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On the approximation of some special functions in Ramanujan's generalized modular equation with signature 3*

Miao-Kun Wang[1] · Hong-Hu Chu[2] · Yu-Ming Chu[3]

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Abstract
We study several special functions in Ramanujan's generalized modular equation with signature 3. Some sharp inequalities for these functions, including the estimates for the solution of Ramanujan's generalized modular equation with signature 3 and triplication inequality for the generalized Grötzsch ring function with two parameters, are derived.

Keywords Gaussian hypergeometric function · Ramanujan's cubic transformation · Ramanujan's generalized modular equation · Generalized Grötzsch ring function

Mathematics Subject Classification 33C05 · 11F03

1 Introduction

For $a \in (0,1)$, $r \in (0,1)$, $p > 0$, Ramanujan's generalized modular equation with signature $1/a$ and degree p [2,5] is defined by

$$\frac{F(a, 1-a; 1; 1-s^2)}{F(a, 1-a; 1; s^2)} = p\frac{F(a, 1-a; 1; 1-r^2)}{F(a, 1-a; 1; r^2)}, \tag{1.1}$$

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✉ Yu-Ming Chu
chuyuming@zjhu.edu.cn

Miao-Kun Wang
wmk000@126.com

Hong-Hu Chu
chuhonghu2005@126.com

1 Department of Mathematics, Huzhou University, Huzhou 313000, China

2 College of Civil Engineering, Hunan University, Changsha 410082, China

3 College of Science, Hunan City University, Yiyang 413000, China

where

$$F(a,b;c;x) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} \quad (|x|<1) \tag{1.2}$$

is the classical Gaussian hypergeometric function [34–36,40,47,57] with real parameters a, b, and c ($c \neq 0, -1, \ldots$), and (a,n) is the shifted factorial function, namely, $(a,0) = 1$ for $a \neq 0$ and $(a,n) = a(a+1)(a+2)(a+3)\cdots(a+n-1)$ for $n = 1, 2, \ldots$

Ramanujan's generalized modular equation (1.1) and its solution s have been studied over two centuries. At the beginning of the twentieth century, Ramanujan made an intensive study of the classical modular equation ($a = 1/2$), and for several small prime numbers $p = 3, 5, 7, \ldots$, he derived a lot of algebraic identities satisfied by the solutions of classical modular equation [5]. Moreover, in his unpublished notebooks and papers [26,27], he also listed many algebraic identities involving the solutions of some generalized modular equations with signature 3, 4, 6 and prime degree. Unfortunately, Ramanujan did not give their complete proofs.

Ramanujan's theory for generalized modular equations did not develop until 1987. The Borweins [7,8] studied the hypergeometric function $F(1/3, 2/3; 1; x)$, and rediscovered the Ramanujan cubic transformation which can be found on page 258 of Ramanujan's second notebook

$$F\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \left(\frac{1-r}{1+2r}\right)^3\right) = (1+2r)F\left(\frac{1}{3}, \frac{2}{3}; 1; r^3\right). \tag{1.3}$$

Besides, the Borweins [9] also found a beautiful cubic analog of a famous theta-function identity of Jacobi for fourth powers, which brought new ideas on establishing Ramanujan's theory. Later, an important paper titled " Ramanujan's theories of elliptic functions to alternative bases" [6] was written by Berndt, Bhargava, Garvan, in which they studied generalized modular equations with signature 3, 4, and 6, and gave the complete proofs of many algebraic identities involving the solutions of the corresponding modular equations in Ramanujan's notebook. For recent development of Ramanujan's theory see [17,28–31].

On the other hand, in 1996, Vuorinen [32] pointed out that there exists a connection between geometric function theory and Ramanujan's theory. In fact, when $a = 1/2$, (1.1) can be written as

$$\mu(s) = p\mu(r),$$

where $\mu(r) = \pi/2[F(1/2, 1/2; 1; 1-r^2)/F(1/2, 1/2; 1; r^2)]$ [24,45,55] is said to be the plane Grötzsch ring function, which is the conformal modulus of the plane Grötzsch ring $\mathbf{B}^2 \setminus [0, r]$, where $\mathbf{B}^2$ is the open unit disk in the complex plane. Its solution s can be expressed by

$$s = \varphi_K(r) = \mu^{-1}(\mu(r)/K), \quad K = 1/p,$$

where $\varphi_K(r)$ is the Hersch–Pfluger distortion function [12,21,42] in quasiconformal maps [3,20]. Around 2000, in order to study the generalized modular equation (1.1) and its related special functions, Anderson et al. [2] introduced the decreasing homeomorphism $\mu_a : (0,1) \to (0,\infty)$, as a generalization of the Grötzsch ring function $\mu(r) = \mu_{1/2}(r)$, defined by

$$\mu_a(r) = \frac{\pi}{2\sin(\pi a)} \frac{F(a, 1-a; 1; 1-r^2)}{F(a, 1-a; 1; r^2)}, \quad r \in (0,1), \tag{1.4}$$

for $a \in (0, 1/2]$. Using (1.4), the modular equation (1.1) can be rewritten as

$$\mu_a(s) = p\mu_a(r), \quad 0 < r < 1. \tag{1.5}$$

The solution of (1.5) is given by

$$s \equiv \varphi_K(a, r) = {\mu_a}^{-1}(\mu_a(r)/K), \quad K = 1/p,$$

which is a generalization of the Hersch–Pfluger distortion function. Moreover, in [2], several monotonicity theorems for $\mu_a(r)$ and $\varphi_K(a,r)$, and multiplicative functional inequalities for $\varphi_K(a,r)$ were presented. In particular, many remarkable results involving the special functions in Ramanujan's generalized modular equation can be found in the literature [1,10,11,13–16,18,22,23,25,33,37,38,41,43,44,46,51,53,56].

Let $r' = \sqrt[3]{1-r^3}$ for $r \in [0,1]$,

$$\mu_a^*(r) = \mu_a(r^{3/2}) = \frac{\pi}{2\sin(\pi a)} \frac{F(a, 1-a; 1; 1-r^3)}{F(a, 1-a; 1; r^3)}, \tag{1.6}$$

$$\varphi_K^*(a, r) = \left[\varphi_K(a, r^{3/2})\right]^{2/3} = \mu_a^{*-1}(\mu_a^*(r)/K) \tag{1.7}$$

and

$$\mu^*(r) = \mu_{1/3}^*(r), \quad \varphi_K^*(r) = \varphi_K^*(1/3, r). \tag{1.8}$$

Then it is easy to check that for $K_1, K_2 > 0$,

$$\varphi_{K_1}^*(a, \varphi_{K_2}^*(a, r)) = \varphi_{K_1K_2}^*(a, r), \quad \varphi_{K_1}^*(\varphi_{K_2}^*(r)) = \varphi_{K_1K_2}^*(r), \tag{1.9}$$

and from (1.3) one has

$$\mu^*(r) = 3\mu^*\left(\frac{\sqrt[3]{9r(1+r+r^2)}}{1+2r}\right), \quad \mu^*(r) = \frac{1}{3}\mu^*\left(\frac{1-r'}{1+2r'}\right), \tag{1.10}$$

$$\varphi_3^*(r) = \frac{\sqrt[3]{9r(1+r+r^2)}}{1+2r}, \quad \varphi_{1/3}^*(r) = \frac{1-r'}{1+2r'}, \quad \varphi_3^*(r)^3 + \varphi_{1/3}^*(r')^3 = 1. \tag{1.11}$$

(The above symbols in (1.6)–(1.8) have been employed in [39, Eqs. (1.9)–(1.11)]). Making use of (1.10), (1.11), and Ramanujan's cubic transformation (1.3), Wang et al. [39] derived some new analytic properties for special functions $\mu_a^*(r)$ and $\varphi_K^*(a,r)$

in Ramanujan's generalized modular equation with signature 3, instead of $\mu_a(r)$ and $\varphi_K(a, r)$. In particular, a infinite-product representation for $\mu^*(r)$ which only contains r and an analog of the triplication formula (1.10) for the modular function $\mu_a^*(r)$ were established.

