathrm{GL}_m(\mathfrak{f})$, $X = \left(x_{ij}\right) \in \mathcal{N}_m^-(\mathfrak{f})$ is a lower triangular nilpotent matrix, such that*

$$\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\sigma_{2m}^{-1} \in N_{2m}(\mathfrak{f})\begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix} N_{2m}(\mathfrak{f}),$$

*for $1 \le l \le m$. Then*

1. $g \in N_m(\mathfrak{f})\, w_l N_m(\mathfrak{f})$.

2. *If* $g \in w_l N_m(\mathfrak{f})$, *then* $x_{ij} = 0$ *for every* $j < i$ *such that* $\tau_l^{-1}(j) < \tau_l^{-1}(i)$, *or equivalently* $X \in \mathcal{N}_m^-(\mathfrak{f}) \cap \left(w_l \mathcal{N}_m^+(\mathfrak{f}) w_l^{-1}\right)$, *where* $\mathcal{N}_m^+(\mathfrak{f})$ *is the subgroup of* $M_m(\mathfrak{f})$ *consisting of upper triangular nilpotent matrices.*

*Furthermore, for* $g \in w_l N_m(\mathfrak{f})$ *and such* $X$,

$$\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1} = \begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix} v,$$

*where* $v \in N_{2m}(\mathfrak{f})$ *is an upper triangular unipotent matrix, having zeros right above its diagonal.*

***Proof*** The lemma is proved in [19, Lemma 2.33] for the case that $g = wdu$, where $w$ is a permutation matrix, $d$ is a diagonal matrix, and $u \in N_m(\mathfrak{f})$. Therefore, we need only to show the first part for general $g$. By the Bruhat decomposition, we can write $g = u_1 w d u_2$, where $u_1, u_2 \in N_m(\mathfrak{f})$, $w$ is a permutation matrix, and $d$ is a diagonal matrix. Denote $g' = wdu_2$. We have

$$\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1} = \sigma_{2m} \begin{pmatrix} u_1 & \\ & u_1 \end{pmatrix} \sigma_{2m}^{-1} \sigma_{2m} \begin{pmatrix} I_m & u_1^{-1} X u_1 \\ & I_m \end{pmatrix} \begin{pmatrix} g' & \\ & g' \end{pmatrix} \sigma_{2m}^{-1}.$$

We have that $\sigma_{2m} \left(\begin{smallmatrix} u_1 & \\ & u_1 \end{smallmatrix}\right) \sigma_{2m}^{-1} \in N_{2m}(\mathfrak{f})$. Write $u_1^{-1} X u_1 = L + U$, where $L \in \mathcal{N}_m^-(\mathfrak{f})$ is a lower triangular nilpotent matrix and $U \in \mathcal{B}_m(\mathfrak{f})$ is an upper triangular matrix. Then we have that

$$\sigma_{2m} \begin{pmatrix} I_m & u_1^{-1} X u_1 \\ & I_m \end{pmatrix} = \sigma_{2m} \begin{pmatrix} I_m & U \\ & I_m \end{pmatrix} \sigma_{2m}^{-1} \sigma_{2m} \begin{pmatrix} I_m & L \\ & I_m \end{pmatrix}.$$

Since $\sigma_{2m} \left(\begin{smallmatrix} I_m & U \\ & I_m \end{smallmatrix}\right) \sigma_{2m}^{-1} \in N_{2m}(\mathfrak{f})$, and since $\sigma_{2m} \left(\begin{smallmatrix} u_1 & \\ & u_1 \end{smallmatrix}\right) \sigma_{2m}^{-1} \in N_{2m}(\mathfrak{f})$, we get that

$$\sigma_{2m} \begin{pmatrix} I_m & L \\ & I_m \end{pmatrix} \begin{pmatrix} g' & \\ & g' \end{pmatrix} \sigma_{2m}^{-1} \in N_{2m}(\mathfrak{f}) \begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix} N_{2m}(\mathfrak{f}).$$

Since $g' = wdu_2$, we get from [19, Lemma 2.33] that $wd = w_l$, as required. □

**Lemma 3.2** ([19, Lemma 2.34])

1. *Let* $d \in \mathrm{GL}_m(\mathfrak{f})$ *be a diagonal matrix. Then* $|N_m(\mathfrak{f}) w_l d N_m(\mathfrak{f})| = q^{\binom{m}{2} - \binom{l}{2} - \binom{m-l}{2}} \cdot |N_m(\mathfrak{f})|$. *Here* $\binom{k}{2} = \frac{k(k-1)}{2}$, *for any non-negative integer* $k$.
2. *The set*

$$\begin{aligned} &\mathcal{N}_m^-(\mathfrak{f}) \cap \left(w_l \mathcal{N}_m^+(\mathfrak{f}) w_l^{-1}\right) \\ &\quad = \left\{ \left(x_{ij}\right) \in \mathcal{N}_m^-(\mathfrak{f}) \mid x_{ij} = 0,\ \forall j < i \text{ s.t. } w_l^{-1}(j) < w_l^{-1}(i) \right\} \end{aligned}$$

*is of cardinality* $q^{\binom{m}{2}-\binom{l}{2}-\binom{m-l}{2}}$. *Here,* $\mathcal{N}_m^+(\mathfrak{f})$ *is the subgroup of* $M_m(\mathfrak{f})$ *consisting of upper triangular nilpotent matrices (i.e., upper triangular matrices with zeros on their diagonal).*

***Remark 3.1*** In [19, Lemma 2.34] the lemma is stated only for $d$ a diagonal matrix of a certain form, but its proof only uses the fact that $d$ is a diagonal matrix.

**Lemma 3.3** *Suppose that* $a = \operatorname{diag}(a_1, \ldots, a_m)$ *is an invertible diagonal matrix, and* $X \in \mathcal{N}_m^-(F)$ *is a lower nilpotent matrix, such that*

$$\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} a & \\ & a \end{pmatrix} = \lambda \cdot u \cdot g_{2m}^r \cdot k, \tag{3.1}$$

*where* $\lambda \in F^*$, $u \in N_{2m}$, $1 \le r \le 2m$, $k \in K_{2m}$. *Then*

1. $r = 2l$ *is even, for some* $1 \le l \le m$.
2. $|a_1| = \cdots = |a_{m-l}| = |\lambda|$.
3. $|a_{m-l+1}| = \cdots = |a_m| = |\lambda| \cdot |\varpi|$.
4. $d_l^{-1} X d_l \in \mathcal{N}_m^-(\mathfrak{o})$.

***Proof*** (1) Taking the absolute value of the determinant of both sides of Eq. (3.1), we get $|\det a|^2 = |\lambda|^{2m} \cdot |\det g_{2m}|^r$, and since $|\det g_{2m}| = \left|-t^{-1}\varpi\right| = q^{-1}$, we must have that $r$ is even. Thus, $r = 2l$, for some $1 \le l \le m$. Then

$$g_{2m}^{2l} = \begin{pmatrix} & I_{2m-2l} \\ t^{-1}\varpi I_{2l} & \end{pmatrix} = \begin{pmatrix} I_{2m-2l} & \\ & t^{-1}\varpi I_{2l} \end{pmatrix}\begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix}.$$

(2 & 3) Denote $Z = a^{-1}Xa$, $u_Z = \sigma_{2m}\begin{pmatrix} I_m & Z \\ & I_m \end{pmatrix}\sigma_{2m}^{-1}$. Denote

$$b = \sigma_{2m}\begin{pmatrix} a & \\ & a \end{pmatrix}\sigma_{2m}^{-1} = \operatorname{diag}(a_1, a_1, \ldots, a_m, a_m).$$

Then $bu_Z\sigma_{2m} = \lambda u g_{2m}^{2l} k$.
Let $u_Z = n_Z t_Z k_Z$ be an Iwasawa decomposition ($n_Z \in N_{2m}$, $t_Z \in A_{2m}$, $k_Z \in K_{2m}$). Then we have $\lambda^{-1} b t_Z = \left(b n_Z^{-1} b^{-1} u\right) g_{2m}^{2l} \left(k \sigma_{2m}^{-1} k_Z^{-1}\right)$. Denote $u' = b n_Z^{-1} b^{-1} u \in N_{2m}$. Then we get

$$\begin{pmatrix} I_{2m-2l} & \\ & \left(t^{-1}\varpi\right)^{-1} I_{2l} \end{pmatrix} u'^{-1} \lambda^{-1} b t_Z \in K_{2m}.$$

Writing $t_Z = \operatorname{diag}(t_1, \ldots, t_{2m})$, we get that $|\lambda|^{-1}|a_i||t_{2i}| = 1$ and $|\lambda|^{-1}|a_i||t_{2i-1}|$
$= 1$, for every $1 \le i \le m - l$, and that $|\lambda|^{-1}|a_i||t_{2i}| = |\varpi|$ and $|\lambda|^{-1}|a_i||t_{2i-1}| = |\varpi|$, for every $m - l + 1 \le i \le m$. By [8, Section 5, Proposition 4], $|t_i| \ge 1$ for odd $i$ and $|t_i| \le 1$ for even $i$. Thus, we get that $|t_i| = 1$ for every $i$. Hence, $|a_1| = \cdots = |a_{m-l}| = |\lambda|$ and $|a_{m-l+1}| = \cdots = |a_m| = |\lambda| \cdot |\varpi|$.

