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Sets of values of equivalent almost periodic functions

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Abstract
This paper presents a full generalization of Bohr's equivalence theorem for the case of almost periodic functions, which improves a recent result that was uniquely formulated in the case of existence of an integral basis for the set of exponents of the associated Dirichlet series.

Keywords Almost periodic functions · Exponential sums · Bohr equivalence theorem · Dirichlet series · Bohr-equivalence relation

Mathematics Subject Classification 42A75 · 30D20 · 30B50 · 11K60 · 30Axx

1 Introduction

In the beginnings of the 20th century, the Danish mathematician Bohr gave important steps in the understanding of general Dirichlet series, which consist of those exponential sums that take the form

$$\sum_{n\geq 1} a_n e^{-\lambda_n s},\ a_n \in \mathbb{C},\ s = \sigma + it,$$

where $\{\lambda_n\}$ is a strictly increasing sequence of positive numbers tending to infinity. As a result of his investigations on these functions, he introduced an equivalence relation among them that led to the so-called Bohr's equivalence theorem, which shows that

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equivalent Dirichlet series take the same values in certain vertical lines or strips in the complex plane (e.g. see [1,4,10,14]).

On the other hand, Bohr also developed during the 1920s the theory of almost periodic functions, which opened a way to study a wide class of trigonometric series of the general type and even exponential series (see for example [3,5–8]). The space of almost periodic functions in a vertical strip $U \subset \mathbb{C}$, which will be denoted in this paper as $AP(U, \mathbb{C})$, coincides with the set of the functions that can be approximated uniformly in every reduced strip of U by exponential polynomials $a_1 e^{\lambda_1 s} + a_2 e^{\lambda_2 s} + \ldots + a_n e^{\lambda_n s}$ with complex coefficients a_j and real exponents λ_j (see for example [7, Theorem 3.18]). These approximating finite exponential sums can be found by Bochner–Fejér's summation (see, in this regard, [3, Chap. 1, Sect. 9]).

Concerning this subject we recall that exponential polynomials and general Dirichlet series constitute a particular family of exponential sums or, in other words, expressions of the type

$$P_1(p)e^{\lambda_1 p} + \cdots + P_j(p)e^{\lambda_j p} + \cdots ,$$

where the λ_j's are complex numbers and the $P_j(p)$'s are polynomials in the parameter p. In this respect, we established in [13, Definition 2] (see also [11, Definition 3]) a generalization of Bohr's equivalence relation on the classes $\mathcal{S}_\Lambda$ (which we will refer to as Bohr-equivalence) consisting of exponential sums of the form

$$\sum_{j \geq 1} a_j e^{\lambda_j p},\ a_j \in \mathbb{C},\ \lambda_j \in \Lambda, \tag{1}$$

where $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ is an arbitrary countable set of distinct real numbers (not necessarily unbounded) that are called exponents or frequencies. Based on this equivalence relation, and under the assumption of existence of an integral basis for the set of exponents (whose condition is defined in Sect. 2), we proved in [13, Theorem 18] that two Bohr-equivalent almost periodic functions, whose associated Dirichlet series could be assumed to have the same set of exponents, take the same values on every open vertical strip included in their strip of almost periodicity U, which constituted a first extension of Bohr's equivalence theorem for this case of functions.

In this paper, we will consider a new approach to the question of proving that a result analogous to that of Bohr's equivalence theorem holds in the case of almost periodic functions in $AP(U, \mathbb{C})$, not only for those whose set of exponents has an integral basis. The main ingredient is the equivalence relation introduced in Definition 2, denoted as $\overset{*}{\sim}$ (and which we will refer to as *-equivalence), on the classes $\mathcal{S}_\Lambda$ of exponential sums of type (1) and later adapted to the case of the almost periodic functions in $AP(U, \mathbb{C})$. Based on this equivalence relation, which is less restrictive than Bohr-equivalence, we will show that every equivalence class in $AP(U, \mathbb{C})/\overset{*}{\sim}$ is connected with a certain auxiliary function that generates all the sets of values taken by any almost periodic function in the equivalence class along a given vertical line included in the strip of almost periodicity (see Proposition 4 in this paper). This improves [13, Propositions 12 and 13] whose condition of existence of an integral basis was really necessary (see

[13, Remark 14]), and it leads us to formulate and prove Theorem 1 (and Corollary 2 for the Bohr-equivalence), which is the main result of this paper and constitutes a full generalization of Bohr's equivalence theorem for the case of almost periodic functions. Hence with our new equivalence relation it is possible to overcome the restriction of the integral basis (which is not merely a technical difficulty but it is inherently a limit of Bohr's definition) and to obtain a more general result.

2 Definitions and preliminary results

We first recall the following equivalence relation, inspired by that of [1, p.173] for the case of general Dirichlet series, which was already defined in [11, Definition 1] and [13, Definition 1].