This paper is a continuation of the work in [39] and deals with the inequalities for the special functions in Ramanujan's generalized modular equation with signature 3. In Sect. 2, we shall present some estimates for the function $\varphi_K^*(r)$. Employing the monotonicity criterion for the quotient of power series [48, Theorem 2.1], Ramanujan's cubic transformation inequalities for a class of hypergeometric functions will be established in Sect. 3. In Sect. 4, we will extend the triplication formula (1.10) to the generalized Grötzsch ring function with two positive parameters a and b defined by

$$\mu_{a,b}^*(r) = \frac{B(a,b)}{2}\frac{F(a,b;(a+b+1)/2;1-r^3)}{F(a,b;(a+b+1)/2;r^3)}, \quad r \in (0,1), \tag{1.12}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t$ is the classical beta function.

Before ending this section, we shall recall some known results for the functions $F(a,b;c;x)$ and $\mu_{a,b}^*(r)$, which will be used afterwards. The behavior of the Gaussian hypergeometric function $F(a,b;c;x)$ near $x=1$ can be divided into the following three cases $a+b<c$, $a+b=c$, and $a+b>c$ (see [3, Theorems 1.19 and 1.48]):

$$\begin{cases} F(a,b;c;1) = \Gamma(c)\Gamma(c-a-b)/[\Gamma(c-a)\Gamma(c-b)], & a+b<c, \\ B(a,b)F(a,b;c;x) + \log(1-x) & \\ = R(a,b) + O((1-x)\log(1-x)), & a+b=c, \\ F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x), & a+b>c, \end{cases} \tag{1.13}$$

where $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t$ $(\mathrm{Re}(x)>0)$ [49,50,52,54,58–60] is the gamma function,

$$R(a,b) = -\psi(a) - \psi(b) - 2\gamma, \quad R(a) = R(a,1-a), \quad R(1/3) = \log 27,$$

$\psi(x) = \Gamma'(x)/\Gamma(x)$ is the Psi function and $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^n 1/k - \log n\right) = 0.577\ldots$ is the Euler constant [19].

According to (1.12), it is clear to see that $\mu_a^*(r) = \mu_{a,1-a}^*(r)$ and $\mu^*(r) = \mu_{1/3,2/3}^*(r)$. By (1.13), one has

$$\mu_{a,b}^*(0^+) = \lim_{r\to 0^+}\frac{B(a,b)}{2}F\left(a,b;\frac{a+b+1}{2};1-r^3\right) = \begin{cases} \frac{B(a,b)}{2}H(a,b), & a+b<1; \\ +\infty, & a+b\geq 1, \end{cases} \tag{1.14}$$

$$\mu_{a,b}^*(1^-) = \lim_{r\to 1^-}\frac{B(a,b)}{2F\left(a,b;\frac{a+b+1}{2};r^3\right)} = \begin{cases} \frac{B(a,b)}{2H(a,b)}, & a+b<1; \\ 0, & a+b\geq 1, \end{cases} \tag{1.15}$$

where and in what follows we denote by

$$H(a,b)=\frac{B\left(\frac{a+b+1}{2},\frac{1-a-b}{2}\right)}{B\left(\frac{1+b-a}{2},\frac{1+a-b}{2}\right)}.$$

Moreover, due to [4, Lemmas 4.5, 4.6, and 4.8], $\mu^*_{a,b}(r)$ satisfies the following derivative formula

$$\frac{\mathrm{d}\mu^*_{a,b}(r)}{\mathrm{d}r}=-\frac{3\Gamma\left(\frac{a+b+1}{2}\right)^2}{2\Gamma(a+b)}\frac{1}{r^{\frac{3(a+b)-1}{2}}r'^{\frac{3(a+b+1)}{2}}F\left(a,b;\frac{a+b+1}{2};r^3\right)^2}. \tag{1.16}$$

2 Ramanujan cubic transformation and $\varphi^*_K(r)$

In this section, by employing the ascending and descending Ramanujan cubic transformations, we will find some estimates for the solution of Ramanujan's generalized modular equation with signature 3.

Lemma 2.1 [2, Theorem 5.5(4),(6)] *Let* $a\in(0,1/2]$ *and* $r\in(0,1)$. *Then one has*

(1) *The function* $r\mapsto\mu_a(r)/\log(1/r)$ *is strictly increasing from* $(0,1)$ *onto* $(1,\infty)$;
(2) *The function* $r\mapsto\mu_a(r)/\log\left[\mathrm{e}^{R(a)/2}/r\right]$ *is strictly decreasing from* $(0,1)$ *onto* $(0,1)$.

If $a=1/3$, Lemma 2.1 reduces to the following Lemma 2.2.

Lemma 2.2 *Let* $r\in(0,1)$. *Then*

(1) *The function* $r\mapsto\mu^*(r)/\log(1/r)$ *is strictly increasing from* $(0,1)$ *onto* $(3/2,\infty)$;
(2) *The function* $r\mapsto\mu^*(r)/\log(3/r)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,3/2)$.

Theorem 2.3 *For* $K>0$ *and* $r\in(0,1)$, *let* $\lambda=\varphi^*_K(r)$. *For* $x\in(0,1)$, *let* $\{x_n\}$ *be the descending Ramanujan sequence given by* $x_0=x$, $x_{p+1}=(1-x'_p)/(1+2x'_p)=\varphi^*_{1/3}(x_p)=\varphi^*_{1/3^{p+1}}(x)$, $p=0,1,2,\ldots$ *Then* $\lambda_p=\varphi^*_K(r_p)$. *Moreover,*

(1) $r_p^{1/K}\le\lambda_p\le 3^{1-1/K}r_p^{1/K}$, $K\ge1$;
(2) $3^{1-1/K}r_p^{1/K}<\lambda_p<r_p^{1/K}$, $0<K<1$.

Proof It is easy to check that

$$\mu^*(\lambda_p)=\mu^*\left(\frac{1-\lambda'_{p-1}}{1+2\lambda'_{p-1}}\right)=3\mu^*(\lambda_{p-1})=3^p\mu^*(\lambda)=3^p\frac{\mu^*(r)}{K}=\frac{\mu^*(r_p)}{K},$$

so that $\lambda_p=\varphi^*_K(r_p)$.

For part (1), since $K\ge1$ and the function μ^* is strictly decreasing on $(0,1)$, $\lambda_p\ge r_p$. It follows from Lemma 2.2 that

$$\frac{\mu^*(\lambda_p)}{\log(3/\lambda_p)}\le\frac{\mu^*(r_p)}{\log(3/r_p)}\quad\text{and}\quad\frac{\mu^*(\lambda_p)}{\log(1/\lambda_p)}\ge\frac{\mu^*(r_p)}{\log(1/r_p)},$$

equivalently,

$$\frac{1/K}{\log(3/\lambda_p)} \le \frac{1}{\log(3/r_p)} \quad \text{and} \quad \frac{1/K}{\log(1/\lambda_p)} \ge \frac{1}{\log(1/r_p)},$$

which lead to part (1). The proof of part (2) is similar. □

Corollary 2.4 *For $K > 1$ and $r \in (0, 1)$, let $r_p = \varphi^*_{1/3^p}(r)$ $(p = 0, 1, 2, \ldots)$ be the descending Ramanujan sequence defined in Theorem* 2.3. *Then*

(1) $r^{1/K} \le \varphi^*_K(r) \le 3^{1-1/K} r^{1/K}$;
(2) $\varphi^*_K(r) \ge \varphi^*_{3^p}(r_p^{1/K})$;
(3) $\varphi^*_{3^p}(3^{1-K} r_p^K) < \varphi^*_{1/K}(r) < \varphi^*_{3^p}(r_p^K)$;

If p_0 is so large such that $3^{1-1/K} r_{p_0}^{1/K} < 1$, then

(4) $\varphi^*_K(r) \le \varphi^*_{3^p}(3^{1-1/K} r_p^{1/K})$ *for* $p = p_0, p_0 + 1, \ldots$

Proof Part (1) directly follows from Theorem 2.3(1) with $p = 0$. For part (2), by (1.9) and Theorem 2.3(1), we have

$$\varphi^*_K(r) = \varphi^*_K(\varphi^*_{3^p}(r_p)) = \varphi^*_{3^p}(\varphi^*_K(r_p)) \ge \varphi^*_{3^p}\left(r_p^{1/K}\right).$$