(4) By [8, Section 5, Proposition 5], there exists $\alpha > 0$, such that if $Z = (z_{ij})$, then $\max_{1\le i,j\le m} |z_{ij}|^{\alpha} \le \prod_{\substack{1\le i\le 2m \\ i \text{ odd}}} |t_i|$. This implies that $Z = a^{-1}Xa \in M_m(\mathfrak{o})$ since $|t_i| = 1$. We have $a = \lambda \cdot d_l \cdot k'$, where $k' \in \mathrm{GL}_m(\mathfrak{o}) \cap A_m = (\mathfrak{o}^*)^m$, and this implies $d_l^{-1} X d_l \in M_m(\mathfrak{o})$, and therefore, in $\mathcal{N}_m^-(F) \cap M_m(\mathfrak{o}) = \mathcal{N}_m^-(\mathfrak{o})$. □

**Lemma 3.4** *Let $g = ak$, where $a = \mathrm{diag}(a_1, \dots, a_m)$ is an invertible matrix, $k \in \mathrm{GL}_m(\mathfrak{o})$, $d^\times g = \delta_{B_m}^{-1}(a)\, d^\times k \prod_{i=1}^m d^\times a_i$, and $X \in \mathcal{N}_m^-(F)$ be a lower triangular nilpotent matrix. If*

$$\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1} \in N_{2m} H'_{2m},$$

*then there exists $1 \le l \le m$, such that $\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1} \in F^* N_{2m} g_{2m}^{2l} I_{2m}^+$. Moreover, if $\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1} \in \lambda N_{2m} g_{2m}^{2l} I_{2m}^+$ for $1 \le l \le m$ and $\lambda \in F^*$, then*

1. *$a = \lambda d_l \mathrm{diag}(u_1, \dots, u_m)$, where $u_1, \dots, u_m \in \mathfrak{o}^*$, $\delta_{B_m}^{-1}(a) = \delta_{B_m}^{-1}(d_l) = q^{-l(m-l)}$.*
2. *Let $k'' = \mathrm{diag}(u_1, \dots, u_m) k$. Then $\nu(k'') \in N_m(\mathfrak{f}) w_l N_m(\mathfrak{f})$, $d^\times k = d^\times k''$.*
3. *If $\nu(k'') \in w_l N_m(\mathfrak{f})$, then $X = d_l Z d_l^{-1}$ and $dX = \delta_{B_m}^{-1}(d_l)\, dZ$, where $Z \in \mathcal{N}_m^-(\mathfrak{o})$ satisfies $\nu(Z) \in \mathcal{N}_m^-(\mathfrak{f}) \cap \left(w_l \mathcal{N}_m^+(\mathfrak{f}) w_l^{-1}\right)$. Moreover, in this case,*

$$\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1} = \lambda g_{2m}^{2l} v,$$

*where $v \in \mathrm{GL}_{2m}(\mathfrak{o})$ satisfies $\nu(v) \in N_{2m}(\mathfrak{f})$, $\nu(v)$ has zeros right above its diagonal, and $v$ has zero at its bottom-left corner.*

***Proof*** 1. Suppose that

$$\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} a & \\ & a \end{pmatrix} \begin{pmatrix} k & \\ & k \end{pmatrix} \sigma_{2m}^{-1} = \lambda u g_{2m}^r k', \tag{3.2}$$

where $\lambda \in F^*$, $u \in N_{2m}(F)$, $r \in \mathbb{Z}$, $k' \in I_{2m}^+$. Since $g_{2m}^{2m} = t^{-1}\varpi I_{2m}$, we may assume (by modifying $\lambda$) that $1 \le r \le 2m$. By theorem 3.3, we have that $r = 2l$, $X = d_l Z d_l^{-1}$, where $Z \in \mathcal{N}_m^-(\mathfrak{o})$, and $a = \lambda d_l \cdot \mathrm{diag}(u_1, \dots, u_m)$, for some $u_1, \dots, u_m \in \mathfrak{o}^*$.

2. Let $k'' = \mathrm{diag}(u_1, \dots, u_m) \cdot k$ and $d'_l = \begin{pmatrix} I_{2m-2l} & \\ & t^{-1}\varpi I_{2l} \end{pmatrix}$. Using these notations and part 1, we have that

$$\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1} = \lambda \sigma_{2m} \begin{pmatrix} d_l & \\ & d_l \end{pmatrix} \begin{pmatrix} I_m & Z \\ & I_m \end{pmatrix} \begin{pmatrix} k'' & \\ & k'' \end{pmatrix} \sigma_{2m}^{-1}. \tag{3.3}$$

Since $d_l' = \sigma_{2m}\begin{pmatrix} d_l & \\ & d_l \end{pmatrix}\sigma_{2m}^{-1}$, we get from Eq. (3.3)

$$\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\sigma_{2m}^{-1} = \lambda d_l' \sigma_{2m}\begin{pmatrix} I_m & Z \\ & I_m \end{pmatrix}\begin{pmatrix} k'' & \\ & k'' \end{pmatrix}\sigma_{2m}^{-1}. \tag{3.4}$$

Recall that $r = 2l$. Writing $g_{2m}^{2l} = d_l' \begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix}$, we get by combining Eqs. (3.2) and (3.4) that

$$\begin{aligned} \sigma_{2m}\begin{pmatrix} I_m & Z \\ & I_m \end{pmatrix}\begin{pmatrix} k'' & \\ & k'' \end{pmatrix}\sigma_{2m}^{-1} &= d_l'^{-1} u g_{2m}^{2l} k' \\ &= \left(d_l'^{-1} u d_l'\right)\begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix} k', \end{aligned} \tag{3.5}$$

which implies that $d_l'^{-1} u d_l' \in N_{2m}(\mathfrak{o})$, as $\begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix}, k' \in K_{2m}$, and the left-hand side of Eq. (3.5) is in $K_{2m}$. Since $d_l'^{-1} u d_l' \in N_{2m}(\mathfrak{o}) \subseteq I_{2m}^{+}$ and $k' \in I_{2m}^{+}$, we get from Eq. (3.5) that

$$\nu\left(\sigma_{2m}\begin{pmatrix} I_m & Z \\ & I_m \end{pmatrix}\begin{pmatrix} k'' & \\ & k'' \end{pmatrix}\sigma_{2m}^{-1}\right) \in N_{2m}(\mathfrak{f})\begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix} N_{2m}(\mathfrak{f}).$$

Since $Z \in \mathcal{N}_m^{-}(\mathfrak{o})$, $\nu(Z) \in \mathcal{N}_m^{-}(\mathfrak{f})$, and by applying Theorem 3.1, we have that $\nu\left(k''\right) \in N_m(\mathfrak{f})\, w_l N_m(\mathfrak{f})$.

3. Assume that $\nu\left(k''\right) \in w_l N_m(\mathfrak{f})$, then by Theorem 3.1, we have $\nu(Z) \in \mathcal{N}_m^{-}(\mathfrak{f}) \cap \left(w_l \mathcal{N}_m^{+}(\mathfrak{f})\, w_l^{-1}\right)$, and

$$\nu\left(\sigma_{2m}\begin{pmatrix} I_m & Z \\ & I_m \end{pmatrix}\begin{pmatrix} k'' & \\ & k'' \end{pmatrix}\sigma_{2m}^{-1}\right) = \begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix} v',$$

where $v' \in N_{2m}(\mathfrak{f})$ is an upper triangular matrix, having zeros right above its diagonal. Therefore,

$$\sigma_{2m}\begin{pmatrix} I_m & Z \\ & I_m \end{pmatrix}\begin{pmatrix} k'' & \\ & k'' \end{pmatrix}\sigma_{2m}^{-1} = \begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix} v, \tag{3.6}$$

where $v \in \mathrm{GL}_{2m}(\mathfrak{o})$ satisfies $\nu(v) = v'$. Combining Eq. (3.6), Eq. (3.4), and the fact that $g_{2m}^{2l} = d_l' \begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix}$, we get

$$\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\sigma_{2m}^{-1} = \lambda d_l' \begin{pmatrix} & I_{2m-2l} \\ I_{2l} & \end{pmatrix} v = \lambda g_{2m}^{2l} v.$$

Finally, suppose that $l < m$. Note that a non-zero scalar multiple of the last row of $v$ appears as the $2m-2l$ row of $\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\sigma_{2m}^{-1}$. The $(2m-2l, 1)$ coordinate of $\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\sigma_{2m}^{-1}$ is the $(2m-l,1)$ coordinate of $\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}$, and this is zero, as $2m-l>m$. If $l=m$, then $g_{2m}^{2m}=t^{-1}\varpi I_{2m}$, and therefore, the last row of $v$ is a scalar multiple of the last row of $\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\sigma_{2m}^{-1}$, which has zero as its first coordinate.

□

### 3.2 The even case

In this section, we compute the twisted exterior square factors for the even case $n=2m$.

**Theorem 3.5** *Let $\pi$ be a simple supercuspidal representation of $\mathrm{GL}_{2m}(F)$ with central character $\omega$, associated with the data $(t_0,\zeta)$. Let $\mu : F^* \to \mathbb{C}^*$ be a unitary tamely ramified character, i.e., $\mu\restriction_{1+\mathfrak{p}}=1$. Denote $\xi=\zeta^2\cdot\mu\left((-1)^{m-1}t^{-1}\varpi\right)$. Let $(\omega\cdot\mu^m)_\xi : F^*\to\mathbb{C}^*$ be the character defined by $(\omega\cdot\mu^m)_\xi\left(\varpi^j u\right)=\xi^j\omega(u)\mu(u)^m$, for $j\in\mathbb{Z}$, $u\in\mathfrak{o}^*$. Then*

$$\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=\left(\xi q^{-\left(s-\frac{1}{2}\right)}\right)^{m-1}\gamma\left(s,(\omega\cdot\mu^m)_\xi,\psi\right).$$

*Explicitly,*

1. *If $(\omega\cdot\mu^m)\restriction_{\mathfrak{o}^*}\neq 1$, then*

$$\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=\left(\xi q^{-\left(s-\frac{1}{2}\right)}\right)^{m-1}\frac{1}{\sqrt{q}}\sum_{\lambda\in\mathfrak{f}^*}\psi(\lambda)\,\omega\left(\lambda^{-1}\right)\mu\left(\lambda^{-m}\right).$$

   *In this case, $L\left(s,\pi,\wedge^2\otimes\mu\right)=1$, $\epsilon\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)$.*
2. *If $(\omega\cdot\mu^m)\restriction_{\mathfrak{o}^*}=1$, then*

$$\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=\left(\xi q^{-\left(s-\frac{1}{2}\right)}\right)^{m-2}\frac{1-\xi q^{-s}}{1-\xi^{-1}q^{-(1-s)}}.$$

   *In this case, $L\left(s,\pi,\wedge^2\otimes\mu\right)=\frac{1}{1-\xi q^{-s}}$,*

$$\epsilon\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=\xi^{m-2}q^{-(m-2)\left(s-\frac{1}{2}\right)}.$$

***Proof*** We will compute the twisted exterior square gamma factor by computing the twisted Jacquet–Shalika integrals $J\left(s,\pi\left(\sigma_{2m}^{-1}\right)W,\phi,\mu,\psi\right)$ and $\tilde{J}\left(s,\pi\left(\sigma_{2m}^{-1}\right)W,\phi,\mu,\psi\right)$, where $W$ is the Whittaker function introduced in Sect. 2.3, and $\phi : F^m\to\mathbb{C}$ is the function defined by

$$\phi(x)=\begin{cases}\psi(-\nu(x_1)) & x=(x_1,\dots,x_m)\in\mathfrak{o}^m,\\ 0 & \text{otherwise.}\end{cases} \tag{3.7}$$

Then

$$\mathcal{F}_{\psi}\phi\left(x\right)=\begin{cases}q^{\frac{m}{2}}\delta_{\varepsilon_1}\left(\nu\left(x\right)\right) & x\in\mathfrak{o}^m,\\ 0 & \text{otherwise,}\end{cases}$$

where $\delta_{\varepsilon_1}(x)$ is the indicator function of $\varepsilon_1=(1,0,\ldots,0)\in\mathfrak{f}^m$.

