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Simultaneous cubic and quadratic diagonal equations in 12 prime variables

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Abstract

The system of equations

$$u_1 p_1^2 + \cdots + u_s p_s^2 = 0,$$
$$v_1 p_1^3 + \cdots + v_s p_s^3 = 0$$

has prime solutions $(p_1, \ldots, p_s)$ for $s \geq 12$, assuming that the system has solutions modulo each prime p. This is proved via the Hardy–Littlewood circle method, building on Wooley's work on the corresponding system over the integers and recent results on Vinogradov's mean value theorem. Additionally, a set of sufficient conditions for local solvability is given: If both equations are solvable modulo 2, the quadratic equation is solvable modulo 3, and for each prime p at least 7 of each of the u_i, v_i are not zero modulo p, then the system has solutions modulo each prime p.

Keywords Diophantine equations · Hardy–Littlewood circle method · Waring–Goldbach problem · Diagonal forms

Mathematics Subject Classification 11P05 · 11P32

1 Introduction

Much work has been done in applying the Hardy–Littlewood circle method to find integral solutions to systems of simultaneous equations (see [2,3,10], and [12] for examples). In particular, recent progress on Vinogradov's mean value theorem (see [1, 9]) has enabled progress on questions of this type. Here we consider the question of solving systems of equations with prime variables, generalizing the Waring–Goldbach problem in the same way existing work on integral solutions of systems of equations generalizes Waring's problem. Following Wooley [12], we address here the simplest

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nontrivial case: one quadratic equation and one cubic equation. We find that under suitable local conditions, 12 variables will suffice for us to establish an eventually positive asymptotic formula guaranteeing solutions to the system of equations.

Consider a pair of equations of the form

$$\begin{aligned} u_1 p_1^2 + \cdots + u_s p_s^2 &= 0, \\ v_1 p_1^3 + \cdots + v_s p_s^3 &= 0, \end{aligned} \tag{1}$$

where $u_1, \ldots, u_s, v_1, \ldots, v_s$ are nonzero integer constants and $p_1, \ldots, p_s$ are variables restricted to prime values. We seek to prove the following theorem:

Theorem 1.1 *If*

1. *the system* (1) *has a nontrivial real solution,*
2. $s \geq 12$, *and*
3. *for every prime* p, *the corresponding local system*

$$\begin{aligned} u_1 x_1^2 + \cdots + u_s x_s^2 &\equiv 0 \pmod{p}, \\ v_1 x_1^3 + \cdots + v_s x_s^3 &\equiv 0 \pmod{p} \end{aligned} \tag{2}$$

has a solution $(x_1, \ldots, x_s)$ *with all* $x_i \not\equiv 0 \pmod{p}$,

then the system has a solution $(p_1, \ldots, p_s)$ *with all* p_i *prime. Moreover, if we let* $R(P)$ *be the number of solutions* $(p_1, \ldots, p_s)$ *with each* $p_i \leq P$, *each weighted by* $(\log p_1) \ldots (\log p_s)$, *then we have* $R(P) \sim C P^{s-5}$ *for some constant* $C > 0$ *uniformly over all choices of* $u_1, \ldots, u_s, v_1, \ldots, v_s$.

In Sect. 9 we give a sufficient condition for (2) to be satisfied, giving us the explicit theorem

Theorem 1.2 *Consider the system*

$$\begin{aligned} u_1 p_1^2 + \cdots + u_s p_s^2 &= U, \\ v_1 p_1^3 + \cdots + v_s p_s^3 &= V, \end{aligned} \tag{3}$$

where $u_1, \ldots, u_s, v_1, \ldots, v_s$, *are nonzero integer constants and* U, V *are integer constants. If*

1. *the system has a nontrivial real solution,*
2. $s \geq 12$,
3. *the quadratic form* $u_1 p_1^2 + \cdots + u_s p_s^2$ *is indefinite,*
4. $\sum_{i=1}^{s} u_i \equiv U \pmod{2}$ *and* $\sum_{i=1}^{s} v_i \equiv V \pmod{2}$,
5. $\sum_{i=1}^{s} u_i \equiv U \pmod{3}$, *and*
6. *for each prime* $p \neq 2$, *at least 7 of each of the* u_i *and the* v_i *are not zero modulo* p,

then the system has a solution $(p_1, \ldots, p_s)$ *with all* p_i *prime. Moreover, if we let* $R(P)$ *be the nu mber of solutions* $(p_1, \ldots, p_s)$, *each weighted by* $(\log p_1) \ldots (\log p_s)$, *then we have* $R(P) \sim C P^{s-5}$ *where* $C > 0$ *uniformly over all choices of* $u_1, \ldots, u_s, v_1, \ldots, v_s$, U, *and* V.

We use the Hardy–Littlewood circle method to prove these results. Section 2 performs the necessary setup for the application of the circle method: defining the relevant functions and the major arc/minor arc dissection. Section 3 consists of a number of preliminary lemmas, which are referenced throughout. Section 4 proves a Hua-type bound necessary for the minor arcs. Section 5 proves a Weyl-type bound on the minor arcs by means of Vaughan's identity. Section 6 is the circle method reduction to the singular series and singular integral. Section 7 shows the convergence of the singular series and Sect. 8 shows that it is eventually positive, contingent on the local solvability of the system (3). Section 9 shows sufficient conditions for the solvability of the local system. This depends on a computer check of local solvability for a finite number of primes. Section 10 discusses several techniques which can be employed to improve the efficiency of this computation. Section 11 finishes the proof of Theorems 1.1 and 1.2. Appendix 1 contains the source code used to run the computations laid out in Sect. 10.

2 Notation and definitions

As is standard in the literature, we use $e(\alpha)$ to denote $e^{2\pi i\alpha}$. The letter p is assumed to refer to a prime wherever it is used, and ε means a sufficiently small positive real number. The symbols Λ and μ are the von Mangoldt and Möbius functions, respectively. Symbols in bold are tuples, with the corresponding symbol with a subscript denoting a component, i.e., $\mathbf{a} = (a_1, \ldots, a_k)$. The letter C is used to refer to a positive constant, with the value of C being allowed to change from line to line. We write $f(x) \ll g(x)$ for $f(x) = O(g(x))$, $f(x) \asymp g(x)$ if both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold, and $f(x) \sim g(x)$ if $f(x)/g(x) \to 1$ as $x \to \infty$. When we refer to a solution of the system under study, we mean an ordered s-tuple of prime numbers $(p_1, \ldots, p_s)$ satisfying (1) or (3), depending on context.

Define the generating function

$$f_i(\alpha_2, \alpha_3) = \sum_{p \le P} (\log p) e(\alpha_2 u_i p^2 + \alpha_3 v_i p^3). \tag{4}$$

Let $\mathcal{A}$ be the unit square $(\mathbb{R}/\mathbb{Z})^2$ and let S_0 be the set of solutions of the system (1). Then

$$\begin{aligned}
&\int_{\mathcal{A}} \prod_{i=1}^{s} f_i(\alpha_2, \alpha_3) d\alpha_2\, d\alpha_3 \\
&\quad = \int_{\mathcal{A}} \sum_{p_1, \ldots, p_s \le P} \prod_{i=1}^{s} \Big((\log p_i) e(\alpha_2 u_i p_i^2 + \alpha_3 v_i p_i^3) \Big) d\alpha_2\, d\alpha_3 \\
&\quad = \sum_{\{p_1, \ldots, p_s\} \in S_0} \prod_{i=1}^{s} (\log p_i) = R(P)
\end{aligned} \tag{5}$$

by orthogonality. Thus $R(P) > 0$ if and only if there is a solution to the system (1).

We divide $\mathcal{A}$ into major and minor arcs. For any T with $1 \leq T \leq P$. and for all $q < T$, $1 \leq a_2 \leq q$, $1 \leq a_3 \leq q$, $(a_2, a_3, q) = 1$, let a typical major arc $\mathfrak{M}(a_2, a_3, q; T)$ consist of all (α_2, α_3) such that

$$|\alpha_2 - a_2/q| \leq \frac{T}{qP^2} \qquad \text{and} \qquad |\alpha_3 - a_3/q| \leq \frac{T}{qP^3}.$$

Let the major arcs $\mathfrak{M}(T)$ be the union of all such $\mathfrak{M}(a_2, a_3, q)$, and let the minor arcs $\mathfrak{m}(T)$ be the complement of $\mathfrak{M}(T)$ in $\mathcal{A}$.

We will use two distinct dissections in our argument: the primary dissection into $\mathfrak{M} = \mathfrak{M}(Q)$ and $\mathfrak{m} = \mathfrak{m}(Q)$ with $Q = (\log P)^A$, where A is a positive constant whose value will be fixed later, and a secondary dissection $\mathfrak{M}(R)$, $\mathfrak{m}(R)$ with $R = P^{\frac{1}{2}+\delta}$ for some sufficiently small positive δ.

3 Preliminary lemmas

We begin by defining the necessary generating functions. Let

$$f(\boldsymbol{\alpha}) = \sum_{P<p\leq 2P} e(\alpha_2 p^2 + \alpha_3 p^3),$$

$$g(\boldsymbol{\alpha}) = \sum_{P<n\leq 2P} e(\alpha_2 n^2 + \alpha_3 n^3),$$

$$S(q, \boldsymbol{a}) = \sum_{n=1}^{q} e\left(\frac{a_2 n^2 + a_3 n^3}{q}\right), \tag{6}$$

$$W(q, \boldsymbol{a}) = \sum_{\substack{n=1 \\ (n,q)=1}}^{q} e\left(\frac{a_2 n^2 + a_3 n^3}{q}\right),$$

$$v(\boldsymbol{\theta}) = \int_P^{2P} e(\theta_2 x^2 + \theta_3 x^3)dx, \tag{7}$$

and for $\boldsymbol{\gamma} \in \mathfrak{M}(R)$ let

$$V(\boldsymbol{\gamma}) = \frac{1}{q} S(q, \boldsymbol{a}) v\left(\gamma_2 - \frac{a_2}{q}, \gamma_3 - \frac{a_3}{q}\right).$$

Lemma 3.1 *We have the bounds*

$$\int_{\mathcal{A}} |g(\boldsymbol{\alpha})|^{10} d\boldsymbol{\alpha} \ll P^{\frac{31}{6}+\varepsilon}$$

and

$$\int_{\mathcal{A}} |g(\boldsymbol{\alpha})|^{12} d\boldsymbol{\alpha} \ll P^{7}.$$

Proof This is the relevant portion of Theorem 1.3 of [12]. □

Lemma 3.2 *We have the bounds*

$$\int_{\mathcal{A}} |f(\boldsymbol{\alpha})|^{10} d\boldsymbol{\alpha} \ll P^{\frac{31}{6}+\varepsilon} d\boldsymbol{\alpha}$$

and

$$\int_{\mathcal{A}} |f(\boldsymbol{\alpha})|^{12} d\boldsymbol{\alpha} \ll P^{7} d\boldsymbol{\alpha}.$$

Proof For any positive integer k,

$$\int_{\mathcal{A}} |g(\boldsymbol{\alpha})|^{2k} d\boldsymbol{\alpha}$$

is the number of positive integer solutions to the system

$$\begin{aligned} x_1^2 + \cdots + x_k^2 &= x_{k+1}^2 + \cdots + x_{2k}^2, \\ x_1^3 + \cdots + x_k^3 &= x_{k+1}^3 + \cdots + x_{2k}^3 \end{aligned}$$

and

$$\int_{\mathcal{A}} |f(\boldsymbol{\alpha})|^{2k} d\boldsymbol{\alpha}$$

is the number of prime solutions to the same system, so this lemma follows from Lemma 3.1. □

Lemma 3.3

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{m}(R)} |g(\boldsymbol{\alpha})| \ll P^{\frac{5}{6}-\frac{\delta}{3}+\varepsilon}.$$

Proof This follows from Lemma 5.2 of [12]. □

Lemma 3.4

$$v(\boldsymbol{\theta}) \ll \frac{P}{(1 + P^3|\theta_3|)^{1/2}}.$$

Proof If $|\theta_3| \le P^{-3}$, the result is immediate. Thus we assume $|\theta_3| > P^{-3}$. Let $K = (|\theta_3|P)^{\frac{1}{2}}$ and let $r(x) = \theta_2 x^2 + \theta_3 x^3$. Then $r'(x) = 2\theta_2 x + 3\theta_3 x^2$ has at most one zero in $[P, 2P]$. Thus we can divide $[P, 2P]$ into subsets I_1 and I_2 such that $|r'(x)| \ge K$ on I_1, where I_1 is the union of at most three intervals such that $r'(x)$ is monotonic on each, and $|r'(x)| \le K$ on I_2, where I_2 is the union of at most two intervals.