Similarly, for part (3), noting that $\varphi^*_{1/K}(r) = \varphi^*_{1/K}\left(\varphi^*_{3^p}(r_p)\right) = \varphi^*_{3^p}\left(\varphi^*_{1/K}(r_p)\right)$, Theorem 2.3(2) yields

$$\varphi^*_{3^p}\left(3^{1-K} r_p^K\right) < \varphi^*_{1/K}(r) < \varphi^*_{3^p}\left(r_p^K\right).$$

For part (4), since the sequence $\{3^{1-1/K} r_p^{1/K}\}_{p=0}^{\infty}$ decreases with p for any fixed $r \in (0, 1)$ and $K > 1$, then we have that

$$3^{1-1/K} r_0^{1/K} = 3 \cdot \left(\frac{r}{3}\right)^{1/K} \ge 3^{1-1/K} r_p^{1/K} > \lim_{p\to\infty} 3^{1-1/K} r_p^{1/K} = 0.$$

Thus there exists a non-negative integer p_0 depending on r and K such that inequality $3^{1-1/K} r_{p_0}^{1/K} < 1$ holds. The proof of the inequality in part (4) is similar to part (2) or part (3). □

Remark 2.5 The bounds in Corollary 2.4(2) and (4) are computable by a simple iterative procedure. Moreover, the following Theorem 2.6 shows that, for $p \ge 1$, the bounds in Corollary 2.4(2) and (4) are the refinements of the left-hand and right-hand inequalities in part (1), respectively.

Theorem 2.6 *For $K > 1$ and $r \in (0, 1)$, let $r_p = \varphi^*_{1/3^p}(r)$ $(p = 0, 1, 2, \ldots)$ be the descending Ramanujan sequence defined in Theorem* 2.3. *Then the following statements hold*:

(1) *Inequality*

$$r^{1/K} < \varphi_{3^p}^*\left(r_p^{1/K}\right) \le \varphi_K^*(r) \tag{2.1}$$

holds for all $(r, K) \in (0, 1) \times (1, +\infty)$ *with* $p = 1, 2, 3, \ldots$*;*

(2) *If* p_0 *is so large such that* $3^{1-1/K} r_{p_0}^{1/K} < 1$*, then inequality*

$$\varphi_K^*(r) \le \varphi_{3^p}^*\left(3^{1-1/K} r_p^{1/K}\right) < 3^{1-1/K} r^{1/K} \tag{2.2}$$

holds for all $(r, K) \in (0, 1) \times (1, +\infty)$ *with* $p = p_0, p_0 + 1, \ldots$.

Before proving Theorem 2.6, we firstly establish two lemmas.

Lemma 2.7 *Let* $t \in (0, 1)$*. Then*

(1) *The function* $t \mapsto \log[(1-t)/(1+2t)] - (1+t+t^2)[\log(1-t^3)]/[t^2(1+2t)]$ *is strictly decreasing from* $(0, 1)$ *onto* $(-2\log 3, 0)$*;*
(2) *The function* $t \mapsto -\log 3 + \log[(1-t)/(1+2t)] - (1+t+t^2)\{\log[(1-t^3)/27]\}/[t^2(1+2t)]$ *is strictly decreasing from* $(0, 1)$ *onto* $(0, +\infty)$*.*

Proof Let $f(t) = \log[(1-t)/(1+2t)] - (1+t+t^2)[\log(1-t^3)]/[t^2(1+2t)]$ and $g(t) = -\log 3 + \log[(1-t)/(1+2t)] - (1+t+t^2)\{\log[(1-t^3)/27]\}/[t^2(1+2t)]$. Then differentiating f and g gives

$$f'(t) = \frac{(7t + 4t^2 + 2t^3 + 2)\log(1-t^3)}{t^3(1+2t)^2}$$

and

$$g'(t) = \frac{(7t + 4t^2 + 2t^3 + 2)[-\log 27 + \log(1-t^3)]}{t^3(1+2t)^2}.$$

It is clear to see that $f'(t) < 0$ and $g'(t) < 0$ for all $t \in (0, 1)$, so that both f and g decrease on $(0, 1)$. Moreover,

$$\lim_{t\to 0^+} f(t) = 0, \quad \lim_{t\to 1^-} f(t) = -2\log 3, \quad \lim_{t\to 0^+} g(t) = +\infty, \quad \lim_{t\to 1^-} g(t) = 0.$$

This completes the proof. □

Lemma 2.8 *For* $r \in (0, 1)$*, let* $r' = \sqrt[3]{1-r^3}$*, and* $h(r) = (1-r')/(1+2r') = \varphi_{1/3}^*(r)$*. Then*

(1) *The function* $K \mapsto [h(r^{1/K})]^K$ *is strictly decreasing on* $K \in (1, +\infty)$ *for any fixed* $r \in (0, 1)$*, and therefore inequality*

$$r^{1/K} < \varphi_3^*\left([\varphi_{1/3}^*(r)]^{1/K}\right) \tag{2.3}$$

holds for $(r, K) \in (0, 1) \times (1, +\infty)$*;*

(2) *If* $(r, K) \in (0, 1) \times (1, +\infty)$ *with* $3^{1-1/K}r^{1/K} < 1$. *Then the function* $K \mapsto 3^{1-K}[h(3^{1-1/K}r^{1/K})]^K$ *is strictly increasing on* $K \in (1, [\log(3/r)]/\log 3)$ *for any fixed* $r \in (0, 1)$, *and therefore*

$$3^{1-1/K}r^{1/K} > \varphi_3^*\left(3^{1-1/K}[\varphi_{1/3}^*(r)]^{1/K}\right) \tag{2.4}$$

for all $(r, K) \in (0, 1) \times (1, +\infty)$ *with* $3^{1-1/K}r^{1/K} < 1$.

Proof For part (1), let $x = r^{1/K}$, and $F_1(K) = h(x)^K (K \in (1, +\infty))$. Then logarithmic differentiation of F_1 yields

$$\begin{aligned}\frac{F_1'(K)}{F_1(K)} &= \log h(x) - \frac{h'(x)x\log x}{h(x)} = \log\left(\frac{1-x'}{1+2x'}\right) - \frac{\dfrac{3x^2}{x'^2(1+2x')^2}\cdot x\log x}{\left(\dfrac{1-x'}{1+2x'}\right)}\\ &= \log\left(\frac{1-x'}{1+2x'}\right) - \frac{(1+x'+x'^2)\log(1-x'^3)}{x'^2(1+2x')}.\end{aligned}$$

Lemma 2.7(1) implies that, for any fixed $r \in (0, 1)$, $F_1'(K) < 0$ for $K \in (1, +\infty)$, so that $F_1(K)$ is strictly decreasing on $(1, +\infty)$. Consequently,

$$F_1(K) = [h(r^{1/K})]^K = \left[\varphi_{1/3}^*(r^{1/K})\right]^K < F_1(1) = \varphi_{1/3}^*(r)$$

for all $(r, K) \in (0, 1) \times (1, +\infty)$, that is, inequality (2.3) takes place.

