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Proof of two conjectures of Guo and Schlosser

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Abstract
Let $[n] = (1-q^n)/(1-q)$ denote the q-integer and $\Phi_n(q)$ the nth cyclotomic polynomial in q. Recently, Guo and Schlosser provided two conjectures: For any odd integer $n > 3$, modulo $[n]\Phi_n(q)(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n+1)/2}[4k+1]\frac{(aq^{-1};q^2)_k(q^{-1}/a;q^2)_k(q;q^2)_k^2}{(aq^4;q^2)_k(q^4/a;q^2)_k(q^2;q^2)_k^2}q^{4k}\equiv 0,$$

and modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n+1)/2}[4k+1]\frac{(aq^{-1};q^2)_k(q^{-1}/a;q^2)_k(q^{-1};q^2)_k(q;q^2)_k}{(aq^4;q^2)_k(q^4/a;q^2)_k(q^4;q^2)_k(q^2;q^2)_k}q^{6k}\equiv 0,$$

where $(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$. In this paper, we confirm these two conjectures and further give their generalizations involving two free parameters. Our proof uses Guo and Zudilin's 'creative microscoping' method and the Chinese remainder theorem for coprime polynomials.

Keywords q-congruence · q-supercongruence · Cyclotomic polynomial · Basic hypergeometric series · The Chinese remainder theorem

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1 Introduction

In the past few years, many q-congruences and q-supercongruences have been established by different authors. See, for example, [2,4–8,11–25]. In particular, Guo and Schlosser [9, Theorem 5.5] proved that, for any odd integer $n > 3$,

$$\sum_{k=0}^{(n+1)/2} [4k+1] \frac{(q^{-1};q^2)_k^2 (q;q^2)_k^2}{(q^4;q^2)_k^2 (q^2;q^2)_k^2} q^{4k} \equiv 0 \pmod{[n]\Phi_n(q)^2}, \tag{1.1}$$

where the *q-shifted factorial* is defined by

$$(a;q)_\infty = \prod_{j\geq 0}(1-aq^j) \quad \text{and} \quad (a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

For convenience, we adopt the notation

$$(a_1, a_2, \ldots, a_r; q)_k = (a_1;q)_k (a_2;q)_k \cdots (a_r;q)_k, \quad k \in \mathbb{C} \cup \infty.$$

The *q-integer* is defined by $[n] = [n]_q = (1-q^n)/(1-q) = 1 + q + \cdots + q^{n-1}$. Moreover, $\Phi_n(q)$ represents the nth *cyclotomic polynomial* in q, which may be defined as

$$\Phi_n(q) = \prod_{\substack{1\leq k\leq n \\ \gcd(n,k)=1}} (q - \zeta^k)$$

with ζ being an nth primitive root of unity.

Guo and Schlosser [9, Conjecture 5.6] also proposed the following conjecture: for any odd integer $n > 3$,

$$\sum_{k=0}^{(n+1)/2} [4k+1] \frac{(aq^{-1};q^2)_k (q^{-1}/a;q^2)_k (q;q^2)_k^2}{(aq^4;q^2)_k (q^4/a;q^2)_k (q^2;q^2)_k^2} q^{4k} \equiv 0 \pmod{[n]\Phi_n(q)(1-aq^n)(a-q^n)}. \tag{1.2}$$

It is known from [9] that, letting $a \to 1$ in (1.2), we are led to

$$\sum_{k=0}^{(n+1)/2} [4k+1] \frac{(q^{-1};q^2)_k^2 (q;q^2)_k^2}{(q^4;q^2)_k^2 (q^2;q^2)_k^2} q^{4k} \equiv 0 \pmod{[n]\Phi_n(q)^3},$$

which is a refinement of the q-supercongruence (1.1).

Additionally, Guo and Schlosser [9, Conjecture 5.8] provided another similar conjecture: for any odd integer $n > 3$,

$$\sum_{k=0}^{(n+1)/2}[4k+1]\frac{(aq^{-1};q^2)_k(q^{-1}/a;q^2)_k(q^{-1},q;q^2)_k}{(aq^4;q^2)_k(q^4/a;q^2)_k(q^4,q^2;q^2)_k}q^{6k}\equiv 0 \pmod{\Phi_n(q)^2(1-aq^n)(a-q^n)}. \tag{1.3}$$

Likewise, when $a\to 1$ in (1.3), we obtain

$$\sum_{k=0}^{(n+1)/2}[4k+1]\frac{(q^{-1};q^2)_k^3(q;q^2)_k}{(q^4;q^2)_k^3(q^2;q^2)_k}q^{6k}\equiv 0 \pmod{\Phi_n(q)^4}.$$

Inspired by Guo and Schlosser's work, we shall prove these two conjectures in this paper. Our proof relies on the following theorem, which may be deemed as a generalization of the q-supercongruences (1.2) and (1.3).