First we consider I_1:

$$\int_{I_1} e(r(x))dx = \int_{I_1} \frac{1}{2\pi i r'(x)} \frac{d}{dx} e(r(x))dx,$$

so, upon integrating by parts,

$$\int_{I_1} e(r(x))dx = \frac{e(r(x))}{2\pi i r'(x)}\Big|_{I_1} + \int_{I_1} \frac{r''(x)}{2\pi i r'(x)^2} e(r(x))dx.$$

The integral on the right is bounded by

$$\int_{I_1} \frac{|r''(x)|}{2\pi r'(x)^2} dx = \left| \int_{I_1} \frac{r''(x)}{2\pi r'(x)^2} dx \right| = \left| \frac{-1}{2\pi r'(x)}\Big|_{I_1} \right| \ll \frac{1}{K},$$

since $r'(x)$ is monotonic on each interval in I_1. Thus

$$\int_{I_1} e(r(x))dx \ll \frac{e(r(x))}{2\pi i r'(x)}\Big|_{I_1} + \frac{1}{K} \ll \frac{1}{K} \ll \frac{P}{(1 + |\theta_3|P^3)^{1/2}}. \tag{8}$$

Next we consider I_2. Given an interval in I_2, let x_0 be one of its endpoints. Then for any x in I_2,

$$|x - x_0||2\theta_2 + 3\theta_3(x + x_0)| = |r'(x) - r'(x_0)| \le 2K.$$

Moreover,

$$|2\theta_2 + 3\theta_3 x_0| = \frac{|r'(x_0)|}{x_0} \le \frac{K}{x_0}. \tag{9}$$

Applying the triangle identity to (9) yields

$$|3\theta_3 x| - \frac{K}{x_0} \le |2\theta_2 + 3\theta_3(x + x_0)|. \tag{10}$$

Also,

$$|3\theta_3 x| - \frac{K}{x_0} \ge 3|\theta_3|P - \frac{K}{P} \ge 2|\theta_3|P. \tag{11}$$

Combining (9), (10), and (11) yields

$$|x - x_0| \le \frac{2K}{2|\theta_3|P} = \frac{P}{(|\theta_3|P^3)^{1/2}}.$$

Thus

$$\begin{aligned}\int_{I_2} e(r(x))dx &\ll |e(r(x))|\,(\text{meas}(I_2))\\ &\ll 2\max_{x\in I_2}|x-x_0|\\ &\ll \frac{P}{(1+|\theta_3|P^3)^{1/2}}.\end{aligned} \tag{12}$$

Combining (8) and (12) now gives the desired result. ◻

Lemma 3.5 *Let $t = 12-\delta$. Then*

$$\int_{\mathcal{A}} |f(\boldsymbol{\alpha})|^{t-1} d\boldsymbol{\alpha} \ll P^{t-6+\frac{1+\delta}{12}+\varepsilon}.$$

Proof By Hölder's inequality

$$\int_{\mathcal{A}} |f(\boldsymbol{\alpha})|^{t-1} d\boldsymbol{\alpha} \le \left(\int_{\mathcal{A}} |f(\boldsymbol{\alpha})|^{12} d\boldsymbol{\alpha}\right)^{\frac{t-11}{2}} \left(\int_{\mathcal{A}} |f(\boldsymbol{\alpha})|^{10} d\boldsymbol{\alpha}\right)^{\frac{13-t}{2}}.$$

Applying Lemma 3.2 gives

$$\int_{\mathcal{A}} |f(\boldsymbol{\alpha})|^{t-1} d\boldsymbol{\alpha} \ll P^{\frac{7t-77}{2}+\frac{403-31t}{12}+\varepsilon} = P^{t-6+\frac{1+\delta}{12}+\varepsilon}.$$

◻

Lemma 3.6 *Let $R = P^{\frac{1}{2}+\delta}$ and let $\boldsymbol{\gamma} \in \mathfrak{M}(R)$. Then*

$$g(\boldsymbol{\gamma}) = V(\boldsymbol{\gamma}) + O\left(P^{\frac{5}{6}-\frac{\delta}{3}}\right).$$

This follows from Theorem 7.2 of [7].

Lemma 3.7 *Let $\kappa(q)$ be the multiplicative function defined by*

$$\kappa(p^j) = \begin{cases} Cp^{-1/2} & j=1,\\ Cp^{-5/8} & j=2,\\ Cp^{-j/4} & j>2.\end{cases}$$

Then there is a positive constant C such that

$$\max_{\substack{\mathbf{a}\\(q,a_2,a_3)=1}} \frac{|S(q,\mathbf{a})|}{q} \le \kappa(q).$$

Proof The case $j=1$ follows from Theorem 2E of [6]. The cases with $j>1$ follow from Theorem 7.1 of [7]. ◻

Let

$$s_k(\mathbf{m}) = m_1^k + m_2^k + m_3^k - m_4^k - m_5^k - m_6^k. \tag{13}$$

Lemma 3.8 *Let $Q > 0$ and let $M(Q)$ be the number of solutions of the system*

$$s_2(\mathbf{m}) = 0,$$
$$s_1(\mathbf{m}) = 0$$

with all $m_j \leq Q$. Then there is a positive constant C such that

$$M(Q) \sim CQ^3 \log Q.$$

This is a result of Rogovskaya [5].

Lemma 3.9 *If $(q, a_2, a_3) = 1$, then*

$$W(q, \mathbf{a}) \ll q^{\frac{1}{2}+\varepsilon}.$$

In addition, if $(p, a_2, a_3) = 1$, then

$$W(p, \mathbf{a}) \ll p^{\frac{1}{2}}.$$

Proof The case where $q = p$ follows from Theorem 2E of [6]. The case for general q follows from Lemma 8.5 of [4]. □

4 Minor arc bounds

The primary purpose of this section is to prove the following theorem, which, together with the result of the next section, will provide the necessary minor arc bounds for our circle method approach.

Recall from the end of Sect. 2 that δ is a small positive number with $R = P^{\frac{1}{2}+\delta}$. Assume $\delta < 1$ and let $t = 12 - \delta$.

Theorem 4.1 *For δ sufficiently small,*

$$\int_{\mathcal{A}} |f(\boldsymbol{\alpha})|^t d\boldsymbol{\alpha} \ll P^{t-5}(\log P).$$

Let

$$I_t(P) = \int_{\mathcal{A}} |f(\boldsymbol{\alpha})|^t d\boldsymbol{\alpha}.$$

Lemma 4.1

$$I_t(P)^2 \ll P^{2t-10} + P \int_{\mathcal{A}}\int_{\mathcal{A}}\limits_{\boldsymbol{\alpha}-\boldsymbol{\beta}\in\mathfrak{M}(R)} |V(\boldsymbol{\alpha}-\boldsymbol{\beta})||f(\boldsymbol{\alpha})|^{t-1}|f(\boldsymbol{\beta})|^{t-1}d\boldsymbol{\alpha}\, d\boldsymbol{\beta}.$$

Proof

$$\begin{aligned} I_t(P) &= \int_{\mathcal{A}} f(\boldsymbol{\alpha}) f(-\boldsymbol{\alpha})|f(\boldsymbol{\alpha})|^{t-2}d\boldsymbol{\alpha} \\ &= \sum_{P<p\le 2P} \int_{\mathcal{A}} e(\alpha_2 p^2+\alpha_3 p^3) f(-\boldsymbol{\alpha})|f(\boldsymbol{\alpha})|^{t-2}d\boldsymbol{\alpha}. \end{aligned} \tag{14}$$

Applying the Cauchy–Schwarz inequality to (14) yields

$$\begin{aligned} I_t(P)^2 &\ll P \sum_{P<n\le 2P} \left|\int_{\mathcal{A}} e(\alpha_2 n^2+\alpha_3 n^3) f(-\boldsymbol{\alpha})|f(\boldsymbol{\alpha})|^{t-2}d\boldsymbol{\alpha}\right|^2 \\ &= P\int_{\mathcal{A}}\int_{\mathcal{A}} g(\boldsymbol{\alpha}-\boldsymbol{\beta}) f(-\boldsymbol{\alpha})|f(\boldsymbol{\alpha})|^{t-1} f(\boldsymbol{\beta})|f(\boldsymbol{\beta})|^{t-1}d\boldsymbol{\alpha}\, d\boldsymbol{\beta} \\ &\le P\int_{\mathcal{A}}\int_{\mathcal{A}} |g(\boldsymbol{\alpha}-\boldsymbol{\beta})||f(\boldsymbol{\alpha})|^{t-1}|f(\boldsymbol{\beta})|^{t-1}d\boldsymbol{\alpha}\, d\boldsymbol{\beta}. \end{aligned} \tag{15}$$

By Lemmas 3.3 and 3.5 and recalling that $t = 12-\delta$, we can bound the minor arc portion of (15):

$$\begin{aligned} &P \int_{\mathcal{A}}\int_{\mathcal{A}}\limits_{\boldsymbol{\alpha}-\boldsymbol{\beta}\in\mathfrak{m}(R)} |g(\boldsymbol{\alpha}-\boldsymbol{\beta})||f(\boldsymbol{\alpha})|^{t-1}|f(\boldsymbol{\beta})|^{t-1}d\boldsymbol{\alpha}\, d\boldsymbol{\beta} \\ &\ll P^{\frac{11}{6}-\frac{\delta}{3}+\varepsilon}\left(\int_{\mathcal{A}}|f(\boldsymbol{\alpha})|^{t-1}d\boldsymbol{\alpha}\right)^2 \\ &\ll P^{2t-10-\frac{\delta}{6}+2\varepsilon} \ll P^{2t-10}. \end{aligned} \tag{16}$$

We now apply Lemma 3.6 to the major arc portion of (15).

$$\begin{aligned} &P \int_{\mathcal{A}}\int_{\mathcal{A}}\limits_{\boldsymbol{\alpha}-\boldsymbol{\beta}\in\mathfrak{M}(R)} |g(\boldsymbol{\alpha}-\boldsymbol{\beta})||f(\boldsymbol{\alpha})|^{t-1}|f(\boldsymbol{\beta})|^{t-1}d\boldsymbol{\alpha}\, d\boldsymbol{\beta} \\ &= P \int_{\mathcal{A}}\int_{\mathcal{A}}\limits_{\boldsymbol{\alpha}-\boldsymbol{\beta}\in\mathfrak{M}(R)} |V(\boldsymbol{\alpha}-\boldsymbol{\beta})||f(\boldsymbol{\alpha})|^{t-1}|f(\boldsymbol{\beta})|^{t-1}d\boldsymbol{\alpha}\, d\boldsymbol{\beta} \\ &\quad + O\left(P^{\frac{11}{6}-\frac{\delta}{3}}\left(\int_{\mathcal{A}}|f(\boldsymbol{\alpha})|^{t-1}d\boldsymbol{\alpha}\right)^2\right) \\ &= P \int_{\mathcal{A}}\int_{\mathcal{A}}\limits_{\boldsymbol{\alpha}-\boldsymbol{\beta}\in\mathfrak{M}(R)} |V(\boldsymbol{\alpha}-\boldsymbol{\beta})||f(\boldsymbol{\alpha})|^{t-1}|f(\boldsymbol{\beta})|^{t-1}d\boldsymbol{\alpha}\, d\boldsymbol{\beta} + O(P^{2t-10}). \end{aligned} \tag{17}$$

Combining (15), (16), and (17) yields the lemma. □

Let $\boldsymbol{\gamma} = \boldsymbol{\alpha} - \boldsymbol{\beta}$,

$$\lambda = \frac{t-6}{2} = 3 - \frac{\delta}{2} \tag{18}$$

(note that $\lambda > 2$), and

$$J(\boldsymbol{\beta}) = \int_{\mathfrak{M}(R)} |V(\boldsymbol{\gamma})|^{\lambda} |f(\boldsymbol{\beta}+\boldsymbol{\gamma})|^{6} d\boldsymbol{\gamma}. \tag{19}$$

Lemma 4.2

$$I_t(P) \ll P^{t-5} + P^{\lambda} \sup_{\boldsymbol{\beta}\in\mathcal{A}} J(\boldsymbol{\beta}).$$

Proof We begin by noting that

$$|V(\boldsymbol{\alpha}-\boldsymbol{\beta})||f(\boldsymbol{\alpha})|^{t-1}|f(\boldsymbol{\beta})|^{t-1}$$

can be rewritten as

$$\begin{aligned} &\left(|V(\boldsymbol{\alpha}-\boldsymbol{\beta})|^{\lambda}|f(\boldsymbol{\alpha})|^{6}|f(\boldsymbol{\beta})|^{t}\right)^{\frac{1}{2\lambda}} \\ &\times \left(|V(\boldsymbol{\alpha}-\boldsymbol{\beta})|^{\lambda}|f(\boldsymbol{\beta})|^{6}|f(\boldsymbol{\alpha})|^{t}\right)^{\frac{1}{2\lambda}} \\ &\times \left(|f(\boldsymbol{\alpha})f(\boldsymbol{\beta})|^{t}\right)^{1-\frac{1}{\lambda}}. \end{aligned} \tag{20}$$