Definition 1 Let Λ be an arbitrary countable subset of distinct real numbers, $\text{span}_{\mathbb{Q}}(\Lambda)$ the $\mathbb{Q}$-vector space generated by Λ, and $\mathcal{F}$ the $\mathbb{C}$-vector space of arbitrary functions $\Lambda \to \mathbb{C}$. We define a relation $\sim$ on $\mathcal{F}$ by $a \sim b$ if there exists a $\mathbb{Q}$-linear map $\psi : \text{span}_{\mathbb{Q}}(\Lambda) \to \mathbb{R}$ such that

$$b(\lambda) = a(\lambda)e^{i\psi(\lambda)} \quad (\lambda \in \Lambda).$$

Concerning the classes S_Λ of exponential sums of type (1), consider the following equivalence relation, which was already introduced in [11, Definition 3′ (mod.)] and [12, Definition 2], and is defined in terms of the equivalence relation above. From now on, we will denote as $\sharp A$ the cardinal of a set A.

Definition 2 Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, consider $A_1(p)$ and $A_2(p)$ two exponential sums in the class $\mathcal{S}_\Lambda$, say $A_1(p) = \sum_{j\geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j\geq 1} b_j e^{\lambda_j p}$. We will say that A_1 is *-equivalent to A_2, and it will be denoted as $A_1 \overset{*}{\sim} A_2$, if for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$, it is satisfied $a_n^* \sim b_n^*$, where $a_n^*, b_n^* : \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \to \mathbb{C}$ are the functions given by $a_n^*(\lambda_j) := a_j$ and $b_n^*(\lambda_j) := b_j$, $j = 1, 2, \ldots, n$ and $\sim$ is in Definition 1.

That is, we will write $A_1 \overset{*}{\sim} A_2$ if for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$, there exists a $\mathbb{Q}$-linear map $\psi_n : \text{span}_{\mathbb{Q}}(\{\lambda_1, \ldots, \lambda_n\}) \to \mathbb{R}$ such that

$$b_j = a_j e^{i\psi_n(\lambda_j)}, \ j = 1, \ldots, n.$$

It is clear that the relation defined in the foregoing definition is an equivalence relation.

Remark 1 It is worth noting that the equivalence relation that was used in [13, Definition 2] is different from the *-equivalence (i.e. that of Definition 2 of this paper). In fact, fixed $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, and given $A_1(p) = \sum_{j\geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j\geq 1} b_j e^{\lambda_j p}$ two exponential sums in the class $\mathcal{S}_\Lambda$, [13, Definition 2] consists of defining $A_1 \sim A_2$ (we will say in this paper that A_1 is Bohr-equivalent to A_2) if there exists a $\mathbb{Q}$-linear map $\psi : \mathrm{span}_{\mathbb{Q}}(\Lambda) \to \mathbb{R}$ such that

$$b_j = a_j e^{i\psi(\lambda_j)},\ j = 1, 2, \ldots$$

As an immediate consequence of this definition, we have that if two exponential sums are Bohr-equivalent then they also are *-equivalent (according to Definition 2).

Now, let $G_\Lambda = \{g_1, g_2, \ldots, g_k, \ldots\}$ be a basis of the $\mathbb{Q}$-vector space generated by a set $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ of exponents (by abuse of notation, we will say that G_Λ is a basis for Λ), which yields that G_Λ is linearly independent over the rational numbers and each λ_j is expressible as a finite linear combination of terms of G_Λ, say

$$\lambda_j = \sum_{k=1}^{i_j} r_{j,k} g_k \quad \text{for some } r_{j,k} \in \mathbb{Q},\ i_j \in \mathbb{N}. \tag{2}$$

We will say that G_Λ is an *integral basis* for Λ when $r_{j,k} \in \mathbb{Z}$ for each j, k, i.e. $\Lambda \subset \mathrm{span}_{\mathbb{Z}}(G_\Lambda)$. Moreover, we will say that G_Λ is the *natural basis* for Λ, and we will denote it as G^*_Λ, when it is constituted by elements in Λ as follows. Firstly, if $\lambda_1 \neq 0$, then $g_1 := \lambda_1 \in G^*_\Lambda$. Secondly, if $\{\lambda_1, \lambda_2\}$ are $\mathbb{Q}$-rationally independent, then $g_2 := \lambda_2 \in G^*_\Lambda$. Otherwise, if $\{\lambda_1, \lambda_3\}$ are $\mathbb{Q}$-rationally independent, then $g_2 := \lambda_3 \in G^*_\Lambda$, and so on. In this way, if $\lambda_j \in G^*_\Lambda$, then $r_{j,m_j} = 1$ and $r_{j,k} = 0$ for $k \neq m_j$, where m_j is such that $g_{m_j} = \lambda_j$. In fact, each element in G^*_Λ is of the form g_{m_j} for j such that λ_j is $\mathbb{Q}$-linear independent of the previous elements in the basis. Furthermore, if $\lambda_j \notin G^*_\Lambda$, then $\lambda_j = \sum_{k=1}^{i_j} r_{j,k} g_k$, where $\{g_1, g_2, \ldots, g_{i_j}\} \subset \{\lambda_1, \lambda_2, \ldots, \lambda_{j-1}\}$.