Theorem 1 *Let $n > 3$ be an odd integer and $r \in \{0, 2\}$. Then, modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$, we have*

$$\begin{aligned}\sum_{k=0}^{(n+1)/2}&[4k+1]\frac{(aq^{-1},q^{-1}/a,q^{1-r},cq,dq,q;q^2)_k}{(q^4/a,aq^4,q^{2+r},q^2/c,q^2/d,q^2;q^2)_k}\left(\frac{q^{5+r}}{cd}\right)^k\\&\equiv[n]Z_q(a,n)\frac{(aq^{-1},q^{-1}/a,q^{1-r},q/cd;q^2)_{(r+4)/2}}{(q,q^{r+2};q^2)_2(q^2/c,q^2/d;q^2)_{(r+4)/2}}\\&\times\sum_{k=0}^{\frac{n-r-3}{2}}\frac{(aq^{r+3},q^{r+3}/a,q^5,q^{r+5}/cd;q^2)_k}{(q^2,q^{6+r},q^{6+r}/c,q^{6+r}/d;q^2)_k}q^{2k},\end{aligned} \tag{1.4}$$

where

$$\begin{aligned}Z_q(a,n)&=(-1)^{\frac{r+2}{2}}q^{\frac{-2n+r^2+10r+26}{4}}\left\{\frac{n(1-aq^2)(1-q^2/a)(1-a)a^{(n-1)/2}}{(aq,q/a;q^2)_{(r+2)/2}(1-a^n)}-\frac{(1-q^2)^2}{(q;q^2)^2_{(r+2)/2}}\right\}\\&\times\frac{(1-aq^n)(a-q^n)}{(1-a)^2}+q^{\frac{(r-1)(n+1)}{2}+r+8}\frac{(q^{n-2};q^4)_2}{(q^{n-r-1};q^2)_{r+2}}.\end{aligned}$$

Further, for the $r=0$ and $a\to 1$ case of Theorem 1, we have the following stronger conclusion.

Theorem 2 *Let $n > 3$ be an odd integer. Then*

$$\begin{aligned}\sum_{k=0}^{M}[4k+1]&\frac{(q^{-1};q^2)_k^2(q;q^2)_k^2(cq,dq;q^2)_k}{(q^4;q^2)_k^2(q^2;q^2)_k^2(q^2/c,q^2/d;q^2)_k}\left(\frac{q^5}{cd}\right)^k\\ &\equiv S_q(n)\frac{(q^{-1};q^2)_2^2(q/cd;q^2)_2}{(q^2,q^2/c,q^2/d;q^2)_2}\\ &\times\sum_{k=0}^{\frac{n-3}{2}}\frac{(q^3;q^2)_k^2(q^5,q^5/cd;q^2)_k}{(q^2,q^6,q^6/c,q^6/d;q^2)_k}q^{2k} \pmod{[n]\Phi_n(q)^3}, \end{aligned} \tag{1.5}$$

where $M \in \{(n+1)/2, n-2\}$ and

$$\begin{aligned}S_q(n)=&\,q^{\frac{13-n}{2}}\frac{n^2(1-q^2)^2-(1+24q+22q^2+24q^3+q^4)}{24}[n]^3\\ &+q^{\frac{15-n}{2}}\frac{[n-2][n+2]}{[n-1][n+1]}[n].\end{aligned}$$

Putting $c \to 1$ and $d \to \infty$ in Theorem 2, we obtain the following result.

Corollary 3 *Let $n > 3$ be an odd integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\begin{aligned}\sum_{k=0}^{M}(-1)^k[4k+1]&\frac{(q^{-1};q^2)_k^2(q;q^2)_k^3}{(q^4;q^2)_k^2(q^2;q^2)_k^3}q^{5k+k^2}\\ &\equiv S_q(n)\frac{(q^{-1};q^2)_2^2}{(q^2;q^2)_2^2}\sum_{k=0}^{\frac{n-3}{2}}\frac{(q^3;q^2)_k^2(q^5;q^2)_k}{(q^2;q^2)_k(q^6;q^2)_k^2}q^{2k}.\end{aligned} \tag{1.6}$$

When $M = (p^l + 1)/2$ and $q \to 1$ in (1.6), we are led to the following: for odd prime $p > 3$,

$$\sum_{k=0}^{(p^l+1)/2}(-1)^k(4k+1)\frac{(-\frac{1}{2})_k^2(\frac{1}{2})_k^3}{k!^3(k+1)!^2}\equiv\frac{p^l}{4}\sum_{k=0}^{(p^l-3)/2}\frac{(\frac{3}{2})_k^2(\frac{5}{2})_k}{k!(k+2)!^2} \pmod{p^{l+3}},$$

where l is a positive integer, and the notation will be used frequently in this section.