Let

$$I_t^*(P) = \underset{\boldsymbol{\alpha}-\boldsymbol{\beta}\in\mathfrak{M}(R)}{\int_{\mathcal{A}}\int_{\mathcal{A}}} |V(\boldsymbol{\alpha}-\boldsymbol{\beta})||f(\boldsymbol{\alpha})|^{t-1}|f(\boldsymbol{\beta})|^{t-1} d\boldsymbol{\alpha}\, d\boldsymbol{\beta}$$

be the integral on the right in Lemma 4.1. Using (20) to apply Hölder's inequality to $I_t^*(P)$, we obtain

$$\begin{aligned} I_t^*(P) &\ll \left(\underset{\boldsymbol{\alpha}-\boldsymbol{\beta}\in\mathfrak{M}(R)}{\int_{\mathcal{A}}\int_{\mathcal{A}}} |V(\boldsymbol{\alpha}-\boldsymbol{\beta})|^{\lambda}|f(\boldsymbol{\alpha})|^{6}|f(\boldsymbol{\beta})|^{t} d\boldsymbol{\alpha}\boldsymbol{\beta}\right)^{\frac{1}{\lambda}} \\ &\quad\times \left(\underset{\boldsymbol{\alpha}-\boldsymbol{\beta}\in\mathfrak{M}(R)}{\int_{\mathcal{A}}\int_{\mathcal{A}}} |f(\boldsymbol{\alpha})f(\boldsymbol{\beta})|^{t} d\boldsymbol{\alpha}\boldsymbol{\beta}\right)^{1-\frac{1}{\lambda}} \end{aligned} \tag{21}$$

$$\leq I_t(P)^{2-\frac{1}{\lambda}} \left(\sup_{\boldsymbol{\beta}\in\mathcal{A}} \int_{\mathfrak{M}(R)} |V(\boldsymbol{\gamma})|^{\lambda}|f(\boldsymbol{\beta}+\boldsymbol{\gamma})|^{6} d\boldsymbol{\gamma}\right)^{\frac{1}{\lambda}}. \tag{22}$$

Applying (22) to Lemma 4.1, we have

$$I_t(P)^2 \ll P^{2t-10} + P I_t(P)^{2-\frac{1}{\lambda}} \left(\sup_{\boldsymbol{\beta}\in\mathcal{A}} J(\boldsymbol{\beta}) \right)^{\frac{1}{\lambda}}.$$

Thus either $I_t(P) \ll P^{t-5}$ or

$$I_t(P) \ll P^{\lambda} \sup_{\boldsymbol{\beta}\in\mathcal{A}} J(\boldsymbol{\beta}),$$

which implies the desired result. □

Lemma 4.3 *Let $N(q)$ be the number of solutions of the system*

$$\begin{cases} s_3(\mathbf{p}) \equiv 0 \pmod{q}, \\ s_2(\mathbf{p}) = 0, \\ P < p_j \leq 2P. \end{cases}$$

Then

$$J(\boldsymbol{\beta}) \ll P^{\lambda-3} \sum_{q\leq R} \kappa(q)^{\lambda} q N(q).$$

Proof By (19) and the definition of $\mathfrak{M}(R)$,

$$J(\boldsymbol{\beta}) = \sum_{q\leq R} \sum_{a_2=1}^{q} \sum_{\substack{a_3=1 \\ (q,a_2,a_3)=1}}^{q} \frac{|S(q,\mathbf{a})|^{\lambda}}{q^{\lambda}} \int_{-\frac{R}{qP^2}}^{\frac{R}{qP^2}} \int_{-\frac{R}{qP^3}}^{\frac{R}{qP^3}} |v(\boldsymbol{\theta})|^{\lambda} \left| f\left(\boldsymbol{\beta} + \frac{\mathbf{a}}{q} + \boldsymbol{\theta}\right)\right|^6 d\boldsymbol{\theta}.$$

By Lemmas 3.4, 3.7, and the fact that for a given q, the intervals $\left[\frac{a_2}{q} - \frac{R}{qP^2}, \frac{a_2}{q} + \frac{R}{qP^2}\right]$ are disjoint for distinct a_2,

$$J(\boldsymbol{\beta}) \leq \sum_{q\leq R} \int_{-\frac{R}{qP^3}}^{\frac{R}{qP^3}} \frac{\kappa(q)^{\lambda} P^{\lambda}}{(1+P^3|\theta_3|)^{\lambda/2}} \sum_{a_3=1}^{q} \int_0^1 \left| f\left(\beta_2 + \phi, \beta_3 + \frac{a_3}{q} + \theta_3\right)\right|^6 d\phi\, d\theta_3. \tag{23}$$

We now examine the inner sum and integral.

$$\sum_{a_3=1}^{q}\int_0^1\left|f\left(\beta_2+\phi,\beta_3+\frac{a_3}{q}+\theta_3\right)\right|^6 d\phi$$
$$=\sum_{a_3=1}^{q}\int_0^1\sum_{\substack{\mathbf{p}\\ P<p_j\le 2P}} e\left((\beta_2+\phi)s_2(\mathbf{p})+\left(\beta_3+\frac{a_3}{q}+\theta_3\right)s_3(\mathbf{p})\right)d\phi$$
$$=\sum_{\substack{\mathbf{p}\\ P<p_j\le 2P}} e(\beta_2 s_2(\mathbf{p})+(\beta_3+\theta_3)s_3(\mathbf{p}))\sum_{a_3=1}^{q} e\left(\frac{a_3}{q}s_3(\mathbf{p})\right)\int_0^1 e(\phi s_2(\mathbf{p}))d\phi.$$

Now

$$\sum_{a_3=1}^{q} e\left(\frac{a_3}{q}s_3(\mathbf{p})\right)=\begin{cases}0 & s_3(\mathbf{p})\not\equiv 0 \pmod q,\\ q & s_3(\mathbf{p})\equiv 0 \pmod q\end{cases}$$

and

$$\int_0^1 e(\phi s_2(\mathbf{p}))d\phi=\begin{cases}0 & s_2(\mathbf{p})\neq 0,\\ 1 & s_2(\mathbf{p})=0,\end{cases}$$

so

$$\sum_{a_3=1}^{q}\int_0^1\left|f\left(\beta_2+\phi,\beta_3+\frac{a_3}{q}+\theta_3\right)\right|^6 d\phi\ll qN(q). \tag{24}$$

Substituting (24) into (23) yields

$$J(\boldsymbol{\beta})\ll\sum_{q\le R}\kappa(q)^\lambda qN(q)\int_{-\frac{R}{qP^3}}^{\frac{R}{qP^3}}\frac{P^\lambda}{(1+P^3|\theta_3|)^{\lambda/2}}d\theta_3.$$

Since $\lambda>2$, this becomes

$$J(\boldsymbol{\beta})\ll P^{\lambda-3}\sum_{q\le R}\kappa(q)^\lambda qN(q).$$

□

Lemma 4.4 *Let $N_1(q)$ be the number of solutions to the system*

$$\begin{cases}s_3(\mathbf{p})\equiv 0 \pmod q,\\ s_2(\mathbf{p})=0,\\ P<p_j\le 2P \quad p_j\nmid q.\end{cases}$$

Then

$$qN(q)\ll q(\log q)^6+qN_1(q).$$

Proof First, note that

$$qN(q) = \sum_{a_3=1}^{q} \int_0^1 \left| f\left(x, \frac{a_3}{q}\right)\right|^6 dx.$$

Let

$$f_{|}(\boldsymbol{\alpha}) = \sum_{\substack{P \le p < 2P \\ p|q}} e(\alpha_2 p^2 + \alpha_3 p^3)$$

and

$$f_{\nmid}(\boldsymbol{\alpha}) = \sum_{\substack{P \le p < 2P \\ p\nmid q}} e(\alpha_2 p^2 + \alpha_3 p^3).$$

Thus

$$f(\boldsymbol{\alpha}) = f_{|}(\boldsymbol{\alpha}) + f_{\nmid}(\boldsymbol{\alpha}).$$

Since $|f_{|}(\boldsymbol{\alpha})| \ll \log q$,

$$|f(\boldsymbol{\alpha})|^6 \ll (\log q)^6 + |f_{\nmid}(\boldsymbol{\alpha})|^6.$$

Now

$$\sum_{a_3=1}^{q} \int_0^1 \left| f_{\nmid}\left(x, \frac{a_3}{q}\right)\right|^6 dx = qN_1(q),$$

so

$$qN(q) \ll q(\log q)^6 + qN_1(q).$$

□

Lemma 4.5 *Let $N_2(q)$ be the number of solutions of the system*

$$\begin{cases} s_3(\mathbf{r}) \equiv 0 \pmod{q}, \\ s_2(q\mathbf{m} + \mathbf{r}) = 0, \\ 1 \le r_j \le q (q, r_j) = 1, \\ \frac{P - r_j}{q} < m_j \le \frac{2P - r_j}{q}. \end{cases} \tag{25}$$

Then

$$N_1(q) \le N_2(q).$$

Proof We classify the solutions $\mathbf{p}$ counted by $N_1(q)$ according to the residue class r_j of each p_j modulo q, and let $m_j = \frac{p_j - r_j}{q}$. Thus

$$0 = s_2(q\mathbf{m} + \mathbf{r}) \equiv s_2(\mathbf{r}) \pmod{q},$$

so $N_1(q) \le N_2(q)$. □

Lemma 4.6 *Let $N_3(q)$ be the number of solutions of the system*

$$\begin{cases} s_3(\mathbf{r}) \equiv 0 \pmod{q}, \\ s_2(\mathbf{r}) \equiv 0 \pmod{q}, \\ 1 \le r_j \le q \quad (q, r_j) = 1. \end{cases}$$

Then

$$N_2(q) \ll N_3(q) P^4 q^{-5} (\log P) \left(\frac{q^2}{P} + 1 \right).$$

Proof Let $\mathbf{r}$, $\mathbf{m}$ be a solution counted in $N_2(q)$, i.e., let $\mathbf{r}$, $\mathbf{m}$ satisfy (25). Expanding the third equation of (25) gives

$$q^2 s_2(\mathbf{m}) + 2q(r_1m_1 + r_2m_2 + r_3m_3 - r_4m_4 - r_5m_5 - r_6m_6) + s_2(\mathbf{r}) = 0.$$

Since $s_2(\mathbf{r}) \equiv 0 \pmod{q}$ by the second equation of (25), this can be rewritten as

$$qs_2(\mathbf{m}) + 2(r_1m_1 + r_2m_2 + r_3m_3 - r_4m_4 - r_5m_5 - r_6m_6) + \frac{s_2(\mathbf{r})}{q} = 0$$

with each term remaining integer-valued. For a fixed $\mathbf{r}$, define

$$H_j(\alpha) = \sum_{\frac{P-r_j}{q} < m \le \frac{2P-r_j}{q}} e\left(\alpha(qm^2 + 2r_jm)\right). \tag{26}$$

Thus the number of $\mathbf{m}$ satisfying (25) for a given $\mathbf{r}$ is

$$\int_0^1 H_1(\alpha)H_2(\alpha)H_3(\alpha)H_4(-\alpha)H_5(-\alpha)H_6(-\alpha)e\left(\frac{s_2(\mathbf{r})}{q}\alpha\right) d\alpha.$$

By Hölder's inequality this is

$$\le \prod_{j=1}^{6} \left(\int_0^1 |H_j(\alpha)|^6 d\alpha \right)^{\frac{1}{6}}.$$

The integral

$$\int_0^1 |H_j(\alpha)|^6 d\alpha$$

counts the number of solutions of

$$qs_2(\mathbf{m}) + 2r_j s_1(\mathbf{m}) = 0. \tag{27}$$

Let $s_2(\mathbf{m}) = u$ and $s_1(\mathbf{m}) = v$. Then (27) becomes $qu + 2r_j v = 0$. For any solution, we have $|v| \leq \frac{6P}{q}$, and since $(q, r_j) = 1$, $v = \frac{v'q}{(q,2)}$. Thus the number of choices for v' is $\leq 1 + 24P/q^2$, and u is determined by v'.