By the Iwasawa decomposition, we have that $J\left(s,\pi\left(\sigma_{2m}^{-1}\right)W,\phi,\mu,\psi\right)$ is given by

$$\begin{aligned}J=\int W\left(\sigma_{2m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}ak & \\ & ak\end{pmatrix}\sigma_{2m}^{-1}\right)\left|\det a\right|^s\mu\left(\det\left(ak\right)\right)\\ \cdot\phi\left(\varepsilon ak\right)\psi\left(-\mathrm{tr}X\right)\delta_{B_m}^{-1}\left(a\right)dXd^{\times}ad^{\times}k,\end{aligned}\tag{3.8}$$

where $W$ is the Whittaker function defined in Sect. 2.3, given by Eq. (2.1), $X$ is integrated over $\mathcal{B}_m\backslash M_m(F)$, $k$ is integrated over $K_m=\mathrm{GL}_m(\mathfrak{o})$, and $a$ is integrated over the diagonal subgroup $A_m$ of $\mathrm{GL}_m(F)$.

Denote by $W_m$ the group of $m\times m$ permutation matrices. By the Bruhat decomposition for $\mathrm{GL}_m(\mathfrak{f})$, we have the disjoint union

$$\mathrm{GL}_m\left(\mathfrak{f}\right)=\bigsqcup_{\substack{w\in W_m\\ d_0\in A_m(\mathfrak{f})}}N_m\left(\mathfrak{f}\right)wd_0N_m\left(\mathfrak{f}\right).$$

We decompose each of the double cosets of the disjoint union into a disjoint union of left cosets: given $w\in W_m$, $d_0\in A_m(\mathfrak{f})$, we can write

$$N_m\left(\mathfrak{f}\right)wd_0N_m\left(\mathfrak{f}\right)=\bigsqcup_{u_0\in C_{wd_0}}u_0wd_0N_m\left(\mathfrak{f}\right),$$

where $C_{wd_0}\subseteq N_m(\mathfrak{f})$ is a subset of $N_m(\mathfrak{f})$ such that the map

$$\begin{aligned}C_{wd_0}&\to\left\{u_0wd_0N_m\left(\mathfrak{f}\right)\mid u_0\in N_m\left(\mathfrak{f}\right)\right\},\\ u_0&\mapsto u_0wd_0N_m\left(\mathfrak{f}\right)\end{aligned}$$

is a bijection. We may assume without loss of generality that $I_m\in C_{wd_0}$. We have that $\left|C_{wd_0}\right|=\frac{|N_m(\mathfrak{f})wd_0N_m(\mathfrak{f})|}{|N_m(\mathfrak{f})|}$.

We obtain the following decomposition:

$$\mathrm{GL}_m\left(\mathfrak{f}\right)=\bigsqcup_{\substack{w\in W_m\\ d_0\in A_m(\mathfrak{f})}}\bigsqcup_{u_0\in C_{wd_0}}u_0wd_0N_m\left(\mathfrak{f}\right).$$

Since $\nu^{-1}(\mathrm{GL}_m(\mathfrak{f})) = \mathrm{GL}_m(\mathfrak{o})$, we can lift the above decomposition to

$$\mathrm{GL}_m(\mathfrak{o}) = \bigsqcup_{\substack{w \in W_m \\ d \in D_m}} \bigsqcup_{u \in C_{wd}} uwd\nu^{-1}(N_m(\mathfrak{f})),$$

where $D_m \subseteq A_m \cap \mathrm{GL}_m(\mathfrak{o}) = (\mathfrak{o}^*)^m$ is a set of representatives for the inverse image $\nu^{-1}(A_m(\mathfrak{f}))$ (i.e., $D_m \subseteq \nu^{-1}(A_m(\mathfrak{f}))$ and $\nu \restriction_{D_m}: D_m \to A_m(\mathfrak{f})$ is a bijection), and for $d \in D_m$ with $\nu(d) = d_0$, $C_{wd} \subseteq N_m(\mathfrak{o})$ is a set of representatives for the inverse image $\nu^{-1}\left(C_{wd_0}\right)$ (i.e., $C_{wd} \subseteq \nu^{-1}\left(C_{wd_0}\right)$ and $\nu \restriction_{C_{wd}}: C_{wd} \to C_{wd_0}$ is a bijection). Without loss of generality, we may assume that the identity matrix belongs to $D_m$ and also belongs to $C_{wd}$, for every $w \in W_m$ and $d \in D_m$.

Using this decomposition for $K_m$ in Eq. (3.8), we decompose the integral $J$ into a sum of integrals

$$J = \sum_{\substack{w \in W_m \\ d \in D_m}} \sum_{u \in C_{wd}} J_{wd,u},$$

where

$$J_{wd,u} = \int W\left(\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} auwdk & \\ & auwdk \end{pmatrix}\sigma_{2m}^{-1}\right)|\det a|^s \mu(\det(auwdk)) \cdot \phi(\varepsilon auwdk)\psi(-\mathrm{tr}X)\delta_{B_m}^{-1}(a)\,dXd^\times ad^\times k,$$

where $X$ is integrated over $\mathcal{B}_m \backslash M_m(F)$, $k$ is integrated over $I_m^+ = \nu^{-1}(N_m(\mathfrak{f}))$, and $a$ is integrated over $A_m$. Writing $au = aua^{-1} \cdot a$, we have that $aua^{-1} \in N_m$, and since the Jacquet–Shalika integrand is invariant under $N_m \backslash \mathrm{GL}_m(F)$, we have $J_{wd,I_m} = J_{wd,u}$ for any $u \in C_{wd}$. Denote $J_{wd} = J_{wd,I_m}$, then we have

$$J = \sum_{\substack{w \in W_m \\ d \in D_m}} |C_{wd}| J_{wd} = \sum_{\substack{w \in W_m \\ d \in D_m}} \frac{|N_m(\mathfrak{f})w\nu(d)N_m(\mathfrak{f})|}{|N_m(\mathfrak{f})|} J_{wd}.$$

Using the isomorphism $\mathcal{B}_m \backslash M_m(F) \cong \mathcal{N}_m^-$, we can write

$$J_{wd} = \int W\left(\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} awdk & \\ & awdk \end{pmatrix}\sigma_{2m}^{-1}\right)|\det a|^s \mu(\det(awdk)) \cdot \phi(\varepsilon awdk)\delta_{B_m}^{-1}(a)\,dXd^\times ad^\times k, \tag{3.9}$$

where the integration is the same as in $J_{wd,u}$, except that this time $X$ is integrated on $\mathcal{N}_m^-$.

By Theorem 3.4, $J_{wd} = 0$ unless $w = w_l$ for some $1 \le l \le m$. In this case by Theorem 3.4, we have that the integrand of $J_{w_l d}$ is supported on $d_l \cdot F^* \cdot A_m(\mathfrak{o})$, where

$A_m(\mathfrak{o}) = A_m \cap \mathrm{GL}_m(\mathfrak{o}) \cong (\mathfrak{o}^*)^m$. We translate $a$ by $d_l$ and write down the expression for the Haar measure for the subgroup $F^* \cdot A_m(\mathfrak{o})$:

$$\begin{aligned} a &= d_l \lambda \mathrm{diag}(u_1, \ldots, u_m), \text{ where } \lambda \in F^*, u_1, \ldots, u_m \in \mathfrak{o}^*, \\ d^\times a &= d^\times \lambda \prod_{i=1}^{m} d^\times u_i, \\ \delta_{B_m}^{-1}(a) &= \delta_{B_m}^{-1}(d_l) = q^{-l(m-l)}. \end{aligned} \tag{3.10}$$

Denote

$$k'' = \mathrm{diag}(u_1, \ldots, u_m)\, w_l dk. \tag{3.11}$$

By Theorem 3.4, $k''$ satisfies $\nu(k'') \in N_m(\mathfrak{f})\, w_l N_m(\mathfrak{f})$. Since $\nu(k) \in N_m(\mathfrak{f})$, then by the Bruhat decomposition of $\nu(k'')$, we must have

$$\nu(\mathrm{diag}(u_1, \ldots, u_m))\, w_l \nu(d) = w_l.$$

Therefore, we have

$$\mathrm{diag}(u_1, \ldots, u_m) = \mathrm{diag}(u'_1, \ldots, u'_m)\, w_l d^{-1} w_l^{-1}, \tag{3.12}$$

where $u'_1, \ldots, u'_m \in 1 + \mathfrak{p}$ and $\prod_{i=1}^m d^\times u_i = \prod_{i=1}^m d^\times u'_i$. Denote $g = a w_l dk$. By Eq. (3.10) and Eq. (3.12), we have

$$g = a w_l dk = \lambda d_l \mathrm{diag}(u'_1, \ldots, u'_m)\, w_l k. \tag{3.13}$$

By Eqs. (3.11) and (3.12), we have $k'' = \mathrm{diag}(u'_1, \ldots, u'_m)\, w_l k$, and therefore, $\nu(k'') \in w_l N_m(\mathfrak{f})$. Hence, by part 3 of Theorem 3.4,

$$\begin{aligned} X &= d_l Z d_l^{-1}, \text{ where } Z \in \nu^{-1}\left(\mathcal{N}_m^-(\mathfrak{f}) \cap \left(w_l \mathcal{N}_m^+(\mathfrak{f})\, w_l^{-1}\right)\right), \\ dX &= \delta_{B_m}^{-1}(d_l)\, dZ = q^{-l(m-l)} dZ. \end{aligned} \tag{3.14}$$