For part (2), for any fixed $r \in (0, 1)$, we denote $y = 3^{1-1/K}r^{1/K}$. Since $y \in (0, 1)$, then $K \in (1, [\log(3/r)]/\log 3)$. Let $F_2(K) = 3^{1-K}[h(y)]^K$ $(K \in (1, [\log(3/r)]/\log 3))$. Then logarithmic differentiation of F_2 leads to

$$\begin{aligned}\frac{F_2'(K)}{F_2(K)} &= -\log 3 + \log h\,(y) + K\frac{h'(y)}{h(y)}\cdot y\log\left(\frac{r}{3}\right)\left(-\frac{1}{K^2}\right)\\ &= -\log 3 + \log\left(\frac{1-y'}{1+2y'}\right) - \frac{(1+y'+y'^2)\log[(1-y'^3)/27]}{y'^2(1+2y')}.\end{aligned}$$

Lemma 2.7(2) shows that, for any fixed $r \in (0, 1)$, $F_2'(K) > 0$ for each $K \in (1, [\log(3/r)]/\log 3)$, so that $F_2(K)$ is strictly increasing on $(1, [\log(3/r)]/\log 3)$. Consequently,

$$F_2(K) = 3^{1-K}[\varphi_{1/3}^*(3^{1-1/K}r^{1/K})]^K > F_2(1) = \varphi_{1/3}^*(r)$$

for all $(r, K) \in (0, 1) \times (1, [\log(3/r)]/\log 3)$, which yields (2.4). □

Proof of Theorem 2.6 For part (1), it follows from (2.3) that inequality

$$[\varphi_{1/3}^*(r)]^{1/K} < \varphi_3^*\left([\varphi_{1/3^2}^*(r)]^{1/K}\right)$$

also holds for all $(r, K) \in (0,1)\times(1,+\infty)$, so that

$$r^{1/K} < \varphi_3^*\left([\varphi_{1/3}^*(r)]^{1/K}\right) < \varphi_{3^2}^*\left([\varphi_{1/3^2}^*(r)]^{1/K}\right) = \varphi_{3^2}^*\left(r_2^{1/K}\right)$$

takes place. With the same procedure, we obtain the first inequality in (2.1) immediately. Combining with Corollary 2.4(2), the assertions in part (1) hold true.

For part (2), since the domain of the function $r \mapsto \varphi_K^*(r)$ is $(0,1)$, it suffices to consider the case $3^{1-1/K}r^{1/K} < 1$. By (2.4) one has that

$$3^{1-1/K}[\varphi_{1/3}^*(r)]^{1/K} > \varphi_3^*\left(3^{1-1/K}[\varphi_{1/3^2}^*(r)]^{1/K}\right),$$

and thereby

$$3^{1-1/K}r^{1/K} > \varphi_3^*\left(3^{1-1/K}[\varphi_{1/3}^*(r)]^{1/K}\right) > \varphi_{3^2}^*\left(3^{1-1/K}[\varphi_{1/3^2}^*(r)]^{1/K}\right)$$

holds for all $(r, K) \in (0,1)\times(1,+\infty)$ with $3^{1-1/K}r^{1/K} < 1$. By iteration, the second inequality in (2.2) follows. Moreover, by Corollary 2.4(4), part (2) is clear. □

3 Ramanujan's cubic transformation inequalities for a class of hypergeometric functions

In this section, we mainly study a class of hypergeometric functions $F(a, b; (a+b+1)/2; x)$ with $a, b > 0$, and prove Ramanujan's cubic transformation inequalities for $F(a, b; (a+b+1)/2; x)$. That is, we shall determine the maximum regions in the first quadrant of ab-plane such that (1.3) becomes a strict inequality.

For conveniences, we firstly introduce some regions in $\{(a,b)\in\mathbb{R}^2 \mid a>0, b>0\}$ which will be frequently used afterwards (see Fig 1).

$$D_1 = \left\{(a,b) \mid a, b > 0, a+b \le 1, ab - \frac{a+b+1}{9} \le 0\right\},$$

$$D_2 = \left\{(a,b) \mid a, b > 0, a+b \ge 1, ab - \frac{a+b+1}{9} \ge 0\right\},$$

$$D_3 = \left\{(a,b) \mid a, b > 0, a+b < 1, ab - \frac{a+b+1}{9} > 0\right\},$$

$$D_4 = \left\{(a,b) \mid a, b > 0, a+b > 1, ab - \frac{a+b+1}{9} < 0\right\},$$

$$E_1 = \left\{(a,b) \mid a, b > 0, a+b \le 1, 2ab + \frac{8(a+b)-12}{9} \le 0\right\},$$

$$E_2 = \left\{(a,b) \mid a, b > 0, a+b \ge 1, 2ab + \frac{8(a+b)-12}{9} \ge 0\right\},$$

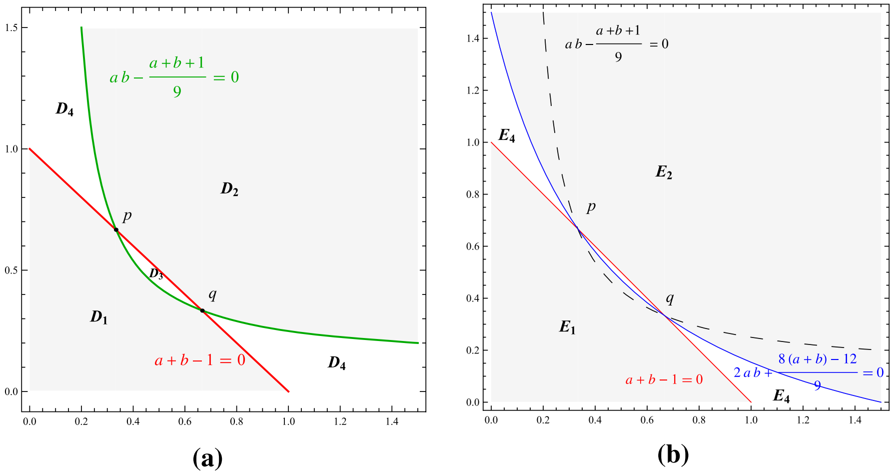

Fig. 1 The regions D_i for $i = 1, 2, 3, 4$, E_1, E_2, E_4 and their boundary curves

$$E_3 = \left\{(a,b)|\ a,b > 0, a+b < 1, 2ab + \frac{8(a+b)-12}{9} > 0\right\},$$

$$E_4 = \left\{(a,b)|\ a,b > 0, a+b > 1, 2ab + \frac{8(a+b)-12}{9} < 0\right\}.$$

Clearly, $\bigcup_{i=1}^{4} D_i = \bigcup_{i=1}^{4} E_i = \{(a,b) \in \mathbb{R}^2 |\ a > 0, b > 0\}$ and $D_i \cap D_j (E_i \cap E_j) = \emptyset$ for $i \neq j \in \{1,2,3,4\}$ except that $D_1 \cap D_2 (E_1 \cap E_2) = \{(1/3, 2/3), (2/3, 1/3)\}$. Moreover, $D_1 \subset E_1$ and $D_2 \subset E_2$.

Theorem 3.1 *The inequality*

$$F\left(a, b; \frac{a+b+1}{2}; \frac{9r(1+r+r^2)}{(1+2r)^3}\right) \leq (1+2r) F\left(a, b; \frac{a+b+1}{2}; r^3\right) \quad (3.1)$$

holds for all $r \in (0,1)$ *with* $a, b > 0$ *if and only if* $(a,b) \in D_1$ *and the reversed inequality*

$$F\left(a, b; \frac{a+b+1}{2}; \frac{9r(1+r+r^2)}{(1+2r)^3}\right) \geq (1+2r) F\left(a, b; \frac{a+b+1}{2}; r^3\right) \quad (3.2)$$

holds for all $r \in (0,1)$ *if and only if* $(a,b) \in D_2$. *The above inequalities become equalities only for* $(a,b) = (1/3, 2/3)$ *or* $(2/3, 1/3)$.

In the remaining case $(a,b) \in D_3 \cup D_4$, *neither of the above inequalities holds for all* $r \in (0,1)$.

Theorem 3.2 *Let* $a, b > 0$ *with* $(a,b) \neq (1/3, 2/3)$ *and* $(2/3, 1/3)$, $L_1 = \{(a,b)|\ a + b = 1, 0 < a < 1/3$ *or* $2/3 < a < 1\}$, $L_2 = \{(a,b)|\ a+b = 1, 1/3 < a < 2/3\}$ *and* *the function* $J(r)$ *be defined on* $(0,1)$ *by*

$$J(r)=(1+2\sqrt[3]{r})F\left(a,b;\frac{a+b+1}{2};r\right)\\ -F\left(a,b;\frac{a+b+1}{2};\frac{9\sqrt[3]{r}(1+\sqrt[3]{r}+\sqrt[3]{r^2})}{(1+2\sqrt[3]{r})^3}\right).$$

Then we have

(1) *If* $(a,b)\in L_1$(*or* L_2), *then* $J(r)$ *is strictly increasing* (*or decreasing*) *from* $(0,1)$ *onto* $(0, 2[R(a,b)-\log 27]/B(a,b))$ (*or* $(2[R(a,b)-\log 27]/B(a,b),0)$);
(2) *If* $(a,b)\in D_1\setminus L_1$, *then* $J(r)$ *is strictly increasing from* $(0,1)$ *onto* $(0,2H(a,b))$*;*
(3) *If* $(a,b)\in D_2\setminus L_2$, *then* $J(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(-\infty,0)$.