Let

$$h(\boldsymbol{\alpha}) = \sum_{\frac{P-r_j}{q} < m \leq \frac{2P-r_j}{q}} e(\alpha_1 m + \alpha_2 m^2).$$

For fixed pair u, v, the number of choices of $\mathbf{m}$ is

$$\int_{\mathcal{A}} |h(\boldsymbol{\alpha})|^6 e(-\alpha_1 v - \alpha_2 u) d\boldsymbol{\alpha} \leq \int_{\mathcal{A}} |h(\boldsymbol{\alpha})|^6 d\boldsymbol{\alpha}.$$

But this is the number of solutions of the system

$$s_2(\mathbf{m}) = 0,$$
$$s_1(\mathbf{m}) = 0,$$

so by Lemma 3.8,

$$\int_{\mathcal{A}} |h(\boldsymbol{\alpha})|^6 d\boldsymbol{\alpha} \ll \left(\frac{P}{q}\right)^3 \log P.$$

So, given $\mathbf{r}$ satisfying the first two equations of (25) and $(q, r_j) = 1$, the number of solutions to the third equation of (25) is

$$\ll \left(1 + \frac{P}{q^2}\right) \frac{P^3}{q^3} \log P = P^4 q^{-5} \left(1 + \frac{q^2}{P}\right) \log P.$$

Thus

$$N_2(q) \ll N_3(q) P^4 q^{-5} (\log P) \left(1 + \frac{q^2}{P}\right).$$

$\square$

Lemma 4.7 *Let $N_3(q)$ be as defined in Lemma 4.6 above. Then there exists a positive constant C such that*

$$N_3(q) \ll q^4 \prod_{p|q} \left(1 + \frac{C}{p}\right).$$

Proof We begin by observing that $N_3(q)$ is a multiplicative function, and that by orthogonality,

$$N_3(p^k) = p^{-2k} \sum_{b_2=1}^{p^k} \sum_{b_3=1}^{p^k} |W(p^k, b_2, b_3)|^6.$$

Sorting the terms of this sum by the value of $(p^k, b_2, b_3) = p^{k-j}$, where $0 \le j \le k$, gives

$$N_3(p^k) = p^{-2k} \sum_{j=0}^{k} \sum_{\substack{a_2=1 \\ (p^j, a_2, a_3)=1}}^{p^j} \sum_{a_3=1}^{p^j} |W(p^k, p^{k-j}a_2, p^{k-j}a_3)|^6.$$

If $j = 0$, then

$$W(p^k, p^{k-j}a_2, p^{k-j}a_3) = \phi(p^k) = p^k(1 - 1/p)$$

and if $j > 0$, then

$$W(p^k, p^{k-j}a_2, p^{k-j}a_3) = p^{k-j} W(p^j, a_2, a_3).$$

Thus

$$N_3(p^k) = p^{4k}(1 - 1/p)^6 + p^{4k} \sum_{j=1}^{k} \sum_{\substack{a_2=1 \\ (p^j, a_2, a_3)=1}}^{p^j} \sum_{a_3=1}^{p^j} p^{-6j} |W(p^j, a_2, a_3)|^6.$$

By Lemma 3.9,

$$\sum_{\substack{\mathbf{a} \\ (p, a_2, a_3)=1}} p^{-6} |W(p, a_2, a_3)|^6 \ll p^{-1},$$

and for $j \ge 2$,

$$\sum_{\substack{\mathbf{a} \\ (p^j, a_2, a_3)=1}} p^{-6j} |W(p^j, a_2, a_3)|^6 \ll p^{-4j+6j/2+j\varepsilon} \ll p^{-j+j\varepsilon}.$$

Thus

$$N_3(p^k) \leq p^{4k}\left(1+\frac{C}{p}\right)$$

and the lemma follows by multiplicativity. $\square$

Proof of Theorem 4.1 By Lemma 4.2,

$$I_t(P) \ll P^{t-5} + P^{\lambda} \sup_{\boldsymbol{\beta}\in\mathcal{A}} J(\boldsymbol{\beta}).$$

Bounding $J(\boldsymbol{\beta})$ with Lemma 4.3 yields

$$I_t(P) \ll P^{t-5} + P^{2\lambda-3} \sum_{q\leq R} \kappa(q)^{\lambda} q N(q). \tag{28}$$

Lemmas 4.4, 4.5, and 4.6 successively bound $N(q)$ in terms of $N_1(q)$, then $N_2(q)$, then $N_3(q)$, and Lemma 4.7 bounds $N_3(q)$. Collecting these bounds and applying them to (28) gives

$$I_t(P) \ll P^{t-5} + P^{2\lambda+1}(\log P) \sum_{q\leq R} \kappa(q)^{\lambda} \Bigg(P^{-4}q(\log q)^6 + \left(\frac{q^2}{P}+1\right)\prod_{p|q}\left(1+\frac{C}{p}\right)\Bigg).$$

Since $q \leq R = P^{\frac{1}{2}+\delta}$,

$$P^{-4}q(\log q)^6 \ll P^{-3} \ll 1,$$

and

$$\frac{q^2}{P} \leq q^{\frac{4\delta}{1+2\delta}},$$

so we have

$$I_t(P) \ll P^{t-5} + P^{2\lambda+1}(\log P) \sum_{q\leq R} \kappa(q)^{\lambda} q^{\frac{4\delta}{1+2\delta}} \prod_{p|q}\left(1+\frac{C}{p}\right). \tag{29}$$

We now desire a bound on

$$\sum_{q\leq R} \kappa(q)^{\lambda} q^{\frac{4\delta}{1+2\delta}} \prod_{p|q}\left(1+\frac{C}{p}\right).$$

Since κ is multiplicative, it suffices to bound

$$\prod_{p\le R}\left(1+\sum_{j=1}^{\infty}\kappa(p^j)^{\lambda}p^{j\frac{4\delta}{1+2\delta}}\right).$$

We have

$$\sum_{j=1}^{\infty}\kappa(p^j)^{\lambda}p^{j\frac{4\delta}{1+2\delta}} \ll p^{-5/4}+p^{-3/2}+\sum_{j=3}^{\infty}p^{-\frac{2}{3}j}$$
$$\ll p^{-5/4}.$$

Thus

$$\prod_{p\le R}\left(1+\left(1+\frac{C}{p}\right)\sum_{j=1}^{\infty}\kappa(p^j)^{\lambda}p^{j\frac{4\delta}{1+2\delta}}\right) \ll \prod_{p\le R}(1+Cp^{-5/4}) \ll 1,$$

which implies that

$$\sum_{q\le R}\kappa(q)^{\lambda}q^{\frac{4\delta}{1+2\delta}}\prod_{p|q}\left(1+\frac{C}{p}\right)\ll 1. \tag{30}$$

Applying (30) to (29) yields

$$I_t(P)\ll P^{t-5}+P^{2\lambda+1}(\log P),$$

which, upon applying the definition of λ in (18), is

$$I_t(P)\ll P^{t-5}(\log P).$$

$\square$

5 A pointwise minor arc bound sensitive to multiple coefficients

In this section, we will work with a narrower set of minor arcs $\mathfrak{m}(Q)$, where $Q=(\log P)^A$. Henceforth $\mathfrak{m}$ will be assumed to mean $\mathfrak{m}(Q)$ rather than $\mathfrak{m}(R)$ unless otherwise specified. Let $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_k)$ and let

$$F_k(\boldsymbol{\alpha})=\sum_{n\le P}\Lambda(n)e(\alpha_1 n+\alpha_2 n^2+\cdots+\alpha_k n^k).$$

This section consists of the proof of the following theorem and corollary:

Theorem 5.1 *For $D > 0$, where $D = D(A)$ can be made arbitrarily large by increasing A, if $(\alpha_2, \alpha_3) \in \mathfrak{m}(Q)$, then*

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{m}(Q)} F_3(\boldsymbol{\alpha}) \ll P(\log P)^{-D}.$$

Corollary 5.1 *For each i, $1 \le i \le s$,*

$$\sup_{(\alpha_2,\alpha_3) \in \mathfrak{m}(Q)} f_i(\alpha_2, \alpha_3) \ll P(\log P)^{-D}.$$

Proof Take $(0, u_i\alpha_2, v_i\alpha_3)$ as the argument of F_3 in Theorem 5.1 and sum over the dyadic intervals, noting that multiplying α_2 and α_3 by the integer coefficients u_i and v_i does not move them out of $\mathfrak{m}(Q)$, and that there are trivially $\ll P^{1/2} \log P$ prime powers $\le P$ which contribute $\ll P^{1/2}(\log P)^2$ to the sum. □

We begin by citing some known results on Vinogradov's mean value theorem. Let

$$J_{s,k}(P) = \int_{[0,1)^k} |F_k(\boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha}.$$

We cite the bound

$$J_{3,2}(P) \ll P^3 \log P \tag{31}$$

from [5] (cf. [7] chap. 7 exercise 2) and for $s > 6$

$$J_{s,3}(P) \ll P^{2s-6} \tag{32}$$

from equation (7) of [1].

Let $X = (\log P)^B$ for some $B > 0$ to be fixed later. For brevity, we let $h(n) := e(\alpha_1 n + \alpha_2 n^2 + \alpha_3 n^3)$. Then

$$F_3(\boldsymbol{\alpha}) = \sum_{n \le P} \Lambda(n) h(n).$$

Applying Vaughan's identity [8] to this sum yields

$$\sum_{n \le P} \Lambda(n) h(n) = S_1 + S_2 + S_3 + S_4, \tag{33}$$

where

$$S_1 = \sum_{n \le X} \Lambda(n) h(n),$$

$$S_2 = \sum_{n \le P} \left(\sum_{\substack{kl=n \\ k \le X}} \mu(k) \log l \right) h(n),$$

$$S_3 = \sum_{n \le P} \sum_{\substack{kl=n \\ k \le X^2}} \left(\sum_{\substack{m,n \\ mn=k \\ m \le X, n \le X}} \Lambda(m)\mu(n) \right) h(n),$$

$$S_4 = \sum_{n \le P} \left(\sum_{\substack{kl=n \\ k>X, l>X}} a(k)b(l) \right) h(n),$$

with

$$a(k) = \sum_{\substack{l|k \\ l>X}} \Lambda(l),$$

$$b(l) = \begin{cases} \mu(l), & l > X \\ 0, & l \le X. \end{cases}$$

This now enables us to bound each of the sums S_1, S_2, S_3, S_4 individually to obtain the desired bound on $F_3(\boldsymbol{\alpha})$. The bounds on these four sums constitute Lemmas 5.1–5.4.

Lemma 5.1

$$S_1 \ll X. \tag{34}$$

Proof Since $|h(n)| \ll 1$,

$$S_1 = \sum_{n \le X} \Lambda(n)h(n) \ll \sum_{m \le X} \Lambda(n) \ll X,$$

where the last bound is a classical result of Chebyshev. □

Lemma 5.2

$$S_3 \ll P(\log P)^{B-A/12+4}.$$

Proof

$$S_3 = \sum_{n \le P} \sum_{\substack{kl=n \\ k \le X^2}} \left(\sum_{\substack{m_1,m_2 \\ m_1m_2=k \\ m_1 \le X, m_2 \le X}} \Lambda(m_1)\mu(m_2) \right) h(m_2). \tag{35}$$

Let

$$c_3(k) := \sum_{\substack{m_1,m_2 \\ m_1m_2=k \\ m_1 \le X, m_2 \le X}} \Lambda(m_1)\mu(m_2)$$

and note for future reference that

$$|c_3(k)| \leq \sum_{m|k} \Lambda(m) = \log k.$$

Interchanging the order of summation in (35) yields

$$\begin{aligned} S_3 &= \sum_{k \leq X^2} c_3(k) \sum_{l \leq P/k} h(kl) \\ &= \sum_{k \leq X^2} c_3(k) \sum_{l \leq P/k} e(\alpha_1 kl + \alpha_2 k^2 l^2 + \alpha_3 k^3 l^3). \end{aligned} \tag{36}$$

We now use Dirichlet's theorem on Diophantine approximation to obtain integers b_j, q_j for $j \in \{2, 3\}$ such that $(b_j, q_j) = 1$,

$$\begin{aligned} \left| \alpha_j k^j - \frac{b_j}{q_j} \right| &\leq \frac{(\log(P/k))^{A/2}}{q_j (P/k)^j}, \\ q_j &\leq \frac{(P/k)^j}{(\log(P/k))^{A/2}}. \end{aligned} \tag{37}$$

Assume for contradiction that $q_j \leq (\log(P/k))^{A/2}$ for both $j = 2$ and $j = 3$ and rewrite (37) as

$$\left| \alpha_j - \frac{b_j}{k^j q_j} \right| \leq \frac{(\log(P/k))^{A/2}}{q_j P^j}.$$

Let $b'_j = b_j/(k^j, b_j)$, $q'_j = k^j q_j/(k^j, b_j)$. Then

$$\left| \alpha_j - \frac{b_j}{k^j q_j} \right| \leq \frac{(\log(P/k))^{A/2}}{q'_j P^j},$$

$(b'_j, q'_j) = 1$, and $q'_j \leq (\log(P/k))^{A/2}$ for $j \in \{2, 3\}$. Let $q = \operatorname{lcm}(q'_2, q'_3)$ and $a_j = b'_j q/q_j$. Then $(a_2, a_3, q) = 1$, $q \leq (\log(P/k))^A$, and

$$\left| \alpha_j - \frac{a_j}{q} \right| \leq \frac{(\log(P/k))^A}{q P^j}.$$

This implies that $(\alpha_2, \alpha_3) \in \mathfrak{M}(Q)$. However, we have $(\alpha_2, \alpha_3) \in \mathfrak{m}(Q)$, which is the desired contradiction, so we may assume that $q_j > (\log(P/k))^{A/2}$ for at least one $j_0 \in \{2, 3\}$.

Define $\alpha'_j = \alpha_j k^j$ and note that (37) becomes

$$\left| \alpha_j k^j - \frac{b_j}{q_j} \right| \leq \frac{(\log(P/k))^{A/2}}{q_j (P/k)^j}. \tag{38}$$

We now need a bound on

$$H(\boldsymbol{\alpha}', P/k) := \sum_{l \le P/k} e(\alpha_1' l + \alpha_2' l^2 + \alpha_3' l^3).$$

By Theorem 5.2 of [7], using the Diophantine approximation of (38), we have

$$H(\boldsymbol{\alpha}', P/k) \ll (\log P) \left(J_{3,2}(2P/k) \left(\frac{P}{k}\right)^3 \left(\frac{1}{q'_{j_0}} + \frac{k}{P} + \frac{q'_{j_0} k^{j_0}}{P^{j_0}} \right) \right)^{1/6}.$$

Now by (31), we have $J_{3,2}(P) \ll P^3 (\log P)$, so

$$H(\boldsymbol{\alpha}', P/k) \ll \frac{P}{k} (\log P)^2 \prod_{j=1}^{3} \left(\frac{1}{q'_j} + \frac{k}{P} + \frac{q'_j k^j}{P^j} \right)^{1/6}. \tag{39}$$

Now $\frac{k}{P} \ll P^{-1/2}$, and $\frac{q'_j k^j}{P^j} \ll (\log P)^{2jB-A}$, $1/q'_{j_0} \ll (\log P)^{2Bj_0 - A/2}$, so

$$\frac{1}{q'_{j_0}} + \frac{k}{P} + \frac{q'_{j_0} k^{j_0}}{P^{j_0}} \ll (\log P)^{2Bj_0 - A/2}, \tag{40}$$

assuming $2Bj_0 - A/2 < 0$.