Moreover, we have that

$$\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1} = \lambda g_{2m}^{2l} v, \tag{3.15}$$

where $v \in \mathrm{GL}_{2m}(\mathfrak{o})$ satisfies $\nu(v) \in N_{2m}(\mathfrak{f})$, $\nu(v)$ has zeros right above its diagonal, and $v$ has zero at its bottom-left corner. Therefore, by Eqs. (2.1) and (3.15), in this domain

$$W\left(\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1}\right) = \zeta^{2l} \omega(\lambda). \tag{3.16}$$

We have by Eq. (3.13)

$$\det(aw_l dk) = \lambda^m \left(t^{-1}\varpi\right)^l \left(\prod_{i=1}^{m} u_i'\right) (-1)^{l(m-l)} \det k, \tag{3.17}$$

and

$$|\det a|^s = |\lambda|^{ms} q^{-ls}. \tag{3.18}$$

Since $\det k \in 1 + \mathfrak{p}$, $\prod_{i=1}^{m} u_i' \in 1 + \mathfrak{p}$, and since $\mu$ is a tamely ramified character, and since $(-1)^{l^2} = (-1)^l$, we get by Eq. (3.17)

$$\mu(\det(aw_l dk)) = \mu(\lambda)^m \mu\left((-1)^{m-1} t^{-1}\varpi\right)^l. \tag{3.19}$$

We have by Eq. (3.13) that $\varepsilon a w_l dk = u_m' \lambda t^{-1} \varpi \varepsilon_l k$. Since $u_m' \in 1 + \mathfrak{p} \subseteq \mathfrak{o}^*$, $x \in F^m$ satisfies $x \in \mathfrak{o}^m$ if and only if $u_m' x \in \mathfrak{o}^m$. In the case $x \in \mathfrak{o}^m$, we have $x \equiv u_m' x \pmod{\mathfrak{p}}$, which implies that if $x = (x_1, \dots, x_m)$, then $x_1 \equiv u_m' x_1 \pmod{\mathfrak{p}}$. Since $\psi$ has conductor $\mathfrak{p}$, $\psi\left(u_m' x_1\right) = \psi(x_1)$. Therefore, by the definition of $\phi$ in Eq. (3.7), $\phi\left(u_m' x\right) = \phi(x)$ for every $x \in F^m$. It follows that

$$\phi(\varepsilon a w_l dk) = \phi\left(u_m' \lambda t^{-1} \varpi \varepsilon_l k\right) = \phi\left(\lambda t^{-1} \varpi \varepsilon_l k\right). \tag{3.20}$$

Substituting in Eq. (3.9) the equalities Eqs. (3.10), (3.12), (3.14), (3.16), (3.18), (3.19), and (3.20), we get

$$\begin{aligned} J_{w_l d} = \int & \left(\zeta^{2l} \omega(\lambda)\right) \left(|\lambda|^{ms} q^{-ls}\right) \left(\mu(\lambda)^m \mu\left((-1)^{m-1} t^{-1}\varpi\right)^l\right) \\ & \cdot \phi\left(\lambda t^{-1}\varpi \varepsilon_l k\right) q^{-l(m-l)} \left(q^{-l(m-l)} dZ\right) \left(d^\times \lambda \prod_{i=1}^{m} d^\times u_i'\right) d^\times k, \end{aligned} \tag{3.21}$$

where the integral is integrated over $\lambda \in F^*$, $u_1', \dots, u_m' \in 1 + \mathfrak{p}$, $Z \in \nu^{-1}\left(\mathcal{N}_m^-(\mathfrak{f}) \cap \left(w_l \mathcal{N}_m^+(\mathfrak{f})\, w_l^{-1}\right)\right)$, $k \in I_m^+$.

Denote $\xi = \zeta^2 \mu\left((-1)^{m-1} t^{-1}\varpi\right)$. We can now evaluate the integration over $Z, u_1', \dots, u_m'$ in Eq. (3.21) and get

$$\begin{aligned} J_{w_l d} = q^{-2l(m-l)} q^{-ls} \cdot & \frac{\left|\mathcal{N}_m^-(\mathfrak{f}) \cap \left(w_l \mathcal{N}_m^+(\mathfrak{f})\, w_l^{-1}\right)\right|}{\left|\mathcal{N}_m^-(\mathfrak{f})\right|} \frac{1}{|\mathfrak{f}^*|^m} \xi^l \\ & \cdot \int_{I_m^+} \int_{F^*} \omega(\lambda)\, \mu(\lambda)^m |\lambda|^{ms} \phi\left(\lambda t^{-1}\varpi \varepsilon_l k\right) d^\times \lambda d^\times k. \end{aligned} \tag{3.22}$$

Notice that Eq. (3.22) implies that $J_{w_l d}$ does not depend on $d \in D_m$, and we have $J_{w_l} = J_{w_l d}$ for every $d \in D_m$. Denote

$$J_l = \sum_{d \in D_m} \left|C_{w_l d}\right| J_{w_l d} = \sum_{d \in D_m} \frac{\left|N_m(\mathfrak{f})\, w_l \nu(d)\, N_m(\mathfrak{f})\right|}{\left|N_m(\mathfrak{f})\right|} J_{w_l}. \tag{3.23}$$

By Theorem 3.2,

$$\begin{aligned} \frac{\left|N_m(\mathfrak{f})\, w_l \nu(d)\, N_m(\mathfrak{f})\right|}{\left|N_m(\mathfrak{f})\right|} &= \left|\mathcal{N}_m^-(\mathfrak{f}) \cap \left(w_l \mathcal{N}_m^+(\mathfrak{f})\, w_l^{-1}\right)\right| \\ &= q^{\binom{m}{2} - \binom{l}{2} - \binom{m-l}{2}}. \end{aligned} \tag{3.24}$$

We also have $|D_m| = |A_m(\mathfrak{f})| = |\mathfrak{f}^*|^m$. Therefore, by substituting into Eq. (3.23) the equalities Eqs. (3.22) and (3.24), we have

$$J_l = q^{-\binom{m}{2}} q^{-ls} \xi^l \cdot \int_{I_m^+} \int_{F^*} \omega(\lambda)\, \mu(\lambda)^m\, |\lambda|^{ms}\, \phi\left(\lambda t^{-1} \varpi \varepsilon_l k\right) d^\times \lambda\, d^\times k, \tag{3.25}$$

where the expression $q^{-\binom{m}{2}}$ arises from the identity $-2l(m-l) + 2\binom{m}{2} - 2\binom{l}{2} - 2\binom{m-l}{2} - \binom{m}{2} = -\binom{m}{2}$.

Since $k \in I_m^+$, we have that $\varepsilon_l k \in \mathfrak{o}^m$ has 1 as its $l$ coordinate modulo $\mathfrak{p}$. Therefore, if $\lambda t^{-1} \varpi \varepsilon_l k \in \mathfrak{o}^m$, we must have $\left|\lambda t^{-1} \varpi\right| \le 1$, i.e., $\lambda = \left(t^{-1}\varpi\right)^j \cdot u_0$, for some $u_0 \in \mathfrak{o}^*$ and $j \ge -1$. For a fixed $k \in I_m^+$, we decompose

$$\begin{aligned} &\int_{F^*} \omega(\lambda)\, \mu(\lambda)^m\, |\lambda|^{ms}\, \phi\left(\lambda t^{-1} \varpi \varepsilon_l k\right) d^\times \lambda \\ &\quad = \sum_{j=-1}^{\infty} \omega\left(t^{-1}\varpi\right)^j \mu\left(t^{-1}\varpi\right)^{jm} q^{-jms} \\ &\qquad \cdot \int_{\mathfrak{o}^*} \omega(u_0)\, \mu(u_0)^m\, \phi\left(u_0 \left(t^{-1}\varpi\right)^{j+1} \varepsilon_l k\right) d^\times u_0. \end{aligned} \tag{3.26}$$

Since $\xi^m = \zeta^{2m} \mu\left((-1)^{m(m-1)}\right) \mu\left(t^{-1}\varpi\right)^m$, $m(m-1)$ is even, and $\zeta^{2m} = \omega\left(t^{-1}\varpi\right)$, we have $\xi^m = \omega\left(t^{-1}\varpi\right) \mu\left(t^{-1}\varpi\right)^m$. Therefore, we get from Eq. (3.26) that

$$\begin{aligned} &\int_{F^*} \omega(\lambda)\, \mu(\lambda)^m\, |\lambda|^{ms}\, \phi\left(\lambda t^{-1} \varpi \varepsilon_l k\right) d^\times \lambda \\ &\quad = \sum_{j=-1}^{\infty} \left(\xi q^{-s}\right)^{jm} \cdot \int_{\mathfrak{o}^*} \omega(u_0)\, \mu(u_0)^m\, \phi\left(u_0 \left(t^{-1}\varpi\right)^{j+1} \varepsilon_l k\right) d^\times u_0. \end{aligned} \tag{3.27}$$

If $l \ge 2$, then $\varepsilon_l k$ has 0 as its first coordinate modulo $\mathfrak{p}$, so for every $j \ge -1$, the first coordinate of $u_0 \left(t^{-1}\varpi\right)^{j+1} \varepsilon_l k$ is 0 modulo $\mathfrak{p}$. Thus, $\phi\left(u_0 \left(t^{-1}\varpi\right)^{j+1} \varepsilon_l k\right) = 1$. We also have that

$$\int_{\mathfrak{o}^*} \omega\left(u_0\right) \mu\left(u_0\right)^m d^\times u_0 = \begin{cases} 1 & \left(\omega \cdot \mu^m\right) \restriction_{\mathfrak{o}^*} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, from Eqs. (3.25) and (3.27), we get for $l \geq 2$ that,

$$J_l = \begin{cases} \frac{q^{-\binom{m}{2}}}{\left[\mathrm{GL}_m(\mathfrak{f}):N_m(\mathfrak{f})\right]} \left(\xi q^{-s}\right)^{-m} \left(\xi q^{-s}\right)^l \frac{1}{1-\xi^m q^{-ms}} & \left(\omega \cdot \mu^m\right) \restriction_{\mathfrak{o}^*} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $l = 1$ and $j \geq 0$, we have that $\left(t^{-1}\varpi\right)^{j+1} \varepsilon_l u_0 k \equiv 0 \pmod{\mathfrak{p}}$, and therefore, we have again $\phi\left(u_0 \left(t^{-1}\varpi\right)^{j+1} \varepsilon_l k\right) = 1$ and

$$\int_{\mathfrak{o}^*} \omega\left(u_0\right) \mu\left(u_0\right)^m d^\times u_0 = \begin{cases} 1 & \left(\omega \cdot \mu^m\right) \restriction_{\mathfrak{o}^*} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $l = 1$ and $j = -1$, we have that $\varepsilon_1 k$ has 1 as its first coordinate modulo $\mathfrak{p}$, and therefore, $\phi\left(u_0 \left(t^{-1}\varpi\right)^{j+1} \varepsilon_l k\right) = \psi\left(-\nu\left(u_0\right)\right)$, and we have

$$\int_{\mathfrak{o}^*} \omega\left(u_0\right) \mu\left(u_0\right)^m \psi\left(-u_0\right) d^\times u_0 = \frac{1}{|\mathfrak{f}^*|} \sum_{\lambda \in \mathfrak{f}^*} \omega(\lambda) \mu(\lambda)^m \psi(-\lambda).$$