Moreover, the case $c \to 1$, $d \to 1$ of Theorem 2 yields the following q-supercongruence.

Corollary 4 *Let $n>3$ be an odd integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{M}[4k+1]\frac{(q^{-1};q^2)_k^2(q;q^2)_k^4}{(q^4;q^2)_k^2(q^2;q^2)_k^4}q^{5k}$$
$$\equiv S_q(n)\frac{(q^{-1};q^2)_2^2(q;q^2)_2}{(q^2;q^2)_2^3}\sum_{k=0}^{\frac{n-3}{2}}\frac{(q^3;q^2)_k^2(q^5;q^2)_k^2}{(q^2;q^2)_k(q^6;q^2)_k^3}q^{2k}. \tag{1.7}$$

Putting $M=(p^l+1)/2$ and $q\to 1$ in (1.7), we gain the following: for odd prime $p>3$,

$$\sum_{k=0}^{(p^l+1)/2}(4k+1)\frac{(-\frac{1}{2})_k^2(\frac{1}{2})_k^4}{k!^4(k+1)!^2}\equiv\frac{3p^l}{16}\sum_{k=0}^{(p^l-3)/2}\frac{(\frac{3}{2})_k^2(\frac{5}{2})_k^2}{k!(k+2)!^3}\pmod{p^{l+3}}.$$

In addition, letting $r=2$ and $a\to 1$ in Theorem 1, and applying the L'Hospital rule, we arrive at the following conclusion.

Theorem 5 *Let $n>3$ be an odd integer. Then*

$$\sum_{k=0}^{(n+1)/2}[4k+1]\frac{(q^{-1};q^2)_k^3(q,cq,dq;q^2)_k}{(q^4;q^2)_k^3(q^2,q^2/c,q^2/d;q^2)_k}\left(\frac{q^7}{cd}\right)^k$$
$$\equiv T_q(n)\frac{(q^{-1};q^2)_3^3(q/cd;q^2)_3}{(q,q^4;q^2)_2(q^2/c,q^2/d;q^2)_3}$$
$$\times\sum_{k=0}^{\frac{n-5}{2}}\frac{(q^5;q^2)_k^3(q^7/cd;q^2)_k}{(q^2,q^8,q^8/c,q^8/d;q^2)_k}q^{2k}\pmod{\Phi_n(q)^4}, \tag{1.8}$$

where

$$T_q(n)=q^{\frac{25-n}{2}}\frac{(n^2-1)(1+q)^2(1-q^3)^2-24q(1+3q+7q^2+9q^3+7q^4+3q^5+q^6)}{24(1-q^3)^2(1+q+q^2)^2}[n]^3$$
$$+q^{\frac{n+21}{2}}\frac{(q^{n-2};q^4)_2}{(q^{n-3};q^2)_4}[n].$$

Taking $c\to q^{-2}$ and $d\to\infty$ in Theorem 5, we obtain the following result.

Corollary 6 *Let $n > 3$ be an odd integer. Then*

$$\sum_{k=0}^{(n+1)/2}(-1)^k[4k+1]\frac{(q^{-1};q^2)_k^4(q;q^2)_k}{(q^4;q^2)_k^4(q^2;q^2)_k}q^{9k+k^2}$$
$$\equiv T_q(n)\frac{(q^{-1};q^2)_3^3}{(q,q^4;q^2)_2(q^4;q^2)_3}$$
$$\times\sum_{k=0}^{\frac{n-5}{2}}\frac{(q^5;q^2)_k^3}{(q^2,q^8,q^{10};q^2)_k}q^{2k} \pmod{\Phi_n(q)^4}. \tag{1.9}$$

Letting $n = p^l$ be an odd prime power greater than 3 and $q \to 1$ in (1.9), we get

$$\sum_{k=0}^{(p^l+1)/2}(-1)^k(4k+1)\frac{(-\frac{1}{2})_k^4(\frac{1}{2})_k}{k!(k+1)!^4}\equiv\frac{p^l}{8}\sum_{k=0}^{(p^l-5)/2}\frac{(\frac{5}{2})_k^3}{k!(k+3)!(k+4)!} \pmod{p^4}.$$

Likewise, putting $c \to q^{-2}$ and $d \to q^{-2}$ in Theorem 5, we get the following q-supercongruence.