Consequently, *the inequality*

$$F\left(a,b;\frac{a+b+1}{2};\frac{9r(1+r+r^2)}{(1+2r)^3}\right)\leq(1+2r)F\left(a,b,;\frac{a+b+1}{2};r^3\right)\\ \leq F\left(a,b;\frac{a+b+1}{2};\frac{9r(1+r+r^2)}{(1+2r)^3}\right)+2H(a,b) \tag{3.3}$$

holds for all $r\in(0,1)$ *if* $(a,b)\in D_1\setminus L_1$. *And for all* $(a,b)\in L_1$(*or* L_2) *and* $r\in(0,1)$,

$$F\left(a,b;\frac{a+b+1}{2};\frac{9r(1+r+r^2)}{(1+2r)^3}\right)\leq(\geq)(1+2r)F\left(a,b,;\frac{a+b+1}{2};r^3\right)\\ \leq(\geq)F\left(a,b;\frac{a+b+1}{2};\frac{9r(1+r+r^2)}{(1+2r)^3}\right)+\frac{2[R(a,b)-\log 27]}{B(a,b)}. \tag{3.4}$$

Before proving our main results, we need to establish several lemmas.

Lemma 3.3 [48, Theorem 2.1] *Suppose that the power series* $f(x)=\sum_{n=0}^{\infty}a_nx^n$ *and* $g(x)=\sum_{n=0}^{\infty}b_nx^n$ *have the radius of convergence* $r>0$ *with* $b_n>0$ *for all* $n\in\{0,1,2,\ldots\}$. *Let* $h(x)=f(x)/g(x)$ *and* $H_{f,g}=(f'/g')g-f$, *then the following statements hold true*:

(1) *If the non-constant sequence* $\{a_n/b_n\}_{n=0}^{\infty}$ *is increasing* (*decreasing*) *for all* $n>0$*, then* $h(x)$ *is strictly increasing* (*decreasing*) *on* $(0,r)$*;*

(2) *If the non-constant sequence* $\{a_n/b_n\}_{n=0}^{\infty}$ *is increasing* (*decreasing*) *for* $0<n\leq n_0$ *and decreasing* (*increasing*) *for* $n>n_0$, *then* $h(x)$ *is strictly increasing* (*decreasing*) *on* $(0,r)$ *if and only if* $H_{f,g}(r^-)\geq(\leq)0$. *Moreover*, *if* $H_{f,g}(r^-)<(>)0$, *then there exists* $x_0\in(0,r)$ *such that* $h(x)$ *is strictly increasing* (*decreasing*) *on* $(0,x_0)$ *and strictly decreasing* (*increasing*) *on* (x_0,r).

Lemma 3.4 *Let* $p=(1/3,2/3)$, $q=(2/3,1/3)$, $a,b>0$ *such that* $(a,b)\neq p$ *and* q, *the functions* F, G, $\widetilde{F}$, $\widetilde{G}$, ω, *and* $\widetilde{\omega}$ *be defined by*

$$F(x)=F\left(a,b;\frac{a+b+1}{2};x\right),\quad G(x)=F\left(a+1,b+1;\frac{a+b+3}{2};x\right),$$
$$\widetilde{F}(x)=F\left(\frac{1}{3},\frac{2}{3};1;x\right),\quad \widetilde{G}(x)=F\left(\frac{4}{3},\frac{5}{3};2;x\right),$$
$$\omega(x)=\frac{F(x)}{\widetilde{F}(x)},$$
$$\widetilde{\omega}(x)=\frac{G(x)}{\widetilde{G}(x)},$$

respectively. Then the following statements hold true:

(1) *The function* $\omega(x)$ *is strictly decreasing on* $(0,1)$ *if* $(a,b)\in D_1\setminus\{p,q\}$ *and strictly increasing on* $(0,1)$ *if* $(a,b)\in D_2\setminus\{p,q\}$. *Moreover, if* $(a,b)\in D_3$*(or* D_4*), then there exists* $\delta_0\in(0,1)$ *such that* $\omega(x)$ *is strictly increasing (decreasing) on* $(0,\delta_0)$ *and strictly decreasing (increasing) on* $(\delta_0,1)$;

(2) *The function* $\widetilde{\omega}(x)=G(x)/\widetilde{G}(x)$ *is strictly decreasing on* $(0,1)$ *if* $(a,b)\in E_1\setminus\{p,q\}$ *and strictly increasing on* $(0,1)$ *if* $(a,b)\in E_2\setminus\{p,q\}$. *Moreover, if* $(a,b)\in E_3$*(or* E_4*), then there exists* $\delta_0^*\in(0,1)$ *such that* $\omega(x)$ *is strictly increasing (decreasing) on* $(0,\delta_0^*)$ *and strictly decreasing (increasing) on* $(\delta_0^*,1)$.

Proof For part (1), let

$$A_n=\frac{(a,n)(b,n)}{\left(\frac{a+b+1}{2},n\right)n!},\quad A_n^*=\frac{\left(\frac{1}{3},n\right)\left(\frac{2}{3},n\right)}{(1,n)n!},$$

then one has

$$\omega(x)=\frac{F(x)}{\widetilde{F}(x)}=\frac{\sum_{n=0}^{\infty}A_nx^n}{\sum_{n=0}^{\infty}A_n^*x^n}. \tag{3.5}$$

Lemma 3.3 shows that, in order to obtain the monotonicity of $\omega(x)$, it suffices to take into account the monotonicity of $\{A_n/A_n^*\}_{n=0}^{\infty}$. Simple computations yield

$$\frac{A_{n+1}}{A_{n+1}^*}-\frac{A_n}{A_n^*}=\frac{A_n\cdot\Delta_n}{A_n^*\left(\frac{a+b+1}{2}+n\right)\left(\frac{1}{3}+n\right)\left(\frac{2}{3}+n\right)}, \tag{3.6}$$

where

$$\Delta_n=\left(\frac{a+b-1}{2}\right)n^2+\left(ab+\frac{a+b}{2}-\frac{13}{18}\right)n+ab-\frac{a+b+1}{9}. \tag{3.7}$$

Next, we divide the proof into four cases.

Case 1 $(a,b)\in D_1\setminus\{p,q\}$. Then $a+b\le 1$, $ab-(a+b+1)/9\le 0$ and $ab+(a+b)/2-13/18<0$. This together with (3.6) and (3.7) implies that $\{A_n/A_n^*\}_{n=0}^{\infty}$ is strictly decreasing for all $n>0$. Therefore, (3.5) and Lemma 3.3(1) lead to the conclusion that $\omega(x)$ is strictly decreasing on $(0,1)$.

Case 2 $(a,b)\in D_2\setminus\{p,q\}$. Then making use of the similar argument as in Case 1, we obtain that $\Delta_n>0$. Therefore, $\omega(x)$ is strictly increasing on $(0,1)$ follows easily from (3.5), (3.6), (3.7) and Lemma 3.3(1).

CASE 3 $(a,b)\in D_3$. Then from (3.6) and (3.7) we conclude that the sequence $\{A_n/A_n^*\}$ is increasing for $0\le n\le n_0$ and decreasing for $n>n_0$ for some integer n_0. Furthermore, making use of the derivative formula of Gaussian hypergeometric function

$$\frac{\mathrm{d}F(a,b;c;x)}{\mathrm{d}x}=\frac{ab}{c}F(a+1,b+1;c+1;x)$$

and (1.13) for $a+b<1$ yield

$$\begin{aligned}H_{F,\widetilde{F}}(x)=&\frac{9ab}{(a+b+1)}(1-x)^{\frac{1-a-b}{2}}\frac{F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+3}{2};x\right)}{F\left(\frac{2}{3},\frac{1}{3};2;x\right)}\widetilde{F}(x)-F(x)\\ &\to -H(a,b)<0\end{aligned}\tag{3.8}$$

as $x\to 1^-$. Therefore, combining with (3.5), (3.8) and Lemma 3.3(2), we conclude that there exists $x_1\in(0,1)$ such that $\omega(x)$ is strictly increasing on $(0,x_1)$ and strictly decreasing on $(x_1,1)$.