To summarize, we get

$$J_1 = \begin{cases} \frac{q^{-\binom{m}{2}}}{\left[\mathrm{GL}_m(\mathfrak{f}):N_m(\mathfrak{f})\right]} \left(\xi q^{-s}\right)^{-m} \left(\frac{\left(\xi q^{-s}\right)^{m+1}}{1-\xi^m q^{-ms}} - \frac{\xi q^{-s}}{q-1}\right) & \left(\omega \cdot \mu^m\right) \restriction_{\mathfrak{o}^*} = 1, \\ \frac{q^{-\binom{m}{2}}}{\left[\mathrm{GL}_m(\mathfrak{f}):N_m(\mathfrak{f})\right]} \left(\xi q^{-s}\right)^{-m} \left(\xi q^{-s}\right) \frac{1}{|\mathfrak{f}^*|} \sum_{\lambda \in \mathfrak{f}^*} \omega(\lambda) \mu(\lambda)^m \psi(-\lambda) & \text{otherwise.} \end{cases}$$

Summing all the $J_l$ up, we get

$$\begin{aligned} J &= \sum_{l=1}^m J_l \\ &= \begin{cases} \frac{q^{-\binom{m}{2}}}{\left[\mathrm{GL}_m(\mathfrak{f}):N_m(\mathfrak{f})\right]} \left(\xi q^{-s}\right)^{-(m-2)} \cdot q \cdot \frac{1-\xi^{-1} q^{-(1-s)}}{\left(1-\xi q^{-s}\right)(q-1)} & \left(\omega \cdot \mu^m\right) \restriction_{\mathfrak{o}^*} = 1, \\ \frac{q^{-\binom{m}{2}}}{\left[\mathrm{GL}_m(\mathfrak{f}):N_m(\mathfrak{f})\right]} \left(\xi q^{-s}\right)^{-(m-1)} \frac{1}{|\mathfrak{f}^*|} \sum_{\lambda \in \mathfrak{f}^*} \omega(\lambda) \mu(\lambda)^m \psi(-\lambda) & \text{otherwise} \end{cases}. \end{aligned} \tag{3.28}$$

We now move to compute $\tilde{J} = \tilde{J}\left(s, \pi\left(\sigma_{2m}\right)^{-1} W, \phi, \mu, \psi\right)$. Following the same steps as before for the expression in Theorem 2.3, we have

$$\tilde{J} = \sum_{l=1}^m \tilde{J}_l,$$

where

$$\tilde{J}_l = q^{-\binom{m}{2}} q^{-l(s-1)} \xi^l \cdot \int_{I_m^+} \int_{F^*} \omega(\lambda) \mu(\lambda)^m |\lambda|^{m(s-1)} \mathcal{F}_\psi \phi\left(\lambda^{-1} \varepsilon_1 d_l^{-1} w_l k^\iota\right) \times d^\times \lambda d^\times k.$$

Recall that

$$\mathcal{F}_\psi \phi(x) = \begin{cases} q^{\frac{m}{2}} \delta_{\varepsilon_1}(\nu(x)) & x \in \mathfrak{o}^m, \\ 0 & \text{otherwise.} \end{cases}$$

We have that

$$\lambda^{-1} \varepsilon_1 d_l^{-1} w_l k^\iota = \begin{cases} \lambda^{-1} \varepsilon_{l+1} k^\iota, & 1 \leq l \leq m-1, \\ \lambda^{-1} \left(t^{-1}\varpi\right)^{-1} \varepsilon_1 k^\iota, & l = m. \end{cases}$$

If $1 \leq l \leq m-1$, we have that $\lambda^{-1}\varepsilon_1 d_l^{-1} w_l k^\iota$ is a scalar multiple of $\varepsilon_{l+1} k^\iota \in \mathfrak{o}^m$ and the $l+1$ coordinate of $\varepsilon_{l+1} k^\iota$ is 1 modulo $\mathfrak{p}$, and therefore, $\lambda^{-1}\varepsilon_1 d_l^{-1} w_l k^\iota$ cannot be in the support of $\mathcal{F}_\psi \phi$ for any scalar $\lambda$. If $l = m$, $\lambda^{-1}\varepsilon_1 d_m^{-1} w_l k^\iota$ is a scalar multiple of $\varepsilon_1 k^\iota \in \mathfrak{o}^m$, which satisfies $\nu(\varepsilon_1 k^\iota) \equiv \varepsilon_1 \pmod{\mathfrak{p}}$. Therefore, in order for $\lambda^{-1}\varepsilon_1 d_l^{-1} w_l k^\iota$ to be in $\mathfrak{o}^m$ and to be $\varepsilon_1$ modulo $\mathfrak{p}$, we must have $l = m$, and $\lambda^{-1}\left(t^{-1}\varpi\right)^{-1} \in 1 + \mathfrak{p}$. Hence, we have $\tilde{J}_l = 0$ for $1 \leq l \leq m-1$, and for $l = m$, we have that $\lambda$ is integrated on $\left(t^{-1}\varpi\right)^{-1}(1+\mathfrak{p})$ and that

$$\begin{aligned} \tilde{J}_m &= q^{-\binom{m}{2}} q^{-m(s-1)} \xi^m \cdot \int_{I_m^+} \int_{1+\mathfrak{p}} \omega\left(t^{-1}\varpi\right)^{-1} \mu\left(t^{-1}\varpi\right)^{-m} \left|\left(t^{-1}\varpi\right)\right|^{-m(s-1)} \\ &\quad \times q^{\frac{m}{2}} d^\times \lambda d^\times k \\ &= \frac{1}{|\mathfrak{f}^*|} \frac{q^{-\binom{m}{2}}}{[\mathrm{GL}_m(\mathfrak{f}) : N_m(\mathfrak{f})]} \cdot q^{\frac{m}{2}}. \end{aligned}$$

Therefore,

$$\tilde{J} = \tilde{J}_m = \frac{1}{|\mathfrak{f}^*|} \frac{q^{-\binom{m}{2}}}{[\mathrm{GL}_m(\mathfrak{f}) : N_m(\mathfrak{f})]} \cdot q^{\frac{m}{2}}. \tag{3.29}$$

Recalling the fact that when $(\omega \cdot \mu^m) \restriction_{\mathfrak{o}^*} \neq 1$, the Gauss sum

$$G\left(\omega \cdot \mu^m, \psi\right) = \sum_{\lambda \in \mathfrak{f}^*} \omega(\lambda) \mu(\lambda)^m \psi(-\lambda)$$

has absolute value $\sqrt{q}$, we have that

$$G\left(\omega\cdot\mu^m,\psi\right)^{-1}=\frac{\overline{G\left(\omega\cdot\mu^m,\psi\right)}}{q}=\frac{1}{q}\sum_{\lambda\in\mathfrak{f}^*}\omega\left(\lambda^{-1}\right)\mu\left(\lambda^{-m}\right)\psi\left(\lambda\right). \quad (3.30)$$

By Eqs. (3.28), (3.29), and (3.30), we get

$$\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=\frac{\tilde{J}}{J}$$
$$=\begin{cases}\left(\xi q^{-\left(s-\frac{1}{2}\right)}\right)^{m-2}\cdot\frac{1-\xi q^{-s}}{1-\xi^{-1}q^{-(1-s)}} & \left(\omega\cdot\mu^m\right)\restriction_{\mathfrak{o}^*}=1,\\ \left(\xi q^{-\left(s-\frac{1}{2}\right)}\right)^{m-1}\frac{1}{\sqrt{q}}\sum_{\lambda\in\mathfrak{f}^*}\omega\left(\lambda^{-1}\right)\mu\left(\lambda^{-m}\right)\psi\left(\lambda\right) & \text{otherwise.}\end{cases}$$

The formula $\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=\left(\xi q^{-\left(s-\frac{1}{2}\right)}\right)^{m-1}\gamma\left(s,\left(\omega\cdot\mu^m\right)_\xi,\psi\right)$ now follows from a standard computation of the local factors of Tate's functional equation, see for instance [17, Section 7.1] or [13, Proposition 3.8].

The claim about the other twisted exterior square factors now follows from Theorem 2.6 and the fact that $1-\xi Z$ and $1-\xi^{-1}q^{-1}Z^{-1}$ do not have mutual roots. □

***Remark 3.6*** For the choice of test data $\left(\pi\left(\sigma_{2m}^{-1}\right)W,\phi\right)$ as in the proof, we have that $J\left(s,\pi\left(\sigma_{2m}^{-1}\right)W,\phi,\mu,\psi\right)$ is non-zero if and only if $\mu$ is tamely ramified: otherwise, on the right-hand side of Eq. (3.19), we will have a product $\prod_{i=1}^{m}\mu\left(u_i'\right)$, and since we integrate $u_i'$ over $1+\mathfrak{p}$, we have that the integral vanishes unless the restriction of $\mu$ to $1+\mathfrak{p}$ is trivial.

### 3.3 The odd case

In this section, we compute the twisted exterior square factors for the odd case $n=2m+1$.