Applying the bound of (40) to (39) yields

$$\begin{aligned} H(\boldsymbol{\alpha}', P/k) &\ll \frac{P}{k} (\log P)^2 (\log P)^{(2Bj_0 - A/2)/12} \\ &\ll \frac{P}{k} (\log P)^{B - A/12 + 2}, \end{aligned} \tag{41}$$

since $j_0 \le 3$.

Substituting the bound of (41) into (36), we obtain

$$\begin{aligned} S_3 &\ll \sum_{k \le X^2} (\log k) \frac{P}{k} (\log P)^{B - A/12 + 2} \\ &\ll P (\log P)^{B - A/12 + 4}. \end{aligned}$$

□

Lemma 5.3

$$S_2 \ll P (\log P)^{B - A/12 + 4}.$$

Proof

$$
\begin{aligned}
S_2 &= \sum_{n \le P} \left(\sum_{\substack{kl=n \\ k \le X}} \mu(k) \log l \right) h(n) \\
&= \sum_{k \le X} \mu(k) \sum_{l < P/k} h(kl) \int_1^l \frac{dt}{t} \\
&= \sum_{k \le X} \mu(k) \int_1^{P/k} \sum_{l < P/k} h(kl) \frac{dt}{t} \\
&= \int_1^{P/k} \left(\sum_{k \le X} \mu(k) \sum_{l < P/k} h(kl) \right) \frac{dt}{t}. \qquad (42)
\end{aligned}
$$

Now by (41),

$$
\sum_{l < P/k} h(kl) = H(\boldsymbol{\alpha}', P/k) \ll \frac{P}{k} (\log P)^{B - A/12 + 2}.
$$

Substituting this into (42) yields

$$
\begin{aligned}
S_2 &\ll \int_1^{P/k} \sum_{k \le X} \frac{P}{k} (\log P)^{B - A/12 + 2} \frac{dt}{t} \\
&\ll P (\log P)^{B - A/12 + 2} \left(\sum_{k \le X} \frac{\mu(k)}{k} \right) \int_1^{P/k} \frac{dt}{t} \\
&\ll P (\log P)^{2 + B - A/12} (\log X)(\log P/k) \\
&\ll P (\log P)^{4 + B - A/12}.
\end{aligned}
$$

□

Lemma 5.4

$$
S_4 \ll P (\log P)^{4 - \min\{A, B\}/(4b^2)}
$$

Proof We begin by splitting S_4 into dyadic ranges. Let $\mathcal{M} = \{X2^m : 0 \le k, 2^m \le P/X^2\}$. Then

$$
S_4 = \sum_{M \in \mathcal{M}} S_4(M), \qquad (43)
$$

where

$$
S_4(M) = \sum_{M < k \le 2M} \sum_{l \le P/k} a(k) b(l) h(kl).
$$

Our goal is now to replace the sum over the range $l \le P/k$ with one over the range $l \le P/M$. We begin by considering the integral

$$I(x) := \int_{\mathbb{R}} \frac{\sin(2\pi Rt)}{\pi t} e(-xt)dt,$$

where $R > 0$ is a constant. Computing the integral via the residue theorem gives

$$I(x) = \begin{cases} 1, & |x| < R, \\ 0, & |x| > R. \end{cases}$$

Now for $x \ne R$, $t \ge 1$,

$$\int_{|t|>T} \frac{\sin(2\pi Rt)}{\pi t} e(-xt)dt = \int_{|t|>T} \frac{e\big((R-x)t\big) - e\big(-(R+x)t\big)}{2\pi i t} dt. \tag{44}$$

Integrating the right-hand side of (44) by parts gives

$$\int_{|t|>T} \frac{\sin(2\pi Rt)}{\pi t} e(-xt)dt \ll \frac{1}{T|R-x|} + \frac{1}{T|R+x|} + \frac{1}{T^3} \ll \frac{1}{T\big|R-|x|\big|}.$$

Thus we can rewrite $I(x)$ as an integral over $[-T, T]$ with an acceptable error term:

$$I(x) = \int_{-T}^{T} \frac{\sin(2\pi Rt)}{\pi t} e(-xt)dt + O\left(\frac{1}{T\big|R-|x|\big|}\right).$$

We now take $R = \log(\lfloor P \rfloor + \frac{1}{2})$, $x = \log(kl)$, giving us

$$\begin{aligned} S_4(M) &= \sum_{M<k\le 2M} \sum_{l \le P/M} a(k)b(l)h(kl)I(\log(kl)) \\ &= \int_{-T}^{T} \sum_{M<k\le 2M} \sum_{l\le P/M} \frac{a(k)b(l)}{(kl)^{2\pi i t}} h(kl) \frac{\sin(2\pi Rt)}{\pi t} dt + O\left(\frac{P^2 \log P}{T}\right). \end{aligned}$$

Now

$$\frac{\sin(2\pi Rt)}{\pi t} \ll \frac{1}{\pi t} \ll \frac{1}{|t|}$$

and

$$\frac{\sin(2\pi Rt)}{\pi t} \ll \frac{2\pi Rt}{\pi t} \ll R,$$

so

$$\frac{\sin(2\pi Rt)}{\pi t} \ll \min(R, 1/|t|).$$

Take $T = P^3$, $a(k,t) = a(k)k^{-2\pi it}$, $b(l,t) = b(l)l^{-2\pi it}$, and let

$$S_4(M,t) = \sum_{M<k\le 2M} \sum_{l\le P/k} a(k,t)b(l,t)h(kl). \tag{45}$$

Then

$$\begin{aligned} S_4(M) &\ll \sup_{|t|<T} |S_4(M,t)| \int_{-T}^{T} \frac{\sin(2\pi Rt)}{\pi t} dt \\ &\ll 1 + (\log P) \sup_{|t|<T} |S_4(M,t)|. \end{aligned}$$

We now consider $S_4(M,t)$. Let $b > 6$. By Hölder's inequality

$$S_4(M,t)^{2b} \ll \left(\sum_{M<k\le 2M} |a(k,t)|^{\frac{2b}{2b-1}} \right)^{2b-1} \sum_{M<k\le 2M} \left| \sum_{l\le P/M} b(l,t)h(kl) \right|^{2b}. \tag{46}$$

Now $|a(k,t)| = |a(k)| \le \log k \ll \log M \ll \log P$, so

$$\begin{aligned} S_4(M,t)^{2b} &\ll \left(M(\log P)^{\frac{2b}{2b-1}} \right)^{2b-1} \sum_{M<k\le 2M} \left| \sum_{l\le P/M} b(l,t)h(kl) \right|^{2b} \\ &\ll (\log P)^{2b} M^{2b-1} \sum_{M<k\le 2M} \left| \sum_{l\le P/M} b(l,t)h(kl) \right|^{2b}. \end{aligned} \tag{47}$$

Expanding the $2b$-th power in (47) yields

$$\begin{aligned} &\left| \sum_{l\le P/M} b(l,t)h(kl) \right|^{2b} \\ &\quad = \sum_{\substack{\mathbf{l} \\ l_j \le P/M}} \left(\prod_{i=1}^{b} b(l_i,t) \prod_{i=b+1}^{2b} \overline{b(l_i,t)} \right) \\ &\qquad \times e\left(\alpha_1 k s_1(\mathbf{l}) + \alpha_2 k^2 s_2(\mathbf{l}) + \alpha_3 k^3 s_3(\mathbf{l}) \right), \end{aligned} \tag{48}$$

where

$$s_j(\mathbf{l}) = l_1^j + \cdots + l_b^j - l_{b+1}^j - \cdots - l_{2b}^j.$$

Collecting terms in (48) by values of s_j yields

$$\left|\sum_{l\leq P/M} b(l,t)h(kl)\right|^{2b} = \sum_{\substack{\mathbf{v}\\ |v_j|\leq bP^j}} R_1(\mathbf{v})e(\alpha_1 kv_1+\alpha_2k^2v_2+\alpha_3k^3v_3), \tag{49}$$

where

$$R_1(\mathbf{v}) = \sum_{\substack{\mathbf{l}\\ l_j\leq P/M\\ \mathbf{s}(\mathbf{l})=\mathbf{v}}} \prod_{i=1}^{b} b(l_i,t) \prod_{i=b+1}^{2b} \overline{b(l_i,t)} \ll J_{b,3}(P/M) \ll (P/M)^{2b-6}$$

by (32). Substituting (49) into (47) yields

$$\begin{aligned} S_4(M,t)^{2b} &\ll (\log P)^{2b} M^{2b-1} \\ &\quad\times \sum_{\substack{\mathbf{v}\\ |v_j|\leq bP^jM^{-j}}} R_1(\mathbf{v}) \sum_{M<k\leq 2M} e(\alpha_1 kv_1 \\ &\quad+\alpha_2k^2v_2+\alpha_3k^3v_3) \\ &\ll (\log P)^{2b} M^5 P^{2b-6} \\ &\quad\times \sum_{\substack{\mathbf{v}\\ |v_j|\leq bP^jM^{-j}}} \sum_{M<k\leq 2M} \\ &\quad e(\alpha_1 kv_1+\alpha_2k^2v_2+\alpha_3k^3v_3). \end{aligned} \tag{50}$$

We now repeat the procedure followed from (46) to (50). By Hölder's inequality

$$\begin{aligned} S_4(M,t)|^{4b^2} &\ll \left((\log P)^{2b} M^5 P^{2b-6}\right)^{2b} \left(\sum_{\substack{\mathbf{v}\\ |v_j|\leq bP^jM^{-j}}} 1^{\frac{2b}{2b-1}}\right)^{2b-1} \\ &\quad\times \sum_{\substack{\mathbf{v}\\ |v_j|\leq bP^jM^{-j}}} \left|\sum_{M<k\leq 2M} e(\alpha_1 kv_1+\alpha_2k^2v_2+\alpha_3k^3v_3)\right|^{2b} \qquad (51) \\ &\ll (\log P)^{4b} M^{10b} P^{4b^2-12b} \left(b^3P^6M^{-6}\right)^{2b-1} \\ &\quad\times \sum_{\substack{\mathbf{v}\\ |v_j|\leq bP^jM^{-j}}} \left|\sum_{M<k\leq 2M} e(\alpha_1 kv_1+\alpha_2k^2v_2+\alpha_3k^3v_3)\right|^{2b} \end{aligned}$$

$$\ll (\log P)^{4b^2} M^{6-2b} P^{4b^2-6} \times \sum_{\substack{\mathbf{v} \\ |v_j| \le bP^j M^{-j}}} \left| \sum_{M<k\le 2M} e(\alpha_1 k v_1 + \alpha_2 k^2 v_2 + \alpha_3 k^3 v_3) \right|^{2b}. \tag{52}$$

We expand the $2b$-th power in (52) and collect like terms. Thus

$$\begin{aligned} &\left| \sum_{M<k\le 2M} e(\alpha_1 k v_1 + \alpha_2 k^2 v_2 + \alpha_3 k^3 v_3) \right|^{2b} \\ &= \sum_{\substack{\mathbf{k} \\ M<k_j\le 2M}} e(\alpha_1 s_1(\mathbf{k}) v_1 + \alpha_2 s_2(\mathbf{k}) v_2 + \alpha_3 s_3(\mathbf{k}) v_3) \\ &= \sum_{\substack{\mathbf{u} \\ |u_j| \le b2^j M^j}} R_2(\mathbf{u}) e(\alpha_1 u_1 v_1 + \alpha_2 u_2 v_2 + \alpha_3 u_3 v_3), \end{aligned} \tag{53}$$

where

$$R_2(\mathbf{u}) = \sum_{\substack{\mathbf{k} \\ M<k_j\le 2M \\ \mathbf{s}(\mathbf{k})=\mathbf{u}}} 1 \ll J_{b,3}(2M) \ll M^{2b-6}$$

by (32). Substituting (53) into (52), we obtain

$$S_4(M,t)^{4b^2} \ll (\log P)^{4b^2} P^{4b^2-6} \times \sum_{\substack{\mathbf{u} \\ |u_j| \le b2^j M^j}} \left| \sum_{\substack{\mathbf{v} \\ |v_j| \le bP^j M^{-j}}} e(\alpha_1 u_1 v_1 + \alpha_2 u_2 v_2 + \alpha_3 u_3 v_3) \right|.$$

Summing over each of the v_j gives

$$S_4(M,t)^{4b^2} \ll (\log P)^{4b^2} P^{4b^2-6} \sum_{\substack{\mathbf{u} \\ |u_j| \le b2^j M^j}} \prod_{j=1}^{3} \min\left(\frac{P^j}{M^j}, \frac{1}{\|\alpha_j u_j\|} \right).$$