CASE 4 $(a,b)\in D_4$. Then the assertion about this case directly follows from the similar argument as in CASE 3 together with the fact that

$$\begin{aligned}H_{F,\widetilde{F}}(x)=&\frac{9ab}{(a+b+1)}(1-x)\frac{F\left(a+1,b+1;\frac{a+b+3}{2};x\right)}{F\left(\frac{2}{3},\frac{1}{3};2;x\right)}\widetilde{F}(x)-F(x)\\ =&(1-x)^{\frac{1-a-b}{2}}\left[\frac{9ab}{(a+b+1)}\frac{F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+3}{2};x\right)}{F\left(\frac{2}{3},\frac{1}{3};2;x\right)}F\left(\frac{1}{3},\frac{2}{3};1;x\right)\right.\\ &\left.-F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+1}{2};x\right)\right]\\ &\to +\infty\end{aligned}\tag{3.9}$$

as $x\to 1^-$ since $a+b>1$. Therefore, (3.5), (3.9), and Lemma 3.3(2) lead to the conclusion that there exists $x_2\in(0,1)$ such that $\omega(x)$ is strictly decreasing on $(0,x_2)$ and strictly increasing on $(x_2,1)$.

For part (2), let

$$B_n=\frac{(a+1,n)(b+1,n)}{\left(\frac{a+b+3}{2},n\right)n!},\quad B_n^*=\frac{(4/3,n)(5/3,n)}{(2,n)n!},$$

then

$$\widetilde{\omega}(x)=\frac{G(x)}{\widetilde{G}(x)}=\frac{\sum_{n=0}^{\infty}B_nx^n}{\sum_{n=0}^{\infty}B_n^*x^n}.\tag{3.10}$$

Elementary calculations show that the monotonicity of $\{B_n/B_n^*\}_{n=0}^{\infty}$ depends on the sign of

$$\widetilde{\Delta}_n=\left(\frac{a+b-1}{2}\right)n^2+\left(ab+\frac{3(a+b)}{2}-\frac{31}{18}\right)n+2ab+\frac{8(a+b)-12}{9}.\tag{3.11}$$

Note that

$$
\begin{aligned}
H_{G,\widetilde{G}}(x) &= \frac{9(a+1)(b+1)}{5(a+b+3)} \cdot \frac{F\left(a+2, b+2; \frac{a+b+5}{2}; x\right)}{F\left(\frac{7}{3}, \frac{8}{3}; 3; x\right)} \widetilde{G}(x) - G(x) \\
&= (1-x)^{-\frac{a+b+1}{2}} \zeta(a,b;x),
\end{aligned} \tag{3.12}
$$

where

$$
\begin{aligned}
\zeta(a,b;x) =& \frac{9(a+1)(b+1)}{5(a+b+3)} \frac{F\left(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+5}{2}; x\right)}{F\left(\frac{1}{3}, \frac{2}{3}; 3; x\right)} F\left(\frac{1}{3}, \frac{2}{3}; 2; x\right) \\
& - F\left(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+3}{2}; x\right),
\end{aligned} \tag{3.13}
$$

and

$$
\begin{aligned}
\lim_{x\to 1^-} \zeta(a,b;x) =& \frac{9(a+1)(b+1)}{5(a+b+3)} \frac{\Gamma\left(\frac{a+b+5}{2}\right)\Gamma\left(\frac{a+b+3}{2}\right)}{\Gamma(a+2)\Gamma(b+2)} \frac{\Gamma\left(\frac{8}{3}\right)\Gamma\left(\frac{7}{3}\right)}{\Gamma(3)\Gamma(2)} \frac{\Gamma(2)\Gamma(1)}{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{4}{3}\right)} \\
& - \frac{\Gamma\left(\frac{a+b+3}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma(a+1)\Gamma(b+1)} \\
=& \left(\frac{a+b-1}{2}\right) \frac{\Gamma\left(\frac{a+b+3}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma(a+1)\Gamma(b+1)} = \begin{cases} <0, & a+b<1, \\ >0, & a+b>1. \end{cases}
\end{aligned} \tag{3.14}
$$

Using the similar arguments in part (1), the desired assertions easily follow from (3.10)–(3.14). □

Proof of Theorem 3.1 Assume that $x(r) = [9r(1+r+r^2)]/[(1+2r)^3]$, then one has $x(r) > r^3$ for $0 < r < 1$. It follows from Lemma 3.4 that $\omega(x(r)) < \omega(r^3)$ for $(a,b) \in D_1 \backslash \{p,q\}$ and $\omega(x(r)) > \omega(r^3)$ for $(a,b) \in D_2 \backslash \{p,q\}$. This in conjunction with Ramanujan cubic transformation (1.3) yields

$$
F(x(r)) < \frac{\widetilde{F}(x(r))}{\widetilde{F}(r^3)} F(r^3) = (1+2r)F(r^3)
$$

for $(a,b) \in D_1 \setminus \{p,q\}$, and it reduces to equality only for $(a,b) = p$ (or q). This completes the proof of (3.1).

Inequality (3.2) can be derived analogously and the remaining case follows easily from Lemma 3.4. □

Proof of Theorem 3.2 Let $z = z(r) = [9\sqrt[3]{r}(1+\sqrt[3]{r}+\sqrt[3]{r^2})]/(1+2\sqrt[3]{r})^3$, then

$$
\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{3(1-\sqrt[3]{r})^2}{\sqrt[3]{r^2}(1+2\sqrt[3]{r})^4} = \frac{3(1-z)}{\sqrt[3]{r^2}(1+2\sqrt[3]{r})(1-\sqrt[3]{r})}.
$$

Differentiating J yields

$$\sqrt[3]{r^2}J'(r) = \frac{2}{3}F(r) + \sqrt[3]{r^2}\left(1+2\sqrt[3]{r}\right)\frac{2ab}{a+b+1}G(r) - \frac{2ab}{a+b+1}\cdot\frac{3(1-z)}{(1+2\sqrt[3]{r})(1-\sqrt[3]{r})}G(z), \tag{3.15}$$

where the functions F, G in (3.15) and $\widetilde{F}$, $\widetilde{G}$ in the following Eq. (3.16) are defined as in Lemma 3.4.

It follows from (1.3) that

$$\frac{\mathrm{d}F\left(\frac{1}{3},\frac{2}{3};1;1-\left(\frac{1-\sqrt[3]{r}}{1+2\sqrt[3]{r}}\right)^3\right)}{\mathrm{d}r} = \frac{\mathrm{d}\left[(1+2\sqrt[3]{r})F\left(\frac{1}{3},\frac{2}{3};1;r\right)\right]}{\mathrm{d}r},$$

which leads to

$$\frac{2}{9}\widetilde{G}(z)\frac{3(1-z)}{\sqrt[3]{r^2}(1+2\sqrt[3]{r})(1-\sqrt[3]{r})} = \frac{2}{3}\frac{1}{\sqrt[3]{r^2}}\widetilde{F}(r) + (1+2\sqrt[3]{r})\frac{2}{9}\widetilde{G}(r),$$

equivalently,

$$\frac{3(1-z)}{(1+2\sqrt[3]{r})(1-\sqrt[3]{r})}\frac{\widetilde{G}(z)}{\widetilde{G}(r)} = \frac{3\widetilde{F}(r)}{\widetilde{G}(r)} + \sqrt[3]{r^2}\left(1+2\sqrt[3]{r}\right). \tag{3.16}$$