**Theorem 3.7** *Let $\pi$ be a simple supercuspidal representation of $\mathrm{GL}_{2m+1}(F)$ with central character $\omega$, associated with the data $(t_0,\zeta)$. Let $\mu:F^*\to\mathbb{C}^*$ be a unitary tamely ramified character, i.e., $\mu\restriction_{1+\mathfrak{p}}=1$. Then*

$$\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=\left(\mu\left(t^{-1}\varpi\right)\zeta^2q^{-\left(s-\frac{1}{2}\right)}\right)^m.$$

*Furthermore, in this case,* $L\left(s,\pi,\wedge^2\otimes\mu\right)=1$, $\epsilon\left(s,\pi,\wedge^2\otimes\mu,\psi\right)=\gamma\left(s,\pi,\wedge^2\otimes\mu,\psi\right)$.

***Proof*** We compute $J\left(s, \pi\left(\sigma_{2m+1}^{-1}\right) W, \phi, \mu, \psi\right)$ and $\tilde{J}\left(s, \pi\left(\sigma_{2m+1}^{-1}\right) W, \phi, \mu, \psi\right)$, where again $W$ is the Whittaker function introduced in Sect. 2.3, but this time

$$\phi(x) = \begin{cases} \delta_0(\nu(x)) & x \in \mathfrak{o}^m, \\ 0 & \text{otherwise}, \end{cases} \qquad \mathcal{F}_\psi \phi(x) = \begin{cases} q^{-\frac{m}{2}} & x \in \mathfrak{o}^m, \\ 0 & \text{otherwise}, \end{cases}$$

where $\delta_0$ is the indicator function of $0 \in \mathfrak{f}^m$.

By the Iwasawa decomposition, $J\left(s, \pi\left(\sigma_{2m+1}^{-1}\right) W, \phi, \mu, \psi\right)$ is given by

$$J = \int W\left(\sigma_{2m+1}\begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix}\begin{pmatrix} ak & & \\ & ak & \\ & & 1 \end{pmatrix}\begin{pmatrix} I_m & & \\ & I_m & \\ & Z & 1 \end{pmatrix}\sigma_{2m+1}^{-1}\right)\phi(Z) \\ \cdot |\det a|^{s-1} \mu(\det(ak))\, \delta_{B_m}^{-1}(a)\, dX d^\times a d^\times k dZ, \tag{3.31}$$

where $X$ is integrated on $\mathcal{N}_m^-$, $a$ is integrated on the diagonal matrix subgroup $A_m$, $k$ is integrated on $K_m = \mathrm{GL}_m(\mathfrak{o})$, and $Z$ is integrated on $M_{1\times m}(F)$. In order for $Z$ to be in the support of $\phi$, we must have $Z \in M_{1\times m}(\mathfrak{o})$, such that $\nu(Z) = 0$, i.e., $Z \in M_{1\times m}(\mathfrak{p})$. For such fixed $Z$, we have that

$$\nu\left(\sigma_{2m+1}\begin{pmatrix} I_m & & \\ & I_m & \\ & Z & 1 \end{pmatrix}\sigma_{2m+1}^{-1}\right) = I_{2m+1},$$

and therefore,

$$\sigma_{2m+1}\begin{pmatrix} I_m & & \\ & I_m & \\ & Z & 1 \end{pmatrix}\sigma_{2m+1}^{-1} \in I_{2m+1}^+.$$

Hence, in order for $X, a, k$ to contribute to the integral, we must have

$$\sigma_{2m+1}\begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix}\begin{pmatrix} ak & & \\ & ak & \\ & & 1 \end{pmatrix}\sigma_{2m+1}^{-1} = \lambda u' g_{2m+1}^l k',$$

where $\lambda \in F^*$, $u' \in N_{2m+1}$, $l \in \mathbb{Z}$, $k' \in I_{2m+1}^+$. Since $g_{2m+1}^{2m+1} = t^{-1}\varpi I_{2m+1}$, we may assume (by modifying $\lambda$) that $1 \le l \le 2m+1$. Notice that

$$\lambda g_{2m+1}^l k' = u'^{-1}\sigma_{2m+1}\begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix}\begin{pmatrix} ak & & \\ & ak & \\ & & 1 \end{pmatrix}\sigma_{2m+1}^{-1}, \tag{3.32}$$

and the right-hand side of Eq. (3.32) has $\varepsilon_{2m+1} = (0, \dots, 0, 1)$ as its last row. On the other hand, the last row of $\lambda g_{2m+1}^l k'$ is $\lambda t^{-1}\varpi \varepsilon_l k'$, where $\varepsilon_l$ is the $l$-th standard row vector. Since $\varepsilon_l k'$ is the $l$-th row of $k'$, we have that $\nu\left(\varepsilon_l k'\right)$ is the $l$-th row of an upper triangular unipotent matrix, and therefore, the equality $\lambda t^{-1}\varpi\varepsilon_l k' = \varepsilon_{2m+1}$ can't hold unless $l = 2m+1$. Thus, we have $l = 2m+1$, and that the last row of $k'$ is a scalar multiple of $\varepsilon_{2m+1}$. Since $k' \in \mathrm{GL}_{2m+1}(\mathfrak{o})$, we may assume (by modifying $\lambda$ by a unit) that the last row of $k'$ is $\varepsilon_{2m+1}$. We write $k' = \begin{pmatrix} k'' & v \\ & 1 \end{pmatrix}$, where $k'' \in I_{2m}^{+}$ and $v$ is a column vector in $M_{2m\times 1}(\mathfrak{o})$. Writing $k' = \begin{pmatrix} I_{2m} & v \\ & 1 \end{pmatrix}\begin{pmatrix} k'' & \\ & 1 \end{pmatrix}$, we may assume (by modifying $u'$) that $k' = \begin{pmatrix} k'' & \\ & 1 \end{pmatrix}$, which implies that $u' = \begin{pmatrix} u'' & \\ & 1 \end{pmatrix}$ for $u'' \in N_{2m}$. Thus, we get that $\lambda t^{-1}\varpi = 1$, and that

$$\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} ak & \\ & ak \end{pmatrix}\sigma_{2m}^{-1} = \lambda u'' g_{2m}^{2m} k'' \in \lambda N_{2m} g_{2m}^{2m} I_{2m}^{+}. \tag{3.33}$$

Since Eq. (3.33) holds, we can apply Theorem 3.4 and use $\lambda t^{-1}\varpi = 1$ to get that

$$\begin{aligned} a = \lambda \cdot d_m \cdot \operatorname{diag}(u_1, \ldots, u_m) = \operatorname{diag}(u_1, \ldots, u_m), \text{ where } u_1, \ldots, u_m \in \mathfrak{o}^*, \\ d^{\times} a = \prod_{i=1}^{m} d^{\times} u_i, \\ \delta_{B_m}^{-1}(a) = 1. \end{aligned} \tag{3.34}$$

Denote

$$\begin{aligned} k_0 &= \operatorname{diag}(u_1, \ldots, u_m)\, k, \\ d^{\times} k &= d^{\times} k_0. \end{aligned} \tag{3.35}$$

Then $k = \operatorname{diag}(u_1, \ldots, u_m)^{-1} k_0$, and by Theorem 3.4

$$k_0 \in \nu^{-1}\left(N_m(\mathfrak{f})\, w_m N_m(\mathfrak{f})\right) = \nu^{-1}\left(N_m(\mathfrak{f})\right) = I_m^{+} \tag{3.36}$$

Furthermore, since $\nu(k_0) \in N_m(\mathfrak{f}) = w_m N_m(\mathfrak{f})$, by Theorem 3.4 we have that $X \in \mathcal{N}_m^{-}(\mathfrak{o})$ and that $X$ satisfies $\nu(X) \in \mathcal{N}_m^{-}(\mathfrak{f}) \cap \left(w_m \mathcal{N}_m^{+}(\mathfrak{f})\, w_m^{-1}\right) = \{0_m\}$, i.e.,

$$X \in \mathcal{N}_m^{-}(\mathfrak{p}). \tag{3.37}$$

Also, since $\nu(k_0) \in w_m N_m(\mathfrak{f})$, by Theorem 3.4 we have for such $Z$, $X$, $a$, and $k$ that

$$\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} ak & \\ & ak \end{pmatrix}\sigma_{2m}^{-1} = v_0,$$

where $\nu(v_0) = v_0'$ is an upper triangular unipotent matrix having zeros right above its diagonal, and therefore,

$$\begin{aligned} Y &= \sigma_{2m+1} \begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix} \begin{pmatrix} ak & & \\ & ak & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_m & & \\ & I_m & \\ & Z & 1 \end{pmatrix} \sigma_{2m+1}^{-1} \\ &= \begin{pmatrix} v_0 & \\ & 1 \end{pmatrix} \sigma_{2m+1} \begin{pmatrix} I_m & & \\ & I_m & \\ & Z & 1 \end{pmatrix} \sigma_{2m+1}^{-1} \end{aligned} \tag{3.38}$$

satisfies $\nu(Y) = \begin{pmatrix} v_0' & \\ & 1 \end{pmatrix}$, which implies that $\nu(Y)$ is an upper unipotent matrix with zeros right above its diagonal. Since $Y$ also has zero at its left-bottom corner, we have from Eq. (2.1)

$$W(Y) = 1. \tag{3.39}$$

From Eqs. (3.34) and (3.35), we have that $ak = k_0 \in I_m^+$, so $\det(ak) = \det(k_0) \in 1 + \mathfrak{p}$, which implies

$$\begin{aligned} |\det a|^s &= 1, \\ \mu(\det(ak)) &= 1 \end{aligned} \tag{3.40}$$

as $\mu$ is tamely ramified.