Applying Lemma 2.2 of [7] yields

$$S_4(M,t)^{4b^2} \ll (\log P)^{4b^2+3} P^{4b^2} \prod_{j=1}^{3} \left(\frac{1}{q_j} + \frac{1}{M^j} + \frac{M^j}{P^j} + \frac{q_j}{P^j} \right). \tag{54}$$

Combining (54) with (43) and (45), we obtain

$$S_4 \ll P(\log P)^4 \prod_{j=1}^{3} \left(\frac{1}{q_j} + \frac{1}{X^j} + \frac{q_j}{P^j} \right)^{1/(4b^2)}.$$

Recalling that $q_j > (\log P)^A$ for some j and $X = (\log P)^B$, this is

$$S_4 \ll P(\log P)^{4-\min(A,B)/(4b^2)} \tag{55}$$

for $b > 6$. □

Proof of Theorem 5.1 Using the Vaughan's identity breakdown of (33) and the estimates for the S_i found in Lemmas 5.1, 5.2, 5.3, and 5.4, we have

$$\begin{aligned} F_3(\boldsymbol{\alpha}) &= S_1 + S_2 + S_3 + S_4 \\ &\ll (\log P)^B + P(\log P)^{B-A/12+4} + P(\log P)^{B-A/12+4} \\ &\quad + P(\log P)^{4-\min(A,B)/(4b^2)}. \end{aligned}$$

So, taking $B > 4b^2 D(D+4)$ and $A > 12(B+D+4)$ for some $D > 0$ yields

$$F_3(\boldsymbol{\alpha}) \ll P(\log P)^{-D}$$

uniformly in $\boldsymbol{\alpha}$. □

6 Major arc approximations

On a typical major arc $\mathfrak{M}(a_2, a_3, q)$, let $\alpha_2 = \frac{a_2}{q} + \theta$, $\alpha_3 = \frac{a_3}{q} + \omega$, with $\theta < \frac{Q}{qP^2}$, $\omega < \frac{Q}{qP^3}$, and $q < Q$. For ease of notation, let $\frac{Q}{qP^2} = \Theta$, $\frac{Q}{qP^3} = \Omega$. Let

$$\begin{aligned} W_i(q, a_2, a_3) &= \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{a_2 u_i r^2 + a_3 v_i r^3}{q} \right), \\ f_i^*(\alpha_2, \alpha_3) &= \frac{1}{\phi(q)} W_i(q, a_2, a_3) \int_0^P e(\theta u_i x^2 + \omega v_i x^3) dx, \\ T_i(x, a_2, a_3) &= \sum_{p \le x} (\log p) e\left(\frac{a_2 u_i p^2 + a_3 v_i p^3}{q} \right), \end{aligned}$$

and for $x > \sqrt{P}$,

$$T_i^\dagger(x, a_2, a_3) = \sum_{\sqrt{P} < p \le x} (\log p) e\left(\frac{a_2 u_i p^2 + a_3 v_i p^3}{q} \right).$$

We begin with preliminary bounds on $T_i(x, a_2, a_3)$ and $T_i^{\dagger}(q, a_2, a_3)$.

Lemma 6.1

$$T_i(x, a_2, a_3) = \frac{x}{\phi(q)} W_i(q, a_2, a_3) + O(x \exp(-C(\log x)^{1/2})).$$

Proof The exponential function $e((a_2u_i p^2 + a_3v_i p^3)/q)$ is only sensitive to the residue class of p modulo q, so

$$\begin{aligned} T_i(x, a_2, a_3) &= \sum_{\substack{r=1 \\ (r,q)=1}}^{q} \sum_{\substack{p \le x \\ p \equiv r \ (\mathrm{mod}\ q)}} (\log p) e\left(\frac{a_2u_ir^2 + a_3v_ir^3}{q}\right) + O(q^{\varepsilon} \log q) \\ &= \sum_{\substack{r=1 \\ (r,q)=1}}^{q} \left(e\left(\frac{a_2u_ir^2 + a_3v_ir^3}{q}\right) \sum_{\substack{p \le x \\ p \equiv r \ (\mathrm{mod}\ q)}} \log p \right) + O(q^{\varepsilon} \log q). \end{aligned}$$

Now by the Siegel–Walfisz theorem we have that

$$\sum_{\substack{p \le x \\ p \equiv r \ (\mathrm{mod}\ q)}} \log p = \frac{x}{\phi(q)} + O(x \exp(-C(\log x)^{1/2})), \tag{56}$$

so

$$\begin{aligned} T_i(x, a_2, a_3) &= \sum_{\substack{r=1 \\ (r,q)=1}}^{q} \left(e\left(\frac{a_2u_ir^2 + a_3v_ir^3}{q}\right) \left(\frac{x}{\phi(q)} + O(x \exp(-C(\log x)^{1/2}))\right)\right) \\ &= \frac{x}{\phi(q)} W_i(q, a_2, a_3) + W_i(q, a_2, a_3) \left(O(x \exp(-C(\log x)^{1/2}))\right) \\ &= \frac{x}{\phi(q)} W_i(q, a_2, a_3) + O(x \exp(-C(\log x)^{1/2})). \end{aligned}$$

□

Corollary 6.1 *For $x > \sqrt{P}$,*

$$T_i^{\dagger}(x, a_2, a_3) = \frac{x}{\phi(q)} W_i(q, a_2, a_3) + O(x \exp(-C(\log x)^{1/2})).$$

Proof

$$\begin{aligned} T_i^{\dagger}(x, a_2, a_3) &= T_i(x, a_2, a_3) - T_i(\sqrt{P}, a_2, a_3) \\ &= \frac{x}{\phi(q)} W_i(q, a_2, a_3) + O(x \exp(-C(\log x)^{1/2})) \end{aligned}$$

$$+O(P^{1/2}\exp(-C(\log P)^{1/2})$$
$$= \frac{x}{\phi(q)} W_i(q, a_2, a_3) + O(x \exp(-C(\log x)^{1/2})).$$

□

Lemma 6.2 *On $\mathfrak{M}(q, a_2, a_3)$,*

$$f_i(\alpha_2, \alpha_3) = f_i^*(\alpha_2, \alpha_3) + O(P \exp(-C(\log P)^{1/2}))$$

for some positive constant C.

Proof First, we isolate the range $(\sqrt{P}, P]$, bounding the remainder immediately.

$$\begin{aligned} &|f_i(\alpha_2, \alpha_3) - f_i^*(\alpha_2, \alpha_3)| \\ &= \Bigg|\sum_{p \le P} (\log p) e(\alpha_2 u_i p^2 + \alpha_3 v_i p^3) \\ &\quad - \frac{1}{\phi(q)} W_i(q, a_2, a_3) \int_0^P e(\theta u_i x^2 + \omega v_i x^3) dx \Bigg| \\ &= \Bigg|\sum_{\sqrt{P} < p \le P} (\log p) e(\alpha_2 u_i p^2 + \alpha_3 v_i p^3) \\ &\quad - \frac{1}{\phi(q)} W_i(q, a_2, a_3) \int_{\sqrt{P}}^P e(\theta u_i x^2 + \omega v_i x^3) dx \Bigg| \\ &\quad + O(P^{1/2} \log P). \end{aligned}$$

Now

$$\begin{aligned} &\Bigg|\sum_{\sqrt{P} < p \le P} (\log p) e(\alpha_2 u_i p^2 + \alpha_3 v_i p^3) \\ &\quad - \frac{1}{\phi(q)} W_i(q, a_2, a_3) \int_{\sqrt{P}}^P e(\theta u_i x^2 + \omega v_i x^3) dx \Bigg| \\ &= \Bigg| W_i(q, a_2, a_3) \sum_{\substack{\sqrt{P} < p \le P \\ p \equiv r \ (\mathrm{mod}\ q)}} (\log p) e(\theta u_i p^2 + \omega v_i p^3) \\ &\quad - \frac{1}{\phi(q)} W_i(q, a_2, a_3) \int_{\sqrt{P}}^P e(\theta u_i x^2 + \omega v_i x^3) dx \Bigg| \qquad (57) \\ &= \sum_{\sqrt{P} < m \le P} \Bigg[(\log m) e\left(\frac{a_2 u_i m^2 + a_3 v_i m^3}{q}\right) \mathbb{1}_{\mathcal{P}} \\ &\quad - \frac{1}{\phi(q)} W_i(q, a_2, a_3) \Bigg] e(\theta u_i m^2 + \omega v_i m^3) + O(|\omega| P^{5/2}), \qquad (58) \end{aligned}$$

where $\mathbb{1}_{\mathcal{P}}$ is the indicator function of the primes.

We now apply Abel summation to (58), with the term in square brackets serving as the coefficient. This yields that

$$\begin{aligned}
&|f_i(\alpha_2,\alpha_3) - f_i^*(\alpha_2,\alpha_3)| \\
&\quad = e(\theta u_i P^2 + \omega v_i P^3)\left(T_i(x,a_2,a_3) - \frac{1}{\phi(q)}\sum_{\sqrt{P}<m\le P} W_i(q,a_2,a_3)\right) \\
&\qquad - \int_{\sqrt{P}}^{P} 2\pi i(2\theta u_i x + 3\omega v_i x^2)\left(T_i(x,a_2,a_3) - \frac{1}{\phi(q)}\sum_{\sqrt{P}<m\le x} W_i(q,a_2,a_3)\right)dx \\
&\qquad + O(|\omega| P^{5/2}).
\end{aligned} \tag{59}$$

Now Corollary 6.1 gives that for $x > \sqrt{P}$,

$$T_i^{\dagger}(x,a_2,a_3) - \frac{1}{\phi(q)}\sum_{\sqrt{P}<m\le x} W_i(q,a_2,a_3) \ll x\exp(-C(\log x)^{1/2}),$$

so

$$\begin{aligned}
&|f_i(\alpha_2,\alpha_3) - f_i^*(\alpha_2,\alpha_3)| \\
&\quad = e(\theta u_i P^2 + \omega v_i P^3)\left(O(\phi(q)P\exp(-C(\log P)^{1/2}))\right) \\
&\qquad -2\pi i\int_0^P (2\theta u_i x + 3\omega v_i x^2)\left(O(\phi(q)x\exp(-C(\log P)^{1/2}))\right)dx \\
&\qquad + O(P^{1/2}\log P) + O(|\omega| P^{5/2}) \\
&\quad \ll (1 + |\theta| P^2 + |\omega| P^3)\phi(q)P\exp(-C(\log P)^{1/2}) \\
&\quad \ll (\log P)^A \frac{\phi(q)}{q} P\exp(-C(\log P)^{1/2}) \\
&\quad \ll P\exp(-C(\log P)^{1/2}).
\end{aligned}$$

□

For clarity of notation, let

$$\begin{aligned}
A(q) &= \sum_{\substack{a_2=1 \\ (a_2,a_3,q)=1}}^{q}\sum_{a_3=1}^{q} \frac{1}{\phi(q)^s}\prod_{i=1}^{s} W_i(q,a_2,a_3), \\
\mathfrak{S}(Q) &= \sum_{q<Q} A(q), \\
J(Q) &= \int_{|\theta|<Q/P^2}\int_{|\omega|<Q/P^3}\prod_{i=1}^{s}\int_0^P e(\theta u_i x^2 + \omega v_i x^3)\,dx\,d\omega\,d\theta.
\end{aligned}$$

We are now able to state the primary lemma of this section:

Lemma 6.3 *For some $E > 0$,*

$$R(P) = \mathfrak{S}(Q)J(Q) + O(P^{s-5}(\log P)^{-E}).$$

Proof We first introduce variant major arcs whose length is independent of q:

$$\mathfrak{B}(q, \mathbf{r}, Q) = \left\{(a_2, a_3) : |\alpha_2 - a_2/q| < \frac{Q}{P^2}, |\alpha_3 - a_3/q| < \frac{Q}{P^3}\right\}$$

for $1 \le Q \le P$, $q < Q$, $1 \le a_2 \le q$, $1 \le a_3 \le q$, and $(a_2, a_3, q) = 1$. Let $\mathfrak{B}$ be the union of all such $\mathfrak{B}(q, \mathbf{r}, Q)$ and note that $\mathfrak{M} \subseteq \mathfrak{B}$ and thus $\mathfrak{B} \setminus \mathfrak{M} \subseteq \mathfrak{m}$.