If $(a,b) \in D_1 \setminus \{p,q\}$, then it follows from Lemma 3.4(2) that $G(x)/\widetilde{G}(x)$ is strictly decreasing on $(0,1)$. This in conjunction with $z > r$ implies that $G(z)/\widetilde{G}(z) < G(r)/\widetilde{G}(r)$, that is,

$$G(z) < \frac{\widetilde{G}(z)}{\widetilde{G}(r)}G(r). \tag{3.17}$$

Equations (3.15), (3.16) and the inequality (3.17) implies

$$\begin{aligned}
\sqrt[3]{r^2}J'(r) &= \frac{2}{3}F(r) + \sqrt[3]{r^2}\left(1+2\sqrt[3]{r}\right)\frac{2ab}{a+b+1}G(r) \\
&\quad - \frac{2ab}{a+b+1}\cdot\frac{3(1-z)}{(1+2\sqrt[3]{r})(1-\sqrt[3]{r})}G(z) \\
&> \frac{2}{3}F(r) + \sqrt[3]{r^2}\left(1+2\sqrt[3]{r}\right)\frac{2ab}{a+b+1}G(r) \\
&\quad - \frac{2ab}{a+b+1}\cdot\frac{3(1-z)}{(1+2\sqrt[3]{r})(1-\sqrt[3]{r})}\frac{\widetilde{G}(z)}{\widetilde{G}(r)}G(r) \\
&= \frac{2}{3}F(r) + \sqrt[3]{r^2}\left(1+2\sqrt[3]{r}\right)\frac{2ab}{a+b+1}G(r) \\
&\quad - \frac{2ab}{a+b+1}\left[\frac{3\widetilde{F}(r)}{\widetilde{G}(r)} + \sqrt[3]{r^2}\left(1+2\sqrt[3]{r}\right)\right]G(r)
\end{aligned}$$

$$= 3\left[\frac{2}{9}F(r) - \frac{2ab}{a+b+1}\frac{\widetilde{F}(r)}{\widetilde{G}(r)}G(r)\right]$$
$$= \frac{3F(r)^2}{\widetilde{G}(r)}\left[\frac{\widetilde{F}(r)}{F(r)}\right]'. \tag{3.18}$$

Since $\widetilde{F}(r)/F(r)$ is strictly increasing on (0, 1) for $(a, b) \in D_1 \setminus \{p, q\}$, inequality (3.18) implies that $J(r)$ is strictly increasing on (0, 1) for $(a, b) \in D_1 \setminus \{p, q\}$.

Analogously, if $(a, b) \in D_2 \setminus \{p, q\}$, then $G(z) > \widetilde{G}(z)G(r)/\widetilde{G}(r)$. An argument similar to the one above shows that

$$\sqrt[3]{r^2}J'(r) < \frac{3F(r)^2}{\widetilde{G}(r)}\left[\frac{\widetilde{F}(r)}{F(r)}\right]' < 0$$

for $(a, b) \in D_2 \setminus \{p, q\}$, thus $J(r)$ is strictly decreasing on (0, 1).

Finally, clearly $J(0^+) = 0$, and by (1.13) we have

$$\lim_{r\to 1^-} J(r) = \begin{cases} 2H(a, b), & a+b<1, \\ \frac{2(R(a,b)-\log 27)}{B(a,b)}, & a+b=1, \\ -\infty, & a+b>1. \end{cases}$$

Thus the desired assertions including inequalities (3.3) and (3.4) follow. □

4 Triplication inequality for $\mu^*_{a,b}(r)$

In this section, we shall derive the analogs of triplication formula (1.10) satisfied by $\mu^*(r)$, which is said to be the triplication inequality for the generalized Grötzsch ring function with two parameters $\mu^*_{a,b}(r)$. The main result is the following Theorem 4.1.

Theorem 4.1 *Let*

$$f(r) = 3\mu_{a,b}\left(\frac{\sqrt[3]{9r(1+r+r^2)}}{1+2r}\right) - \mu_{a,b}(r), \quad r \in (0, 1).$$

Then f is strictly increasing from (0, 1) onto $(-\infty, 0)$ for $(a, b) \in D_0 = \{(a, b)|\, a, b > 0, a+b \ge 17/9, ab \ge a+b-10/9\}$. As a consequence, the inequality

$$3\mu_{a,b}\left(\frac{\sqrt[3]{9r(1+r+r^2)}}{1+2r}\right) < \mu_{a,b}(r) \tag{4.1}$$

holds for all $r \in (0, 1)$ if $(a, b) \in D_0$.

Lemma 4.2 *Let $0 < x < 1$, $x' = \sqrt[3]{1-x^3}$ and*

$$g(x) = \frac{(xx')^{\frac{3(a+b-1)}{4}} F\left(a, b; \frac{a+b+1}{2}; x^3\right)}{F\left(\frac{1}{3}, \frac{2}{3}; 1; x^3\right)}.$$

Then g is strictly increasing on $(0,1)$ if $(a,b)\in D_0$.

Proof Taking the derivative of $g(x)$ yields

$$g'(x)=\frac{(xx')^{\frac{3(a+b)-7}{4}}}{x'^2F\left(\frac{1}{3},\frac{2}{3};1;x^3\right)^2}g_1(x), \tag{4.2}$$

where

$$\begin{aligned} g_1(x)=&\left[\frac{3(a+b-1)}{4}(1-2x^3)F\left(a,b;\frac{a+b+1}{2};x^3\right)\right.\\ &\left.+\frac{6ab}{a+b+1}x^3x'^3F\left(a+1,b+1;\frac{a+b+3}{2};x^3\right)\right]F\left(\frac{1}{3},\frac{2}{3};1;x^3\right)\\ &-\frac{2x^3x'^3}{3}F\left(a,b;\frac{a+b+1}{2};x^3\right)F\left(\frac{4}{3},\frac{5}{3};2;x^3\right). \end{aligned} \tag{4.3}$$

It follows from (1.13) that

$$x'^3F\left(\frac{4}{3},\frac{5}{3};2;x^3\right)=F\left(\frac{1}{3},\frac{2}{3};2;x^3\right)\le F\left(\frac{1}{3},\frac{2}{3};1;x^3\right)$$

for $0<x<1$. This together with (4.3) implies that

$$g_1(x)\ge F\left(\frac{1}{3},\frac{2}{3};1;x^3\right)g_2(x), \tag{4.4}$$

where

$$\begin{aligned} g_2(x)=&\left[\frac{3(a+b-1)}{4}-\left(\frac{9a+9b-5}{6}\right)x^3\right]F\left(a,b;\frac{a+b+1}{2};x^3\right)\\ &+\frac{6ab}{a+b+1}x^3(1-x^3)F\left(a+1,b+1;\frac{a+b+3}{2};x^3\right). \end{aligned}$$

Expanding g_2 into power series, we have

$$\begin{aligned} g_2(x)=&\frac{3(a+b-1)}{4}\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{\left(\frac{a+b+1}{2},n\right)}\frac{x^{3n}}{n!}\\ &-\left(\frac{9a+9b-5}{6}\right)\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{\left(\frac{a+b+1}{2},n\right)}\frac{x^{3n+3}}{n!}\\ &+\frac{6ab}{a+b+1}\left[\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)}{\left(\frac{a+b+3}{2},n\right)}\frac{x^{3n+3}}{n!}\right.\\ &\left.-\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)}{\left(\frac{a+b+3}{2},n\right)}\frac{x^{3n+6}}{n!}\right] \end{aligned}$$