Corollary 7 *Let $n > 3$ be an odd integer. Then*

$$\sum_{k=0}^{(n+1)/2}[4k+1]\frac{(q^{-1};q^2)_k^5(q;q^2)_k}{(q^4;q^2)_k^5(q^2;q^2)_k}q^{11k}$$
$$\equiv T_q(n)\frac{(q^{-1};q^2)_3^3(q^5;q^2)_3}{(q,q^4;q^2)_2(q^4;q^2)_3^2}$$
$$\times\sum_{k=0}^{\frac{n-5}{2}}\frac{(q^5;q^2)_k^3(q^{11};q^2)_k}{(q^2,q^8;q^2)_k(q^{10};q^2)_k^2}q^{2k} \pmod{\Phi_n(q)^4}. \tag{1.10}$$

When $n = p^r$ is an odd prime power greater than 3 and $q \to 1$ in (1.10), we get

$$\sum_{k=0}^{(p^l+1)/2}(4k+1)\frac{(-\frac{1}{2})_k^5(\frac{1}{2})_k}{k!(k+1)!^5}\equiv\frac{315p^l}{64}\sum_{k=0}^{(p^l-5)/2}\frac{(\frac{5}{2})_k^3(\frac{11}{2})_k}{k!(k+3)!(k+4)!^2} \pmod{p^4}.$$

The rest of this paper is arranged as follows. Firstly, we present some lemmas which will be needed in the proof of Theorem 1 in Sect. 2. Then, we prove Theorem 1 and Guo and Schlosser's conjectures (1.2) and (1.3) in Sect. 3. Our proof makes use of the 'creative microscoping' method which was recently introduced by Guo and Zudilin [10], and the Chinese remainder theorem for coprime polynomials. Finally, in Sect. 4, we give a proof of Theorem 2.

2 Some Lemmas

We shall make use of Watson's ${}_8\phi_7$ transformation [1, Appendix (III.18)], which can be expressed as

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d, & e & q^{-N} \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{N+1} \end{matrix}; q, \frac{a^2q^{N+2}}{bcde}\right] \\
&\quad = \frac{(aq, aq/de; q)_N}{(aq/d, aq/e; q)_N}\, {}_4\phi_3\left[\begin{matrix} aq/bc, & d, & e, & q^{-N} \\ & aq/b, & aq/c, & deq^{-N}/a \end{matrix}; q, q\right],
\end{aligned} \tag{2.1}
$$

where the basic hypergeometric series ${}_{s+1}\phi_s$ is defined as

$$
{}_{s+1}\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_{s+1} \\ b_1, b_2, \ldots, b_s \end{matrix}; q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{s+1}; q)_k z^k}{(q, b_1, \ldots, b_s; q)_k}
$$

with $0 < |z| < 1$.

In this section, we shall give three lemmas, which will play an important role in our proof of Theorem 1.

Lemma 1 *Let $n > 3$ be an odd integer and $r \in \{0, 2\}$. Then*

$$
\sum_{k=0}^{M} [4k+1] \frac{(aq^{-1}, q^{-1}/a, q^{1-r}/b, cq, dq, q; q^2)_k}{(q^4/a, aq^4, bq^{2+r}, q^2/c, q^2/d, q^2; q^2)_k} \left(\frac{bq^{5+r}}{cd}\right)^k \equiv 0 \pmod{\Phi_n(q)}, \tag{2.2}
$$

where $M \in \{(n+1)/2, n-2\}$.

Proof Let $c_q(k)$ be the kth term on left-hand side of (2.2), i.e.,

$$
c_q(k) = [4k+1] \frac{(aq^{-1}, q^{-1}/a, q^{1-r}/b, cq, dq, q; q^2)_k}{(q^4/a, aq^4, bq^{2+r}, q^2/c, q^2/d, q^2; q^2)_k} \left(\frac{bq^{5+r}}{cd}\right)^k.
$$

Using the following q-congruence due to Guo and Schlosser [9, Lemma 3.1]

$$
\begin{aligned}
&\frac{(aq; q^2)_{(n-1)/2-k}}{(q^2/a; q^2)_{(n-1)/2-k}} \\
&\qquad \equiv (-a)^{\frac{n-1}{2}-2k} q^{\frac{(n-1)^2}{4}+k} \frac{(aq; q^2)_k}{(q^2/a; q^2)_k} \pmod{\Phi_n(q)},
\end{aligned}
$$

we have

$$
c_q(k) \equiv -c_q((n-1)/2-k) \pmod{\Phi_n(q)}.
$$

This proves that

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(aq^{-1},q^{-1}/a,q^{1-r}/b,cq,dq,q;q^2)_k}{(q^4/a,aq^4,bq^{2+r},q^2/c,q^2/d,q^2;q^d)_k}\left(\frac{bq^{5+r}}{cd}\right)^k\equiv 0 \pmod{\Phi_n(q)}. \tag{2.3}$$

Since the numerator of $c_q(k)$ contains the factor $(q;q^2)_k$, we see that $c_q(k)$ is congruent to 0 modulo $\Phi_n(q)$ for $(n+1)/2\le k\le n-2$. Thus, the proof of Lemma 1 is completed. □