Therefore, we have by substituting in Eq. (3.31), Eqs. (3.34)–(3.40) that

$$\begin{aligned} J\left(s, \pi\left(\sigma_{2m+1}^{-1}\right) W, \phi, \mu, \psi\right) &= \int_{M_{1\times m}(\mathfrak{p})} \int_{I_m^+} \int_{(\mathfrak{o}^*)^m} \int_{\mathcal{N}_m^-(\mathfrak{p})} dX \left(\prod_{i=1}^m d^\times u_i\right) \\ &\quad \times d^\times k_0 dZ \\ &= \frac{1}{|M_{1\times m}(\mathfrak{f})|} \frac{1}{[\mathrm{GL}_m(\mathfrak{f}) : N_m(\mathfrak{f})]} \frac{1}{\left|\mathcal{N}_m^-(\mathfrak{f})\right|}. \end{aligned} \tag{3.41}$$

We now move to compute $\tilde{J}\left(s, \pi\left(\sigma_{2m+1}^{-1}\right) W, \phi, \mu, \psi\right)$. By Theorem 2.3, we need to evaluate the integral

$$\begin{aligned} \tilde{J} = \int W\left(\begin{pmatrix} & 1 \\ I_{2m} & \end{pmatrix} \sigma_{2m+1} \begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix} \begin{pmatrix} ak & & \\ & ak & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_m & & -{}^tZ \\ & I_m & \\ & & 1 \end{pmatrix} \sigma_{2m+1}^{-1}\right) \\ \cdot \mathcal{F}_\psi \phi(Z) |\det a|^s \mu(\det(ak)) \delta_{B_m}^{-1}(a) \, dX d^\times a d^\times k dZ, \end{aligned} \tag{3.42}$$

where again $X \in \mathcal{N}_m^-$, $a \in A_m$, $k \in K_m = \mathrm{GL}_m(\mathfrak{o})$, $Z \in M_{1\times m}(F)$. We notice that for every $Z \in M_{1\times m}(\mathfrak{o})$,

$$\sigma_{2m+1}\begin{pmatrix} I_m & & -{}^tZ \\ & I_m & \\ & & 1 \end{pmatrix}\sigma_{2m+1}^{-1} \in I_{2m+1}^+.$$

Therefore, in order for $X$, $a$, $k$ to support the integrand, we need

$$\begin{pmatrix} & 1 \\ I_{2m} & \end{pmatrix}\sigma_{2m+1}\begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix}\begin{pmatrix} ak & & \\ & ak & \\ & & 1 \end{pmatrix}\sigma_{2m+1}^{-1} = \lambda u' g_{2m+1}^l k', \tag{3.43}$$

where $\lambda \in F^*$, $u' \in N_{2m+1}$, $l \in \mathbb{Z}$, $k' \in I_{2m+1}^+$. Note that the left-hand side of Eq. (3.43) has $e_1 = {}^t(1, 0, \dots, 0)$ as its last column. Therefore, $\lambda u' g_{2m+1}^l k'$ needs to have $e_1$ as its last column, i.e., $\lambda u' g_{2m+1}^l k' e_{2m+1} = e_1$, which implies $\lambda g_{2m+1}^l k' e_{2m+1} = u'^{-1} e_1 = e_1$. Since the $2m+1-l$ coordinate of $\nu\left(g_{2m+1}^l k' e_{2m+1}\right)$ is 1, we must have $l = 2m$. Therefore, from $\lambda g_{2m+1}^{2m} k' e_{2m+1} = e_1$, we get that the last column of $k'$ is a scalar multiple of $e_{2m+1}$. Modifying $\lambda$ by a unit, we may assume that $k'$ has $e_{2m+1}$ as its last column. Write $k' = \left(\begin{smallmatrix} k'' & \\ v & 1 \end{smallmatrix}\right)$, where $k'' \in I_{2m}^+$, $v \in M_{1\times 2m}(\mathfrak{p})$. Writing $g_{2m+1}^{2m} = \left(\begin{smallmatrix} & 1 \\ t^{-1}\varpi I_{2m} & \end{smallmatrix}\right)$, we have

$$\begin{aligned} \lambda u' g_{2m+1}^{2m} k' &= \lambda u' \begin{pmatrix} & 1 \\ t^{-1}\varpi I_{2m} & \end{pmatrix}\begin{pmatrix} k'' & \\ v & 1 \end{pmatrix} \\ &= \lambda u' \begin{pmatrix} 1 & v\left(t^{-1}\varpi k''\right)^{-1} \\ & I_{2m} \end{pmatrix}\begin{pmatrix} & 1 \\ t^{-1}\varpi I_{2m} & \end{pmatrix}\begin{pmatrix} k'' & \\ & 1 \end{pmatrix}. \end{aligned}$$

Therefore, by replacing $u'$ by $u'\left(\begin{smallmatrix} 1 & v(t^{-1}\varpi k'')^{-1} \\ & I_{2m} \end{smallmatrix}\right)$, we may assume $k' = \left(\begin{smallmatrix} k'' & \\ & 1 \end{smallmatrix}\right)$, which implies by Eq. (3.43) that $u' = \left(\begin{smallmatrix} 1 & \\ & u'' \end{smallmatrix}\right)$ for $u'' \in N_{2m}$. Substituting the expressions for $u'$, $k'$, and the expression $g_{2m+1}^{2m} = \left(\begin{smallmatrix} & 1 \\ t^{-1}\varpi I_{2m} & \end{smallmatrix}\right)$ in Eq. (3.43), we get that

$$\sigma_{2m+1}\begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix}\begin{pmatrix} ak & & \\ & ak & \\ & & 1 \end{pmatrix}\sigma_{2m+1}^{-1} = \begin{pmatrix} \lambda t^{-1}\varpi u'' k'' & \\ & \lambda \end{pmatrix},$$

and therefore, $\lambda = 1$ and

$$\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} ak & \\ & ak \end{pmatrix}\sigma_{2m}^{-1} = t^{-1}\varpi u'' k''.$$

By Theorem 3.4, we have that

$$a = d_m \cdot \operatorname{diag}(u_1, \ldots, u_m) = t^{-1}\varpi \operatorname{diag}(u_1, \ldots, u_m), \text{ where } u_1, \ldots, u_m \in \mathfrak{o}^*,$$
$$d^\times a = \prod_{i=1}^{m} d^\times u_i, \tag{3.44}$$
$$\delta_{B_m}^{-1}(a) = 1.$$

Denote

$$\begin{aligned} k_0 &= \operatorname{diag}(u_1, \ldots, u_m)\, k, \\ d^\times k_0 &= d^\times k. \end{aligned} \tag{3.45}$$

Then by Theorem 3.4,

$$k_0 \in \nu^{-1}(N_m(\mathfrak{f})\, w_m N_m(\mathfrak{f})) = \nu^{-1}(N_m(\mathfrak{f})) = I_m^+. \tag{3.46}$$

Since $\nu(k_0) \in w_m N_m(\mathfrak{f}) = N_m(\mathfrak{f})$, by Theorem 3.4, we have $X \in \mathcal{N}_m^-(\mathfrak{o})$ satisfies $\nu(X) \in \mathcal{N}_m^-(\mathfrak{f}) \cap \left(w_m \mathcal{N}_m^+(\mathfrak{f})\, w_m^{-1}\right) = \{0_m\}$, i.e.,

$$X \in \mathcal{N}_m^-(\mathfrak{p}). \tag{3.47}$$

Also, since $\nu(k_0) \in w_m N_m(\mathfrak{f})$, by Theorem 3.4 we have for such elements that

$$\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} ak & \\ & ak \end{pmatrix} \sigma_{2m}^{-1} = g_{2m}^{2m} v_0 = t^{-1}\varpi v_0,$$

where $v_0 \in \mathrm{GL}_{2m}(\mathfrak{o})$ satisfies that $\nu(v_0) = v_0'$ is an upper triangular unipotent matrix with zeros right above its diagonal. Hence, we have that

$$\begin{aligned} Y &= \begin{pmatrix} & 1 \\ I_{2m} & \end{pmatrix} \sigma_{2m+1} \begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix} \begin{pmatrix} ak & & \\ & ak & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_m & & -{}^tZ \\ & I_m & \\ & & 1 \end{pmatrix} \sigma_{2m+1}^{-1} \\ &= g_{2m+1}^{2m} \begin{pmatrix} v_0 & \\ & 1 \end{pmatrix} \sigma_{2m+1} \begin{pmatrix} I_m & & -{}^tZ \\ & I_m & \\ & & 1 \end{pmatrix} \sigma_{2m+1}^{-1}. \end{aligned} \tag{3.48}$$

Denote

$$Y' = \begin{pmatrix} v_0 & \\ & 1 \end{pmatrix} \sigma_{2m+1} \begin{pmatrix} I_m & & -{}^tZ \\ & I_m & \\ & & 1 \end{pmatrix} \sigma_{2m+1}^{-1},$$

then $Y = g_{2m+1}^{2m} Y'$, and $\nu\left(Y'\right)$ is an upper triangular unipotent matrix with zeros right above its diagonal, as it is a product of such. $Y'$ also has zero at its left-bottom corner.

Therefore, by Eq. (2.1)

$$W(Y) = \zeta^{2m}. \tag{3.49}$$

By Eq. (3.44) and Eq. (3.45), we have that $ak = t^{-1}\varpi k_0$, and therefore, $\det(ak) = \left(t^{-1}\varpi\right)^m \det k_0$. Since $k_0 \in I_m^+$, then $\det k_0 \in 1 + \mathfrak{p}$, which implies

$$\begin{aligned} |\det a|^s &= q^{-ms}, \\ \mu(\det(ak)) &= \mu\left(t^{-1}\varpi\right)^m, \end{aligned} \tag{3.50}$$

as $\mu$ is tamely ramified.

By substituting in Eq. (3.42), Eqs. (3.44)–(3.50), we have

$$\begin{aligned} \tilde{J} &= \int_{M_{1\times m}(\mathfrak{o})} \int_{I_m^+} \int_{(\mathfrak{o}^*)^m} \int_{\mathcal{N}_m^-(\mathfrak{p})} \zeta^{2m} q^{-\frac{m}{2}} q^{-ms} \mu\left(t^{-1}\varpi\right)^m \\ &\quad \times dX \left(\prod_{i=1}^m d^\times u_i\right) d^\times k_0 dZ \\ &= \frac{1}{[\mathrm{GL}_m(\mathfrak{f}) : N_m(\mathfrak{f})]} \frac{1}{\left|\mathcal{N}_m^-(\mathfrak{f})\right|} q^{-\frac{m}{2}} \zeta^{2m} q^{-ms} \mu\left(t^{-1}\varpi\right)^m. \end{aligned} \tag{3.51}$$

We get from Eqs. (3.41) and (3.51),

$$\gamma\left(s, \pi, \wedge^2 \otimes \mu, \psi\right) = \frac{\tilde{J}}{J} = \mu\left(t^{-1}\varpi\right)^m \zeta^{2m} q^{-ms} q^{\frac{m}{2}}.$$

The result regarding the other local factors now follows from Theorems 2.4 and 2.5. ◻

## 4 Exterior square gamma factors local converse theorem

In this section, we present and prove a local converse theorem for simple supercuspidal representations. Unlike previous local converse theorems, which are usually based on Rankin–Selberg gamma factors, our theorem is based on twisted exterior square gamma factors.