$$
\begin{aligned}
=&\frac{3(a+b-1)}{4}+\left[\frac{3ab(a+b-1)}{2(a+b+1)}-\left(\frac{9a+9b-5}{6}\right)+\frac{6ab}{a+b+1}\right]x^3\\
&+\frac{3(a+b-1)}{4}\sum_{n=0}^{\infty}\frac{(a,n+2)(b,n+2)}{\left(\frac{a+b+1}{2},n+2\right)}\frac{x^{3n+6}}{(n+2)!}\\
&-\left(\frac{9a+9b-5}{6}\right)\sum_{n=0}^{\infty}\frac{(a,n+1)(b,n+1)}{\left(\frac{a+b+1}{2},n+1\right)}\frac{x^{3n+6}}{(n+1)!}\\
&+3\left[\sum_{n=0}^{\infty}\frac{(a,n+2)(b,n+2)}{\left(\frac{a+b+1}{2},n+2\right)}\frac{x^{3n+6}}{(n+1)!}\right.\\
&\left.-\sum_{n=0}^{\infty}\frac{(a,n+1)(b,n+1)}{\left(\frac{a+b+1}{2},n+1\right)}\frac{x^{3n+6}}{n!}\right]\\
=&\frac{3(a+b-1)}{4}\left[1-\frac{8x^3}{9(a+b+1)}\right]+\frac{9ab(a+b-1)-9(a-b)^2+1}{6(a+b+1)}x^3\\
&+\sum_{n=0}^{\infty}\frac{(a,n+1)(b,n+1)}{12\left(\frac{a+b+1}{2},n+2\right)}\frac{C_n}{(n+2)!}x^{3n+6},
\end{aligned}
\tag{4.5}
$$

where $C_n = 9(a+b-1)(a+n+1)(b+n+1)-(9a+9b-5)\,(a+b+1+2n+2)\,(n+2)+36(a+n+1)(b+n+1)(n+2)-18\,(a+b+1+2n+2)\,(n+1)(n+2) = (9a+9b-17)n^2+(36ab+5(a+b)-37)n+9ab(a+b+3)-9(a-b)^2-(8a+8b+15)$.

For $(a,b)\in D_0$, namely, $a,b>0$, $a+b\geq 17/9$, and $ab\geq a+b-10/9$, it is not difficult to verify that

(1) $9ab(a+b-1)-9(a-b)^2+1\geq 9\,(a+b-10/9)\,(a+b-1)-9(a-b)^2+1 = 36ab-19(a+b)+11\geq 36(a+b-10/9)-19(a+b)+11 = 17(a+b)-29\geq 28/9>0$;
(2) $36ab+5(a+b)-37\geq 36\,(a+b-10/9)+5(a+b)-37 = 41\,(a+b-77/41)\geq 4/9>0$;
(3) $9ab(a+b+3)-9(a-b)^2-(8a+8b+15)\geq 9\,(a+b-10/9)\,(a+b+3)-9(a-b)^2-(8a+8b+15) = 36ab+9(a+b)-45\geq 36(a+b-10/9)+9(a+b)-45 = 45(a+b)-85\geq 0$.

Thus

$$C_n > 0 \tag{4.6}$$

for $a,b\in D_0$, and by (4.5), we conclude that $g_2(x)>0$ for all $x\in(0,1)$. Therefore, it follows from (4.2)-(4.4) that $g(x)$ is strictly increasing on $(0,1)$ for $(a,b)\in D_0$. □

Remark 4.3 In the following text, we will find that the monotonicity of $g(x)$ in Lemma 4.2 is the most important ingredient to prove Theorem 4.1. Thus Lemma 4.2 focuses on searching for $a,b>0$ such that $g(x)$ is strictly monotone on $(0,1)$. In fact, if $a+b<1$, then it is clear to see that $\lim_{x\to 0^+} g(x)=+\infty$, and by (1.13) one has

$$\lim_{x\to 1^-} g(x) = H(a,b) \lim_{x\to 1^-} \frac{(1-x^3)^{(a+b-1)/4}}{F(1/3,2/3;1;x^3)}$$
$$= H(a,b) \lim_{x\to 1^-} \frac{B(1/3,2/3)}{(1-x^3)^{(1-a-b)/4}[R(1/3)-\log(1-x^3)]} = +\infty,$$

and thereby g is not monotone on $(0,1)$; if $a+b=1$, then Lemma 3.4(1) shows that $g(x)$ is strictly decreasing on $(0,1)$ if $(a,b)\in L_1$ and strictly increasing if $(a,b)\in L_2$. In the remaining case $a+b>1$, it follows from (4.3) that $g_1(0^+) = 3(a+b-1)/4 > 0$. Thus by (4.2) we know that $g(x)$ is strictly increasing on $(0,\epsilon)$ for a sufficient small $\epsilon > 0$. This enables us to find a sufficient condition for a, b with $a+b>1$ such that $g(x)$ is strictly increasing on $(0,1)$ in Lemma 4.2.

The following inequality, as another analogs of Ramanujan cubic transformation, can be derived immediately from the monotonicity of $g(x)$ in Lemma 4.2 and the Ramanujan cubic transformation (1.3).

Corollary 4.4 *Let $x = x(r) = \sqrt[3]{9r(1+r+r^2)}/(1+2r)$ and $(a,b)\in D_0$. Then the inequality*

$$(xx')^{\frac{3(a+b-1)}{4}} F\left(a,b;\frac{a+b+1}{2};x^3\right) > (1+2r)(rr')^{\frac{3(a+b-1)}{4}} F\left(a,b;\frac{a+b+1}{2};r^3\right) \tag{4.7}$$

holds for all $r\in(0,1)$.

Proof of Theorem 4.1 Let $x = x(r) = \sqrt[3]{9r(1+r+r^2)}/(1+2r)$ and $x' = \sqrt[3]{1-x^3}$. Then

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{(1-r)^2}{3^{1/3}(1+2r)^2[r(1+r+r^2)]^{2/3}} = \frac{x'^2(1+2x')^2}{3x^2}.$$

Differentiating f gives

$$\begin{aligned} f'(r) =& -\frac{9\Gamma\left(\frac{a+b+1}{2}\right)^2}{2\Gamma(a+b)} \frac{1}{x^{\frac{3(a+b)-1}{2}} x'^{\frac{3(a+b+1)}{2}} F\left(a,b;\frac{a+b+1}{2};x^3\right)^2} \cdot \frac{x'^2(1+2x')^2}{3x^2} \\ &+ \frac{3\Gamma\left(\frac{a+b+1}{2}\right)^2}{2\Gamma(a+b)} \frac{1}{r^{\frac{3(a+b)-1}{2}} r'^{\frac{3(a+b+1)}{2}} F\left(a,b;\frac{a+b+1}{2};r^3\right)^2} \\ =& \frac{3\Gamma\left(\frac{a+b+1}{2}\right)^2}{2\Gamma(a+b)} \cdot \frac{(1+2x')^2}{(1+2r)^2 x^{\frac{3(a+b+1)}{2}} x'^{\frac{3(a+b)-1}{2}} F\left(a,b;\frac{a+b+1}{2};x^3\right)^2} \\ &\times \left[\frac{(xx')^{\frac{3(a+b-1)}{2}} F\left(a,b;\frac{a+b+1}{2};x^3\right)^2}{(rr')^{\frac{3(a+b-1)}{2}} F\left(a,b;\frac{a+b+1}{2};r^3\right)^2} - (1+2r)^2\right]. \end{aligned} \tag{4.8}$$

Thus the monotonicity of f follows immediately from (4.7) and (4.8). Moreover, clearly $f(1^-) = 0$ and

$$
\begin{aligned}
&\lim_{r\to 0^+} f(r)\\
&=\lim_{r\to 0^+}\frac{B(a,b)}{2}\left[3F\left(a,b;\frac{a+b+1}{2};\left(\frac{1-r}{1+2r}\right)^3\right)\right.\\
&\quad\left.-F\left(a,b;\frac{a+b+1}{2};1-r^3\right)\right]\\
&=\frac{B(a,b)}{2}\lim_{r\to 0^+}\left[3\left(\frac{\sqrt[3]{9r(1+r+r^2)}}{1+2r}\right)^{\frac{3(1-a-b)}{2}}\right.\\
&\quad\times F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+1}{2};\left(\frac{1-r}{1+2r}\right)^3\right)\\
&\quad\left.-r^{\frac{3(1-a-b)}{2}}F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+1}{2};1-r^3\right)\right]\\
&=\frac{1}{2}B\left(\frac{a+b+1}{2},\frac{a+b-1}{2}\right)\lim_{r\to 0^+}\left[3\left(\frac{\sqrt[3]{9r(1+r+r^2)}}{1+2r}\right)^{\frac{3(1-a-b)}{2}}-r^{\frac{3(1-a-b)}{2}}\right]\\
&=-\infty.
\end{aligned}
$$

Consequently inequality (4.1) follows. □

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