Lemma 2 *Let $n>3$ be an odd integer and $r\in\{0,2\}$. Then*

$$\begin{aligned}&\sum_{k=0}^{M}[4k+1]\frac{(aq^{-1},q^{-1}/a,q^{1-r}/b,cq,dq,q;q^2)_k}{(q^4/a,aq^4,bq^{2+r},q^2/c,q^2/d,q^2;q^2)_k}\left(\frac{bq^{5+r}}{cd}\right)^k\\&\equiv[n]b^{\frac{n+1}{2}}q^{\frac{(r-1)(n+1)}{2}}\frac{(1-q^{n-2})(1-q^{n+2})(q^{-2-r}/b;q^2)_{(n+1)/2}}{(1-q^{-3})(1-q^{-1})(bq^{2+r};q^2)_{(n+1)/2}}\\&\times\sum_{k=0}^{(n+1)/2}\frac{(aq^{-1},q^{-1}/a,q^{1-r}/b,q/cd;q^2)_k}{(q^2,q^2/c,q^2/d,q^{-2-r}/b;q^2)_k}q^{2k}\pmod{\Phi_n(q)(1-aq^n)(a-q^n)},\end{aligned}\tag{2.4}$$

where $M\in\{(n+1)/2,n-2\}$.

Proof When $a=q^{-n}$ or q^n, the left-hand side of (2.4) equals

$${}_8\phi_7\left[\begin{matrix}q, & q^{\frac52}, & -q^{\frac52}, & cq, & dq, & q^{1-r}/b, & q^{-1-n}, & q^{-1+n}\\ & q^{\frac12}, & -q^{\frac12}, & q^2/c, & q^2/d, & bq^{2+r}, & q^{4+n}, & q^{4-n}\end{matrix};q^2,\frac{bq^{5+r}}{cd}\right].$$

Applying Watson's ${}_8\phi_7$ transformation (2.1), we obtain

$$\begin{aligned}&\sum_{k=0}^{M}[4k+1]\frac{(q^{-1+n},q^{-1-n},q^{1-r}/b,cq,dq,q;q^2)_k}{(q^{4-n},aq^{4+n},bq^{2+r},q^2/c,q^2/d,q^2;q^2)_k}\left(\frac{bq^{5+r}}{cd}\right)^k\\&=\frac{(q^3,bq^{3+r-n};q^2)_{(n+1)/2}}{(bq^{2+r},q^{4-n};q^2)_{(n+1)/2}}\sum_{k=0}^{(n+1)/2}\frac{(q^{-1+n},q^{-1-n},q^{1-r}/b,q/cd;q^2)_k}{(q^2,q^2/c,q^2/d,q^{-2-r}/b;q^2)_k}q^{2k}\\&=[n]b^{\frac{n+1}{2}}\frac{(1-q^{n-2})(1-q^{n+2})(q^{-2-r}/b;q^2)_{(n+1)/2}}{(1-q^{-3})(1-q^{-1})(bq^{2+r};q^2)_{(n+1)/2}}q^{\frac{(r-1)(n+1)}{2}}\\&\times\sum_{k=0}^{(n+1)/2}\frac{(q^{-1+n},q^{-1-n},q^{1-r}/b,q/cd;q^2)_k}{(q^2,q^2/c,q^2/d,q^{-2-r}/b;q^2)_k}q^{2k},\end{aligned}$$

which means that the q-supercongruence (2.4) holds modulo $1-aq^n$ and $a-q^n$. The proof then follows from Lemma 1 and the fact that $\Phi_n(q)$, $1-aq^n$ and $a-q^n$ are pairwise relatively prime polynomials. □

Lemma 3 *Let $n>3$ be an odd integer and $r\in\{0,2\}$. Then*

$$\begin{aligned}&\sum_{k=0}^{M}[4k+1]\frac{(aq^{-1},q^{-1}/a,q^{1-r}/b,cq,dq,q;q^2)_k}{(q^4/a,aq^4,bq^{2+r},q^2/c,q^2/d,q^2;q^2)_k}\left(\frac{bq^{5+r}}{cd}\right)^k\\&\quad\equiv[n]\frac{(bq^2;q^2)_{r/2}(b;q^2)_{(r+4)/2}(q;q^2)^2_{(n-1)/2}}{(q;q^2)_2(q^4/a;q^2)_{(n+r-1)/2}(aq^4;q^2)_{(n+r-1)/2}}\\&\quad\times\sum_{k=0}^{(n+r-1)/2}\frac{(aq^{-1},q^{-1}/a,q^{1-r}/b,q/cd;q^2)_k}{(q^2,q^2/c,q^2/d,q^{-2-r}/b;q^2)_k}q^{2k}\pmod{b-q^n},\end{aligned}$$

where $M\in\{(n+1)/2,n-2\}$.