It follows immediately from Lemma 6.2 that

$$\left|\prod_{i=1}^{s} f_i(\alpha_2, \alpha_3) - \prod_{i=1}^{s} f_i^*(\alpha_2, \alpha_3)\right| \ll P^s \exp(-C(\log P)^{1/2}). \tag{60}$$

Summing (60) over all arcs in $\mathfrak{B}$ gives

$$\begin{aligned}
&\int_{\mathfrak{B}} \left|\prod_{i=1}^{s} f_i(\alpha_2, \alpha_3) - \prod_{i=1}^{s} f_i^*(\alpha_2, \alpha_3)\right| d\alpha_2\, d\alpha_3 \\
&= \sum_{q<Q} \sum_{\substack{a_2=1 \\ (a_2,a_3,q)=1}}^{q} \sum_{a_3=1}^{q} \int_{\mathfrak{B}(a_2,a_3,q)} \left|\prod_{i=1}^{s} f_i(\alpha_2, \alpha_3) - \prod_{i=1}^{s} f_i^*(\alpha_2, \alpha_3)\right| d\alpha_2\, d\alpha_3 \\
&\ll \sum_{q<Q} \sum_{\substack{a_2=1 \\ (a_2,a_3,q)=1}}^{q} \sum_{a_3=1}^{q} \int_{-Q/P^2}^{Q/P^2} \int_{-Q/P^3}^{Q/P^3} P^s \exp(-C(\log P)^{1/2}) d\alpha_2\, d\alpha_3 \\
&\ll Q^3 P^{s-5} \exp(-C(\log P)^{1/2}).
\end{aligned} \tag{61}$$

We now wish to compute

$$\begin{aligned}
&\int_{\mathfrak{B}} \prod_{i=1}^{s} f_i^*(\alpha_2, \alpha_3) d\alpha_2 d\alpha_3 \\
&= \sum_{q<Q} \sum_{\substack{a_2=1 \\ (a_2,a_3,q)=1}}^{q} \sum_{a_3=1}^{q} \prod_{i=1}^{s} \frac{1}{\phi(q)} W_i(q, a_2, a_3) \\
&\times \int_{-Q/P^2}^{Q/P^2} \int_{-Q/P^3}^{Q/P^3} \int_0^P \\
&e(\theta u_i x^2 + \omega v_i x^3) dx\, d\theta\, d\omega
\end{aligned}$$

$$
\begin{aligned}
&+O(P^{s-5}(\log P)^{-E}) \\
&= \mathfrak{S}(Q)J(Q) + O(P^{s-5}(\log P)^{-E}).
\end{aligned} \tag{62}
$$

Combining (61) and (62) yields the bound

$$
\int_{\mathfrak{B}} \prod_{i=1}^{s} f_i(\alpha_2, \alpha_3) = \mathfrak{S}(Q)J(Q) + O(P^{s-5}(\log P)^{-E}). \tag{63}
$$

Combining Theorem 4.1 and Corollary 5.1 yields the minor arc bound

$$
\int_{\mathfrak{m}} \prod_{i=1}^{s} f_i(\alpha_2, \alpha_3) \ll P^{s-5}(\log P)^{-E}, \tag{64}
$$

and moreover, since $\mathcal{A} \setminus \mathfrak{B} \subseteq \mathfrak{m}$, by Corollary 5.1 and Theorem 4.1 we have

$$
\int_{\mathcal{A}\setminus\mathfrak{B}} \prod_{i=1}^{s} f_i(\alpha_2, \alpha_3) \ll P^{s-5}(\log P)^{-E}. \tag{65}
$$

Now by (5), (63), and (65) we have

$$
R(P) = \mathfrak{S}(Q)J(Q) + O(P^{s-5}(\log P)^{-E}). \tag{66}
$$

□

7 Convergence of the singular series

Lemma 7.1 *Let* $(q_1, q_2) = 1$. *Then*

$$
W_i(q_1q_2, a_2, a_3) = W_i(q_2, a_2q_1, a_3q_1^2)W_i(q_1, a_2q_2, a_3q_2^2).
$$

Proof Each residue class r modulo q_1q_2 with $(r, q_1q_2) = 1$ is uniquely represented as $cq_1 + dq_2$ with $1 \leq c \leq q_2$, $(c, q_2) = 1$, $1 \leq d \leq q_1$, and $(d, q_1) = 1$. Also, cq_1, dq_2 run over all residue classes modulo q_2, q_1 with $(cq_1, q_2) = 1$, $(dq_2, q_1) = 1$, respectively. Thus

$$
\begin{aligned}
W_i(q_1q_2, a_2, a_3) &= \sum_{\substack{c=1 \\ (c,q_2)=1}}^{q_2} \sum_{\substack{d=1 \\ (d,q_1)=1}}^{q_1} e\left(\frac{a_2u_i(cq_1 + dq_2)^2 + a_3v_i(cq_1 + dq_2)^3}{q_1q_2}\right) \\
&= \sum_{\substack{c=1 \\ (c,q_2)=1}}^{q_2} \sum_{\substack{d=1 \\ (d,q_1)=1}}^{q_1} e\left(\frac{a_2u_ic^2q_1 + a_3v_ic^3q_1^2}{q_2}\right) e\left(\frac{a_2u_id^2q_2 + a_3v_id^3q_2^2}{q_1}\right) \\
&= W_i(q_2, a_2q_1, a_3q_1^2)W_i(q_1, a_2q_2, a_3q_2^2).
\end{aligned}
$$

□

Lemma 7.2 *A(q) is multiplicative.*

Proof Let $(q_1, q_2) = 1$. Then

$$A(q_1q_2) = \sum_{\substack{a_2=1 \\ (a_2,a_3,q_1q_2)=1}}^{q_1q_2} \sum_{a_3=1}^{q_1q_2} \frac{1}{\phi(q_1q_2)^s} \prod_{i=1}^{s} W_i(q_1q_2, a_2, a_3).$$

Now a_2 and a_3 can be represented by $b_1q_2 + b_2q_1$ and $c_1q_2 + c_2q_1$, respectively, with $1 \leq b_1, c_1 \leq q_1$, $1 \leq b_2, c_2 \leq q_2$. So we can rewrite our sum as

$$A(q_1q_2) = \sum_{\substack{b_1=1 \\ (b_1,c_1,q_1)=1}}^{q_1} \sum_{c_1=1}^{q_1} \sum_{\substack{b_2=1 \\ (b_2,c_2,q_2)=1}}^{q_2} \sum_{c_2=1}^{q_2} \frac{1}{\phi(q_1q_2)^s} \prod_{i=1}^{s} W_i(q_2, b_2q_1^2, c_2q_1^3) W_i(q_1, b_1q_2^2, c_1q_2^3).$$

Now since $(q_1, q_2) = 1$, $(c_2, b_2, q_1) = 1$, and $(b_1, c_1, q_2) = 1$, we have that $b_2q_1^2, c_2q_1^3, b_1q_2^2, c_1q_2^3$ run through complete sets of residue classes modulo q_2, q_2, q_1, q_1, respectively. Thus

$$\begin{aligned} A(q_1q_2) &= \sum_{\substack{b_1=1 \\ (b_1,b_1,q_2)=1}}^{q_2} \sum_{c_1=1}^{q_2} \sum_{\substack{b_2=1 \\ (b_2,c_2,q_1)=1}}^{q_1} \sum_{c_2=1}^{q_1} \frac{1}{\phi(q_1q_2)^s} \prod_{i=1}^{s} W_i(q_2, b_2, c_2) W_i(q_1, b_1, c_1) \\ &= A(q_1)A(q_2). \end{aligned}$$

□

Let $\mathfrak{S}$ be the completed singular series

$$\mathfrak{S} = \sum_{q=1}^{\infty} A(q).$$

Since $A(q)$ is multiplicative,

$$\mathfrak{S} = \prod_{p} \left(1 + \sum_{k=1}^{\infty} A(p^k)\right). \tag{67}$$

Lemma 7.3 $\mathfrak{S}$ *converges absolutely.*

Proof

$$A(p^k) = \sum_{\substack{a_2=1 \\ (a_2,a_3,p^k)=1}}^{p^k} \sum_{a_3=1}^{p^k} \frac{1}{\phi(p^k)^s} \prod_{i=1}^{s} W_i(p^k, a_2, a_3).$$

By Lemma 3.9 and the fact that there are $\ll p^{2k}$ choices for the pair a, b, we have

$$\begin{aligned} A(p^k) &\ll p^{2k}\phi(p^k)^{-s}((p^k)^{\frac{1}{2}+\varepsilon})^s \\ &\ll (p^k)^{2-\frac{1}{2}s+\varepsilon}. \end{aligned}$$

Since $s \geq 7$, we have

$$A(p^k) \ll (p^k)^{-\frac{3}{2}+\varepsilon}. \tag{68}$$

Thus

$$\sum_{k=1}^{\infty} A(p^k) \ll \sum_{k=1}^{\infty} (p^k)^{-\frac{3}{2}+\varepsilon} = \frac{p^{-3/2+\varepsilon}}{1-p^{-3/2+\varepsilon}} \ll p^{-3/2+\varepsilon}.$$

Then

$$\sum_p \sum_{k=1}^{\infty} A(p^k) \ll \sum_p p^{-3/2+\varepsilon}$$

converges, so

$$\mathfrak{S} = \prod_p \left(1 + \sum_{k=1}^{\infty} A(p^k)\right)$$

converges. □

8 Positivity of the singular series

To show that $R(P)$ is eventually positive, we now need to show that $\mathfrak{S}$ is positive.

Lemma 8.1 *There exists $R > 0$ such that*

$$\frac{1}{2} < \prod_{p \geq R} \left(1 + \sum_{k=1}^{\infty} A(p^k)\right).$$

Proof By (68), we have $A(p^k) \ll (p^k)^{-3/2+\varepsilon} \ll (p^k)^{-1/4}$. Choose C, R such that $A(p^k) \leq Cp^{-5/4} < Cp^{-1/4} < \frac{1}{8}$ for all $p \geq R-1$. Then

$$\begin{aligned} \prod_{p \geq R} \left(1 - Cp^{-5/4}\right) &\geq 1 - \sum_{p \geq R} Cp^{-5/4} \\ &\geq 1 - C\int_{R-1}^{\infty} x^{-5/4}dx = 1 - 4C(R-1)^{-1/4} \geq \frac{1}{2}. \end{aligned}$$

□

We now need only show that for $p \leq R$, $1+\sum_{k=1}^{\infty} A(p^k) > 0$. For $1 \leq t \leq s$, define $M_t(q)$ to be the number of solutions $(x_1, \ldots, x_s)$ to the simultaneous congruences

$$\sum_{i=1}^{t} u_i x_i^2 \equiv 0 \pmod{q},$$
$$\sum_{i=1}^{t} v_i x_i^3 \equiv 0 \pmod{q}$$

with $(x_i, q) = 1$ for all i.

Lemma 8.2 *For any positive integer q,*

$$M_s(q) = \frac{\phi(q)^s}{q^2} \sum_{d|q} A(d).$$

Proof

$$\begin{aligned} M_s(q) &= \frac{1}{q^2} \sum_{r_2=1}^{q} \sum_{r_3=1}^{q} \sum_{\substack{x_1=1 \\ (x_1,q)=1}}^{q} \cdots \\ &\quad \times \sum_{\substack{x_s=1 \\ (x_s,q)=1}}^{q} e\left(\frac{r_2(u_1 x_1^2 + \cdots + u_s x_s^2) + r_3(v_1 x_1^3 + \cdots + v_s x_s^3)}{q}\right) \\ &= \frac{1}{q^2} \sum_{r_2=1}^{q} \sum_{r_3=1}^{q} \prod_{i=1}^{s} \sum_{\substack{x_i=1 \\ (x_1,q)=1}}^{q} e\left(\frac{r_2 u_i x_i^2 + r_3 v_i x_i^3}{q}\right). \end{aligned}$$

Let $d = \frac{q}{(r_2,r_3,q)}$, $a_1 = \frac{r_2}{(r_2,r_3,q)}$, and $a_2 = \frac{r_3}{(r_2,r_3,q)}$. Then, rearranging according to the value of d, we have

$$\begin{aligned} M_s(q) &= \frac{1}{q^2} \sum_{d|q} \sum_{\substack{a_2=1 \\ (a_2,a_3,d)=1}}^{d} \sum_{a_3=1}^{d} \prod_{i=1}^{s} \frac{\phi(q)}{\phi(d)} \sum_{\substack{x_i=1 \\ (x_i,d)=1}}^{d} e\left(\frac{a_2 u_i x_i^2 + a_3 v_i x_i^3}{d}\right) \\ &= \frac{\phi(q)^s}{q^2} \sum_{d|q} A(d). \end{aligned}$$

□

Lemma 8.3 *For positive integers t, γ with $t > \gamma$,*

$$M_s(p^t) \geq M_s(p^\gamma) p^{(t-\gamma)(s-2)}.$$

Proof This is [11], Lemma 6.7, with the added observation that (in that paper's notation)

$$\max\{|b_1 - a_1|_p, |b_2 - a_2|_p\} \le p^{-\gamma} \Rightarrow p^{\gamma} | (b_1 - a_1), (b_2 - a_2).$$

So if $a_1, b_1 \not\equiv 0 \pmod{p}$, then $a_2, b_2 \not\equiv 0 \pmod{p}$. Thus the argument lifts solutions over reduced residue classes modulo p^{γ} to solutions over reduced residue classes modulo p^t, so it applies here without modification. □

Theorem 8.1 *If for every prime p there exists a positive integer γ such that $M_s(p^{\gamma}) > 0$, then $\mathfrak{S} > 0$.*

Proof By Lemma 8.2,

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} A(p^k) &= \lim_{t\to\infty} \frac{p^{2t}}{\phi(p^t)^s} M(p^t) \\ &\ge \lim_{t\to\infty} p^{(2-s)t} M(p^t). \end{aligned}$$

By Lemma 8.3, for some positive integer γ,

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} A(p^k) &\ge \lim_{t\to\infty} p^{(2-s)t} M(p^{\gamma}) p^{(t-\gamma)(s-2)} \\ &\ge \lim_{t\to\infty} p^{(-\gamma)(s-2)} > 0. \end{aligned} \tag{69}$$

The lemma now follows from (67), Lemma 8.1, and (69). □

In Sects. 9 and 10 we prove that, under the conditions of Theorem 1.2, for every p there exists a positive integer γ such that $M(p^{\gamma}) > 0$.