**Theorem 4.1** *Let $n = 2m$ or $n = 2m + 1$. Let $(\pi, V_\pi)$, $\left(\pi', V_{\pi'}\right)$ be simple supercuspidal representations of $\mathrm{GL}_n(F)$, with the same central character $\omega = \omega_\pi = \omega_{\pi'}$, such that $\pi, \pi'$ are associated with the data $(t_0, \zeta)$ and $\left(t_0', \zeta'\right)$ correspondingly, where $\zeta^n = \omega\left(t^{-1}\varpi\right)$, $\zeta'^n = \omega\left(t'^{-1}\varpi\right)$, and $t, t' \in \mathfrak{o}^*$ are lifts of $t_0$, $t_0'$ respectively, i.e., $\nu(t) = t_0$, $\nu\left(t'\right) = t_0'$. Assume that*

1. *If $n = 2m$, then* $\gcd(m-1, q-1) = 1$.
2. *If $n = 2m+1$, then* $\gcd(m, q-1) = 1$.

*Suppose that for every unitary tamely ramified character* $\mu : F^* \to \mathbb{C}^*$, *we have*

$$\gamma\left(s, \pi, \wedge^2 \otimes \mu, \psi\right) = \gamma\left(s, \pi', \wedge^2 \otimes \mu, \psi\right). \tag{4.1}$$

*Then* $\zeta = \pm\zeta'$ *and* $t_0 = t_0'$.

***Proof*** Suppose $n = 2m$. Let $\mu$ be a unitary tamely ramified character. Denote $\xi = \zeta^2 \cdot \mu\left((-1)^{m-1} t^{-1}\varpi\right)$, $\xi' = \zeta'^2 \cdot \mu\left((-1)^{m-1} t'^{-1}\varpi\right)$. We claim that $\xi = \xi'$.

If $(\omega \cdot \mu^m) \restriction_{\mathfrak{o}^*} = 1$, we have by Theorem 3.5 and by Eq. (4.1) that

$$\left(\xi q^{-\left(s-\frac{1}{2}\right)}\right)^{m-2} \frac{1 - \xi q^{-s}}{1 - \xi^{-1} q^{-(1-s)}} = \left(\xi' q^{-\left(s-\frac{1}{2}\right)}\right)^{m-2} \frac{1 - \xi' q^{-s}}{1 - \xi'^{-1} q^{-(1-s)}}, \tag{4.2}$$

and we get $\xi = \xi'$ by comparing the poles of both sides of Eq. (4.2).

If $(\omega \cdot \mu^m) \restriction_{\mathfrak{o}^*} \neq 1$, we have by Theorem 3.5 and by Eq. (4.1) that

$$\left(\xi q^{-\left(s-\frac{1}{2}\right)}\right)^{m-1} \frac{1}{\sqrt{q}} \sum_{\lambda \in \mathfrak{f}^*} \psi(\lambda)\, \omega\left(\lambda^{-1}\right) \mu\left(\lambda^{-m}\right)$$
$$= \left(\xi' q^{-\left(s-\frac{1}{2}\right)}\right)^{m-1} \frac{1}{\sqrt{q}} \sum_{\lambda \in \mathfrak{f}^*} \psi(\lambda)\, \omega\left(\lambda^{-1}\right) \mu\left(\lambda^{-m}\right).$$

Therefore,

$$\left(\frac{\xi}{\xi'}\right)^{m-1} = 1. \tag{4.3}$$

On the other hand,

$$\frac{\xi}{\xi'} = \frac{\zeta^2 \mu\left((-1)^{m-1} t^{-1}\varpi\right)}{\zeta'^2 \mu\left((-1)^{m-1} t'^{-1}\varpi\right)} = \frac{\zeta^2}{\zeta'^2} \mu\left(t' t^{-1}\right). \tag{4.4}$$

Since $\zeta^{2m} = \omega\left(t^{-1}\varpi\right)$ and $\zeta'^{2m} = \omega\left(t'^{-1}\varpi\right)$, we get from Eq. (4.4) that

$$\left(\frac{\xi}{\xi'}\right)^m = \frac{\omega\left(t^{-1}\varpi\right)}{\omega\left(t'^{-1}\varpi\right)} \mu\left(t' t^{-1}\right)^m = \omega\left(t' t^{-1}\right) \mu\left(t' t^{-1}\right)^m,$$

which implies that

$$\left(\frac{\xi}{\xi'}\right)^{m(q-1)} = 1 \tag{4.5}$$

as $\omega, \mu$ are tamely ramified. Since $m - 1$ is coprime to $m$ and to $q - 1$, we have that $\gcd(m - 1, m(q - 1)) = 1$, and therefore, $m - 1$ is invertible modulo $m(q - 1)$, which implies from Eqs. (4.3) and (4.5) that $\frac{\xi}{\xi'} = 1$, i.e., $\xi = \xi'$.

We proved $\xi = \xi'$. Therefore by Eq. (4.4), we have $\mu\left(t' t^{-1}\right) = \left(\zeta' \zeta^{-1}\right)^2$ for every unitary tamely ramified character $\mu$. Choosing the trivial character, this implies that $\zeta^2 = \zeta'^2$. Suppose $t_0 \neq t_0'$, then there exists a unitary character $\mu_0 : \mathfrak{f}^* \to \mathbb{C}^*$, such that $\mu_0\left(t_0' t_0^{-1}\right) \neq 1$, and we can lift this to a unitary tamely ramified character $\mu : F^* \to \mathbb{C}^*$ that satisfies $\mu \restriction_{\mathfrak{o}^*} = \mu_0 \circ \nu$ and then $\mu\left(t' t^{-1}\right) = \mu_0\left(t_0' t_0^{-1}\right) \neq 1$, which is a contradiction to $\mu\left(t' t^{-1}\right) = \left(\zeta' \zeta^{-1}\right)^2 = 1$. Therefore, we must have $t_0 = t_0'$.

For $n = 2m + 1$, the proof is similar. We have that

$$\left(\frac{\zeta}{\zeta'}\right)^{2m+1} = \frac{\omega\left(t^{-1}\varpi\right)}{\omega\left(t'^{-1}\varpi\right)} = \omega\left(t' t^{-1}\right),$$

and therefore,

$$\left(\frac{\zeta}{\zeta'}\right)^{(2m+1)(q-1)} = 1, \tag{4.6}$$

as $\omega$ is tamely ramified. By Eq. (4.1) and Theorem 3.7, we have for any unitary tamely ramified character $\mu$

$$\left(\mu\left(t^{-1}\varpi\right)\zeta^2 q^{-\left(s-\frac{1}{2}\right)}\right)^m = \left(\mu\left(t'^{-1}\varpi\right)\zeta'^2 q^{-\left(s-\frac{1}{2}\right)}\right)^m,$$

which implies that

$$\left(\frac{\zeta}{\zeta'}\right)^{2m} = \mu\left(t'^{-1}t\right)^m. \tag{4.7}$$

Substituting the trivial character in Eq. (4.7), one gets $\zeta^{2m} = \zeta'^{2m}$, which implies that

$$\left(\frac{\zeta^2}{\zeta'^2}\right)^m = 1. \tag{4.8}$$

Since $\gcd(m, q-1) = \gcd(m, 2m+1) = 1$, we get that $\gcd(m, (2m+1)(q-1)) = 1$, and therefore, $m$ is invertible modulo $(2m+1)(q-1)$. By Eq. (4.6), Eq. (4.8), this implies $\frac{\zeta^2}{\zeta'^2} = 1$. By Eqs. (4.7) and (4.8), we have for every unitary tamely ramified character $\mu : F^* \to \mathbb{C}^*$, $\mu\left(t'^{-1}t\right)^m = 1$. Since $\mu\left(t'^{-1}t\right)^{q-1} = 1$, and since $m$ is coprime to $q-1$, $m$ is invertible modulo $q-1$, and therefore, we have that for every unitary tamely ramified character $\mu$, $\mu\left(t'^{-1}t\right) = 1$. As in the even case, this implies $t_0 = t_0'$. □

***Remark 4.2*** 1. In the even case, although we cannot prove $\pi \cong \pi'$, we get $\zeta = \pm\zeta'$. On the other hand, if $\pi$ and $\pi'$ are simple supercuspidal representations with the same central character $\omega$ associated to the data $(t_0, \zeta)$ and $(t_0, -\zeta)$, then we must have $\gamma\left(s, \pi, \wedge^2 \otimes \mu, \psi\right) = \gamma\left(s, \pi', \wedge^2 \otimes \mu, \psi\right)$ for all tamely ramified character $\mu$. Because by Theorem 3.5,

$$\gamma\left(s, \pi, \wedge^2 \otimes \mu, \psi\right) = \left(\xi q^{-\left(s-\frac{1}{2}\right)}\right)^{m-1} \gamma\left(s, \left(\omega \cdot \mu^m\right)_{\xi}, \psi\right),$$

$$\gamma\left(s, \pi', \wedge^2 \otimes \mu, \psi\right) = \left(\xi' q^{-\left(s-\frac{1}{2}\right)}\right)^{m-1} \gamma\left(s, \left(\omega \cdot \mu^m\right)_{\xi'}, \psi\right),$$

and $\xi = \zeta^2 \cdot \mu\left((-1)^{m-1} t^{-1} \varpi\right) = \xi'$.

2. In the odd case, we actually get that $\zeta = \zeta'$, since $\zeta^2 = \zeta'^2$ and $\zeta^{2m+1} = \zeta'^{2m+1} = \omega\left(t^{-1}\varpi\right)$, and then

$$\zeta = \frac{\zeta^{2m+1}}{\left(\zeta^2\right)^m} = \frac{\zeta'^{2m+1}}{\left(\zeta'^2\right)^m} = \zeta'.$$

As a consequence, when the hypotheses in Theorem 4.1 are met, we have $t_0 = t_0'$ and $\zeta = \zeta'$, so $\pi \cong \pi'$.

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