Proof Letting $b=q^n$ in the left-hand side of the above relation, we have

$$\begin{aligned}&\sum_{k=0}^{M}[4k+1]\frac{(aq^{-1},q^{-1}/a,q^{1-r-n},cq,dq,q;q^2)_k}{(q^4/a,aq^4,q^{2+r+n},q^2/c,q^2/d,q^2;q^2)_k}\left(\frac{q^{5+r+n}}{cd}\right)^k\\&\quad=\frac{[n](q^{2+n};q^2)_{r/2}(q^n;q^2)_{(r+4)/2}(q;q^2)^2_{(n-1)/2}}{(q;q^2)_2(q^4/a;q^2)_{(n+r-1)/2}(aq^4;q^2)_{(n+r-1)/2}}\\&\quad\times\sum_{k=0}^{\frac{n+r-1}{2}}\frac{(aq^{-1},q^{-1}/a,q^{1-r-n},q/cd;q^2)_k}{(q^2,q^2/c,q^2/d,q^{-2-r-n};q^2)_k}q^{2k},\end{aligned}$$

which follows from the substitutions $a=q, q\mapsto q^2, b=cq, c=dq, d=aq^{-1}, e=q^{-1}/a$, and $N=(n+r-1)/2$ in Watson's ${}_8\phi_7$ transformation (2.1). Namely, Lemma 3 is true. □

3 Proof of Theorem 1

Firstly, we need the following two q-congruences:

$$\begin{aligned}&\frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)}\equiv 1\pmod{(1-aq^n)(a-q^n)},\\&\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)}\equiv 1\pmod{(b-q^n)},\end{aligned}$$

which can be found in Guo [3]. Employing the Chinese remainder theorem for coprime polynomials and combining Lemmas 2 and 3, we conclude that, modulo $\Phi_n(q)(1-aq^n)(a-q^n)(b-q^n)$,

$$\sum_{k=0}^{(n+1)/2}[4k+1]\frac{(aq^{-1},q^{-1}/a,q^{1-r}/b,cq,dq,q;q^2)_k}{(q^4/a,aq^4,bq^{2+r},q^2/c,q^2/d,q^2;q^2)_k}\left(\frac{bq^{5+r}}{cd}\right)^k$$
$$\equiv [n]W_q(a,b,n)\sum_{k=0}^{\frac{n+1}{2}}\frac{(aq^{-1},q^{-1}/a,q^{1-r}/b,q/cd;q^2)_k}{(q^2,q^2/c,q^2/d,q^{-2-r}/b;q^2)_k}q^{2k}, \tag{3.1}$$

where

$$W_q(a,b,n)=b^{\frac{n+1}{2}}\frac{(1-q^{n-2})(1-q^{n+2})(q^{-2-r}/b;q^2)_{(n+1)/2}}{(1-q^{-3})(1-q^{-1})(bq^{2+r};q^2)_{(n+1)/2}}q^{\frac{(r-1)(n+1)}{2}}$$
$$\times\frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)}$$
$$+\frac{(bq^2;q^2)_{r/2}(b;q^2)_{(r+4)/2}(q;q^2)^2_{(n-1)/2}}{(q;q^2)_2(q^4/a;q^2)_{(n+r-1)/2}(aq^4;q^2)_{(n+r-1)/2}}\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)}.$$

Obviously, the q-supercongruence (3.1) can be expressed as modulo $\Phi_n(q)(1-aq^n)(a-q^n)(b-q^n)$,

$$\sum_{k=0}^{(n+1)/2}[4k+1]\frac{(aq^{-1},q^{-1}/a,q^{1-r}/b,cq,dq,q;q^2)_k}{(q^4/a,aq^4,bq^{2+r},q^2/c,q^2/d,q^2;q^2)_k}\left(\frac{bq^{5+r}}{cd}\right)^k$$
$$\equiv [n]W_q(a,b,n)\sum_{k=0}^{(r+2)/2}\frac{(aq^{-1},q^{-1}/a,q^{1-r}/b,q/cd;q^2)_k}{(q^2,q^2/c,q^2/d,q^{-2-r}/b;q^2)_k}q^{2k}$$
$$+[n]W_q(a,b,n)\sum_{k=0}^{(n-r-3)/2}\frac{(aq^{-1},q^{-1}/a,q^{1-r}/b,q/cd;q^2)_{k+(r+4)/2}}{(q^2,q^2/c,q^2/d,q^{-2-r}/b;q^2)_{k+(r+4)/2}}q^{2k+r+4}. \tag{3.2}$$