9 Solvability of the local problem

We now consider the local system

$$\begin{aligned} u_1 x_1^2 + \cdots + u_s x_s^2 &\equiv 0 \pmod{p}, \\ v_1 x_1^3 + \cdots + v_s x_s^3 &\equiv 0 \pmod{p} \end{aligned} \tag{70}$$

with $x_i \ne 0$ in $\mathbb{Z}/p\mathbb{Z}$.

We will prove the following result:

Theorem 9.1 *The system*

$$\begin{aligned} u_1 x_1^2 + \cdots + u_s x_s^2 &\equiv U \pmod{p}, \\ v_1 x_1^3 + \cdots + v_s x_s^3 &\equiv V \pmod{p} \end{aligned} \tag{71}$$

has a solution $(x_1, \ldots, x_s)$ with all $x_i \ne 0$ modulo every prime p if

1. $\sum_{i=1}^{s} u_i \equiv U \pmod 2$ *and* $\sum_{i=1}^{s} v_i \equiv V \pmod 2$,
2. $\sum_{i=1}^{s} u_i \equiv U \pmod 3$, *and*
3. *for each prime* p *at least 7 of the* u_i, v_i *are not zero modulo* p.

Observe that if the system

$$\begin{aligned} u_1x_1^2 + \cdots + u_tx_t^2 &\equiv U \pmod p, \\ v_1x_1^3 + \cdots + v_tx_t^3 &\equiv V \pmod p \end{aligned} \tag{72}$$

has a solution for all $u_1, \ldots, u_t, v_1, \ldots, v_t \neq 0$, then so does the system

$$\begin{aligned} u_{i_1}x_{i_1}^2 + \cdots + u_{i_t}x_{i_t}^2 &\equiv U \pmod p, \\ v_{j_1}x_{j_1}^3 + \cdots + v_{j_t}x_{j_t}^3 &\equiv V \pmod p \end{aligned} \tag{73}$$

for any $\{i_1, \ldots, i_t\}, \{j_1, \ldots, j_t\} \subset \{1, \ldots, s\}$. Also observe that the conditions of Theorem 9.1 guarantee solvability modulo $p = 2$ and $p = 3$: $p = 2$ is immediate and for $p = 3$, the condition guarantees that the quadratic equation is satisfied and each term $v_ix_i^3$ of the cubic equation can be independently set to 1 or -1, allowing us to set $v_1x_1^3 = V$ if $V \not\equiv 0 \pmod 3$ and partition the remainder of $\{1, \ldots, t\}$ into groups of 2 and 3, which can be zeroed by setting them to $\{1, -1\}$ and $\{1, 1, 1\}$.

Thus we have reduced Theorem 9.1 to this lemma:

Lemma 9.1 *For all* u_i, $v_i \neq 0 \pmod p$, $p \geq 5$, $t \geq 7$, U, V, *there exist* $\{x_1, \ldots, x_s\}$ *with* $x_i \neq 0 \pmod p$ *such that*

$$\begin{aligned} u_1x_1^2 + \cdots + u_tx_t^2 &\equiv U \pmod p, \\ v_1x_1^3 + \cdots + v_tx_t^3 &\equiv V \pmod p. \end{aligned} \tag{74}$$

Lemma 9.2 *Suppose* $p > 3$, *and that* a *and* b *are not both equal to* p. *Then* $|W_i(p, a_2, a_3)| \leq 2\sqrt{p} + 1$.

Proof Corollary 2F of [6] gives

$$\left| \sum_{r=0}^{p-1} e\left(\frac{a_2u_ir^2 + a_3v_ir^3}{p} \right) \right| \leq 2p^{1/2}.$$

Now

$$
\begin{aligned}
|W_i(p, a_2, a_3)| &= \left|\sum_{r=1}^{p-1} e\left(\frac{a_2u_ir^2 + a_3v_ir^3}{p}\right)\right| \\
&\le \left|\sum_{r=0}^{p-1} e\left(\frac{a_2u_ir^2 + a_3v_ir^3}{p}\right)\right| + 1 \le 2\sqrt{p} + 1.
\end{aligned}
$$

□

Lemma 9.3 $M_t(p) \ge \frac{1}{p^2}\big((p-1)^t - (p^2-1)(2\sqrt{p}+1)^t\big)$.

Proof

$$
M_t(p) = \frac{1}{p^2}\sum_{r_2=1}^{p}\sum_{r_3=1}^{p}\prod_{i=1}^{t} W_i(p, r_2, r_3).
$$

We have $W_i(p, p, p) = p - 1$ and for r_2, r_3 not both p, $W_i(p, r_2, r_3) \le 2\sqrt{p} + 1$ by Lemma 9.2. Thus

$$
\begin{aligned}
\left|M_t(p) - \frac{(p-1)^t}{p^2}\right| &\le \frac{1}{p^2}\sum_{r_2=1}^{p}\sum_{\substack{r_3=1\\ \{r_2,r_3\}\ne\{p,p\}}}^{p}\prod_{i=1}^{t}(2\sqrt{p}+1) \\
&\le \frac{1}{p^2}(p^2-1)(2\sqrt{p}+1)^t.
\end{aligned}
$$

So we have

$$
M_t(p) \ge \frac{1}{p^2}\big((p-1)^t - (p^2-1)(2\sqrt{p}+1)^t\big).
$$

□

Taking $t = 7$, we get

$$
M_7(p) \ge \frac{1}{p^2}\big((p-1)^7 - (p^2-1)(2\sqrt{p}+1)^7\big).
$$

This gives that $M_7(p) > 0$ for $p > 40.58$. This means that we now need only check that Lemma 9.1 holds for each prime smaller than 41. This is now a finite number of cases to check and thus can be verified by computer. In the following section, we note several techniques that may be employed to bring the computational difficulty of the task into the realm of feasibility, and in Appendix 1 we provide Sage code for performing the computation.

It is worth noting that $t = 7$ appears to only be required for $p = 7$. It seems highly probable that $t = 5$ will suffice for all other primes; however, reducing t to 5 weakens the bound of Lemma 9.3 to requiring us to check all primes less than 1193, which would require more computation than is feasible, since even after the optimizations of Sect. 10, the algorithm checks $O(p^7)$ distinct forms for solvability to verify Lemma 9.3 for all primes up through p.

10 Computational techniques

First, we note that if every pair U, V modulo p can be represented by the form in t_0 variables, then every pair can be represented by t variables for $t > t_0$. So we will start our search with $t = 3$ and store the forms that represent all pairs (U, V) of residue classes mod p. We then need only search higher values of t for the forms that failed to represent all pairs of residue classes with a smaller t.

(The methods in this paragraph are closely modeled after those of [10].) By independently substituting $c_i x_i$ for each x_i, we can assume each x_i is either 1 or a fixed quadratic nonresidue c modulo p. By rearranging and multiplying by b^{-1} as needed, we can assume that $u_1, \ldots, u_r = 1, u_{r+1}, \ldots, u_t = c$ with $r \geq \lceil t/2 \rceil$. By multiplying the cubic equation by v_1^{-1} and rearranging, we may assume $1 = v_1 \leq v_2 \leq \cdots \leq v_t$. By substituting $-x_i$ for x_i as needed, we can assume $1 \leq v_i \leq (p-1)/2$ for each v_i without affecting the u_i.

As a final optimization, we note that if the system of congruences

$$\begin{aligned} u_1 x_1^2 + \cdots + u_t x_t^2 &\equiv U \pmod{p}, \\ v_1 x_1^3 + \cdots + v_t x_t^3 &\equiv V \pmod{p} \end{aligned} \tag{75}$$

represents $p^2 - 1$ of the possible p^2 pairs of residue classes (U, V) modulo p, then

$$\begin{aligned} u_1 x_1^2 + \cdots + u_{t+1} x_{t+1}^2 &\equiv U \pmod{p}, \\ v_1 x_1^3 + \cdots + v_{t+1} x_{t+1}^3 &\equiv V \pmod{p} \end{aligned} \tag{76}$$

will necessarily represent all p^2 residue classes, since $(u_{t+1} x_{t+1}^2, v_{t+1} x_{t+1}^3)$ must represent at least two distinct pairs of residue classes, so

$$\begin{aligned} u_1 x_1^2 + \cdots + u_t x_t^2 &= U - u_{t+1} x_{t+1}^2 \pmod{p}, \\ v_1 x_1^3 + \cdots + v_t x_t^3 &= V - v_{t+1} x_{t+1}^3 \pmod{p} \end{aligned} \tag{77}$$

will be solvable for some (u_{t+1}, v_{t+1}). This turns out to be quite useful: a substantial number of forms represent exactly $p^2 - 1$ pairs of residue classes modulo p.

Using these techniques to minimize the computation needed, running the Sage code in Appendix 1 verifies that Lemma 9.1 holds for $p < 41$. This allows us to conclude the following unconditional form of Theorem 8.1.

Lemma 10.1 $\mathfrak{S} > 0$.

11 Conclusion

We have that $R(P) = \mathfrak{S}(Q)J(Q) + O(P^{s-5}(\log P)^{-E})$ by Lemma 6.3. Lemma 10.1, in conjunction with Lemma 8.1, shows that $\mathfrak{S}(Q) > 0$ uniformly over all u_i, v_i satisfying the conditions of Theorem 1.1 or Theorem 1.2.

The singular integral $J(Q)$ is the same as the one Wooley obtains in the corresponding problem over the integers, so by Lemma 7.4 of [12], there exists a positive constant C such that

$$J(Q) = CP^{s-5} + O(P^{s-5}Q^{-1/2}).$$

In addition, we have the asymptotic upper bound $\mathfrak{S}(Q) \ll 1$ from Lemma 7.3. So we have

$$R(P) = CP^{s-5} + O(P^{s-5}(\log P)^{-E})$$

for $E > 0$, $C > 0$ uniformly.

Thus $R(P)$ is eventually positive. This can only be true if there is a solution of (1) over the primes, so we can conclude Theorems 1.1 and 1.2.

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Appendix 1: Sage code

Code: (SageMath 8.6)

```
for p in prime_range(5,41):

    # Find a quadratic non-residue modulo p
    for i in range(1,p):
        if i not in quadratic_residues(p):
            c = i
            break

    uv_done = []
    print("p = " + str(p))

    for t in range(3,8):
        u = [0] * t
        v = [0] * t
        for number_of_c in range(floor(t/2) + 1): # Set u
            for u_index in range(t):
                if u_index < t - number_of_c:
                    u[u_index] = 1
                else:
                    u[u_index] = c
            skip_v = False
            for v_counter in range(((p-1)/2)^(t-1)): # Set v
                v[0] = 1
                for v_index in range(1,t):
                    v[v_index] = floor(v_counter % ((p-1)/2)^(v_index) /
```

```
                        ((p-1)/2)^(v_index-1)) + 1
                    if u[v_index] == u[v_index-1] and v[v_index] < v[v_index-1]:
                        skip_v = True
                if skip_v == True:
                    skip_v = False
                else:
                    # If removing the last coefficients yields a smaller form that
                    # has already passed, add this form to that list and continue
                    if (u[:t-1], v[:t-1]) in uv_done:
                        uv_done.append((deepcopy(u),deepcopy(v)))
                    else:
                        L = []
                        done = False
                        for i in range((p-1)^t):
                            if done:
                                break;
                            x = [None] * t
                            for j in range(t): # Set x
                                x[j] = floor(i % (p-1)^(j+1) / (p-1)^j) + 1
                            a = 0
                            b = 0
                            for k in range(t):
                                a = mod(a + u[k]*x[k]^2, p)
                                b = mod(b + v[k]*x[k]^3, p)
                            inL = False
                            for pair in L:
                                if (pair[0] == a and pair[1] == b):
                                    inL = True
                                    break;
                            # If the pair (a, b) has not already been represented
                            # by this form, store that it can be
                            if inL == False:
                                L.append((a,b))
                                if len(L) == p^2:
                                    done = True

                        # Uncomment this line to print information on each form
                        #print("u: " + str(u) + " v: " + str(v) + " " + str(len(L)))

                        # If the form represents all pairs (a, b), add it to the list
                        if done:
                            uv_done.append((deepcopy(u), deepcopy(v)))
                        # If the form represents all pairs (a, b) but one, add it
                        elif len(L) == p^2-1 and t < 7:
                            uv_done.append((deepcopy(u), deepcopy(v)))
                        else:
                            if t == 7:
                                print("u: " + str(u) + " v: " + str(v) + "fails.")
print("Search complete")
```

Output:

```
p = 5
p = 7
p = 11
p = 13
p = 17
p = 19
p = 23
p = 29
p = 31
p = 37
Search complete
```

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