Letting $b\to 1$ in (3.2) and using the relation

$$(1-q^n)(1+a^2-a-aq^n)=(1-a)^2+(1-aq^n)(a-q^n),$$

we attain modulo $\Phi_n(q)(1-aq^n)(a-q^n)(b-q^n)$,

$$\sum_{k=0}^{(n+1)/2}[4k+1]\frac{(aq^{-1},q^{-1}/a,q^{1-r},cq,dq,q;q^2)_k}{(q^4/a,aq^4,q^{2+r},q^2/c,q^2/d,q^2;q^2)_k}\left(\frac{q^{5+r}}{cd}\right)^k$$
$$\equiv [n]Y_q(a,n)\frac{(aq^{-1},q^{-1},q^{1-r},q/cd;q^2)_{(r+4)/2}}{(q,q^{r+2};q^2)_2(q^2/c,q^2/d;q^2)_{(r+4)/2}}q^{r+4}$$
$$\times\sum_{k=0}^{(n-r-3)/2}\frac{(aq^{r+3},q^{r+3}/a,q^{1-r},q^{r+5}/cd;q^5)_k}{(q^2,q^{6+r}/c,q^{6+r}/d,q^{6+r};q^2)_k}q^{2k}, \tag{3.3}$$

where

$$Y_q(a,n)=\left\{\frac{(q^{n-2};q^4)_2}{(q^{n-r-1};q^2)_{r+2}}q^{\frac{(r-1)(n+1)}{2}+4}+(-1)^{(r+2)/2}\frac{(q;q^2)^2_{(n-1)/2}}{(q^4/a,aq^4;q^2)_{(n+r-1)/2}}\right\}$$
$$\times\frac{(1-aq^n)(a-q^n)}{(1-a)^2}+\frac{(q^{n-2};q^4)_2}{(q^{n-r-1};q^2)_{r+2}}q^{\frac{(r-1)(n+1)}{2}+4}.$$

Noting $q^n\equiv 1 \pmod{\Phi_n(q)}$ and recalling

$$(aq^2,q^2/a;q^2)_{(n-1)/2}\equiv(-1)^{(n-1)/2}\frac{(1-a^n)q^{-(n-1)^2/4}}{(1-a)a^{(n-1)/2}} \pmod{\Phi_n(q)},$$
$$(aq,q/a;q^2)_{(n-1)/2}\equiv(-1)^{(n-1)/2}\frac{(1-a^n)q^{(1-n^2)/4}}{(1-a)a^{(n-1)/2}} \pmod{\Phi_n(q)},$$

which were first observed by Guo [3, Lemma 2.1], we are led to the following q-congruence: modulo $\Phi_n(q)$,

$$\frac{(q^{n-2};q^4)_2}{(q^{n-r-1};q^2)_{r+2}}q^{\frac{(r-1)(n+1)}{2}+4}+(-1)^{(r+2)/2}\frac{(q;q^2)^2_{(n-1)/2}}{(q^4/a,aq^4;q^2)_{(n+r-1)/2}}$$
$$\equiv(-1)^{\frac{r+2}{2}}q^{\frac{-2n+r^2+6r+10}{4}}\left\{\frac{n(1-aq^2)(1-q^2/a)(1-a)a^{(n-1)/2}}{(aq,q/a;q^2)_{(r+2)/2}(1-a^n)}-\frac{(1-q^2)^2}{(q;q^2)^2_{(r+2)/2}}\right\}. \tag{3.4}$$

The proof then follows from (3.3) and (3.4).

Obviously, Theorem 1 also holds true when the summation in the left-hand side of (1.4) is from 0 to $n-2$.

We now prove (1.2) and (1.3) which were conjectured by Guo and Schlosser.

Proof of (1.2) When $cd=q$ and $r=0$ in Theorem 1, we have

$$\sum_{k=0}^{(n+1)/2}[4k+1]\frac{(aq^{-1};q^2)_k(q^{-1}/a;q^2)_k(q;q^2)_k^2}{(aq^4;q^2)_k(q^4/a;q^2)_k(q^2;q^2)_k^2}q^{4k}\equiv 0 \pmod{\Phi_n(q)^2(1-aq^n)(a-q^n)}.$$

It remains to show that

$$\sum_{k=0}^{(n+1)/2}[4k+1]\frac{(aq^{-1};q^2)_k(q^{-1}/a;q^2)_k(q;q^2)_k^2}{(aq^4;q^2)_k(q^4/a;q^2)_k(q^2;q^2)_k^2}q^{4k}\equiv 0 \pmod{[n]}. \tag{3.5}$$

For $n>3$, let $\zeta\neq 1$ be an nth root of unity, not necessarily primitive. Then ζ must be a primitive mth root of unity with $m|n$. Let $\alpha_q(k)$ denote the kth term on the left-hand side of (3.5):

$$\alpha_q(k)=[4k+1]\frac{(aq^{-1};q^2)_k(q^{-1}/a;q^2)_k(q;q^2)_k^2}{(aq^4;q^2)_k(q^4/a;q^2)_k(q^2;q^2)_k^2}q^{4k}.$$

Putting $n = m$, $cd = q$, $r = 0$ and $b \to 1$ in Lemma 1, and since $\alpha_\zeta(k) = 0$ for $(m+1)/2 < k \le m-1$, we are led to

$$\sum_{k=0}^{(m+1)/2} \alpha_\zeta(k) = \sum_{k=0}^{m-1} \alpha_\zeta(k) = 0.$$

Since

$$\frac{\alpha_\zeta(lm+k)}{\alpha_\zeta(lm)} = \lim_{q\to\zeta} \frac{\alpha_q(lm+k)}{\alpha_q(lm)} = \alpha_\zeta(k),$$

we immediately obtain

$$\sum_{k=0}^{(n+1)/2} \alpha_\zeta(k) = \sum_{l=0}^{(n/m-3)/2} \alpha_\zeta(lm) \sum_{k=0}^{m-1} \alpha_\zeta(k) + \sum_{k=0}^{(m+1)/2} \alpha_\zeta((n-m)/2+k) = 0,$$

which shows that the cyclotomic polynomial $\Phi_m(q)$ divides the sum $\sum_{k=0}^{(n+1)/2} \alpha_q(k)$. In view of

$$\prod_{m|n, m>1} \Phi_m(q) = [n],$$

the proof of (3.5) is completed and therefore (1.2) is true. □

Proof of (1.3) Likewise, letting $cd = q$ and $r = 2$ in Theorem 1, we get (1.3) immediately. □

4 Proof of Theorem 2

Through the L'Hospital rule, we have

$$\lim_{a\to 1} \left\{ \frac{n(1-aq^2)(1-q^2/a)(1-a)a^{(n-1)/2}}{(1-aq)(1-q/a)(1-a^n)} - (1+q)^2 \right\} \times \frac{(1-aq^n)(a-q^n)}{(1-a)^2}$$
$$= -\frac{n^2(1-q^2)^2 - (1+24q+22q^2+24q^3+q^4)}{24}[n]^2.$$

Hence, when $r = 0$ and $a \to 1$ in Theorem 1, we know that (1.5) holds modulo $\Phi_n(q)^4$. It remains to show that

$$\sum_{k=0}^{M} [4k+1] \frac{(q^{-1};q^2)_k^2 (q;q^2)_k^2 (cq,dq;q^2)_k}{(q^4;q^2)_k^2 (q^2;q^2)_k^2 (q^2/c,q^2/d;q^2)_k} \left(\frac{q^5}{cd}\right)^k \equiv 0 \pmod{[n]}, \quad (4.1)$$

where $M \in \{(n+1)/2, n-2\}$. Just like the proof of (3.5), let $\zeta \neq 1$ be a primitive mth root of unity with $m|n$ and let $\beta_q(k)$ be the kth term on the left-hand side of (4.1), i.e.,

$$\beta_q(k) = [4k+1]\frac{(q^{-1};q^2)_k^2(q;q^2)_k^2(cq,dq;q^2)_k}{(q^4;q^2)_k^2(q^2;q^2)_k^2(q^2/c,q^2/d;q^2)_k}\left(\frac{q^5}{cd}\right)^k.$$

Fixing $n=m$, $a\to 1$, $b\to 1$, and $x=0$ in Lemma 1, and noticing that $\beta_\zeta(k)=0$ for $(m+1)/2<k\le m-1$, we have

$$\sum_{k=0}^{(m+1)/2}\beta_\zeta(k)=\sum_{k=0}^{m-1}\beta_\zeta(k)=0.$$

Similarly as before, we get

$$\sum_{k=0}^{n-2}\beta_\zeta(k)=\sum_{l=0}^{n/m-2}\sum_{k=0}^{m-1}\beta_\zeta(lm+k)+\sum_{k=0}^{m-2}\beta_\zeta((n-m)+k)=0,$$
$$\sum_{k=0}^{(n+1)/2}\beta_\zeta(k)=\sum_{l=0}^{(n/m-3)/2}\beta_\zeta(lm)\sum_{k=0}^{m-1}\beta_\zeta(k)+\sum_{k=0}^{(m+1)/2}\beta_\zeta((n-m)/2+k)=0,$$

which means that the cyclotomic polynomial $\Phi_m(q)$ divides the sums $\sum_{k=0}^{(n+1)/2}\beta_q(k)$ and $\sum_{k=0}^{n-2}\beta_q(k)$. Since

$$\prod_{m|n,m>1}\Phi_m(q)=[n],$$

we immediately obtain (4.1). The q-supercongruence (1.5) then follows from the fact that $[n]$ is coprime with the denominator of the right-hand side of (1.5).

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