In terms of a prefixed basis for the set of exponents Λ, we next quote a first characterization of the *-equivalence of two exponential sums in $\mathcal{S}_\Lambda$ (see [11, Proposition 1' (mod.)] or [12, Proposition 1]).

Proposition 1 *Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, consider $A_1(p)$ and $A_2(p)$ two exponential sums in the class $\mathcal{S}_\Lambda$, say $A_1(p) = \sum_{j\geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j\geq 1} b_j e^{\lambda_j p}$. Fixed a basis G_Λ for Λ, for each $j = 1, 2, \ldots$ let $\mathbf{r}_j \in \mathbb{R}^{\sharp G_\Lambda}$ be the vector of rational components verifying (2). Then $A_1 \overset{*}{\sim} A_2$ if and only if for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$, there exists a vector $\mathbf{x}_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,k}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_n> i}$ for $j = 1, 2, \ldots, n$. Furthermore, if G_Λ is an integral basis for Λ, then the following three statements are equivalent:*

(i) *$A_1 \overset{*}{\sim} A_2$;*
(ii) *$A_1 \sim A_2$;*
(iii) *There exists $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{\langle \mathbf{r}_j, \mathbf{x}_0\rangle i}$ for each $j \geq 1$.*

From Proposition 1, it is now clear that Definition 2 and the definition of Bohr-equivalence (see Remark 1) are equivalent in the case that it is feasible to obtain an integral basis for the set of exponents Λ.

In terms of the natural basis for a set of exponents Λ, we next provide a second characterization of the *-equivalence of two exponential sums in $\mathcal{S}_\Lambda$.

Proposition 2 *Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, consider $A_1(p)$ and $A_2(p)$ two exponential sums in the class $\mathcal{S}_\Lambda$, say $A_1(p) = \sum_{j\geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j\geq 1} b_j e^{\lambda_j p}$. Fixed the natural basis $G_\Lambda^* = \{g_1, g_2, \ldots, g_k, \ldots\}$ for Λ, for each $j = 1, 2, \ldots$ let $\mathbf{r}_j \in \mathbb{R}^{\sharp G_\Lambda^*}$ be the vector of rational components verifying (2). Then $A_1 \overset{*}{\sim} A_2$ if and only if there exists $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in [0, 2\pi)^{\sharp G_\Lambda^*}$ such that for each $j = 1, 2, \ldots$ it is satisfied*

$$b_j = a_j e^{\langle \mathbf{r}_j, \mathbf{x}_0 + \mathbf{p}_j \rangle i}$$

for some $\mathbf{p}_j = (2\pi n_{j,1}, 2\pi n_{j,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*}$, with $n_{j,k} \in \mathbb{Z}$.

Proof Suppose that $A_1 \overset{*}{\sim} A_2$. Consider $I = \{1, 2, \ldots, k, \ldots : \lambda_k \in G_\Lambda^*\}$ and $I_n = \{1, 2, \ldots, k, \ldots, n : \lambda_k \in G_\Lambda^*\}$. Let $j \in I$, then $r_{j,m_j} = 1$ and $r_{j,k} = 0$ for $k \neq m_j$, where m_j is such that $g_{m_j} = \lambda_j$. Thus, by Proposition 1, let $\mathbf{x}_j = (x_{j,1}, x_{j,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*}$ be a vector such that

$$b_j = a_j e^{i\langle \mathbf{r}_j, \mathbf{x}_j \rangle} = a_j e^{i \sum_{k=1}^{i_j} r_{j,k} x_{j,k}} = a_j e^{i r_{j,m_j} x_{j,m_j}} = a_j e^{i x_{j,m_j}}. \tag{3}$$

Define $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*} = \mathbb{R}^{\sharp I}$ as $x_{0,m_j} := x_{j,m_j}$ for $j \in I$. Thus, by taking $\mathbf{p}_j = (0, 0, \ldots)$, the result trivially holds for those j's such that $\lambda_j \in G_\Lambda^*$, i.e. for $j \in I$. Now, let j be such that $\lambda_j \notin G_\Lambda^*$, i.e. $j \notin I$. By Proposition 1, let $\mathbf{x}_j = (x_{j,1}, x_{j,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*}$ be a vector such that

$$b_p = a_p e^{i\langle \mathbf{r}_p, \mathbf{x}_j \rangle} = a_p e^{i \sum_{k=1}^{i_j} r_{p,k} x_{j,k}}, \ \ p = 1, 2, \ldots, j.$$

Note that if $p = 1, 2, \ldots, j$ is such that $\lambda_p \in G_\Lambda^*$, then

$$b_p = a_p e^{i r_{p,m_p} x_{j,m_p}} = a_p e^{i x_{j,m_p}},$$

which necessarily implies, by (3), that $x_{j,m_p} = x_{p,m_p} + 2\pi n_{j,p}$ for some $n_{j,p} \in \mathbb{Z}$. Hence

$$\begin{aligned} b_j &= a_j e^{i\langle \mathbf{r}_j, \mathbf{x}_j \rangle} = a_j e^{i \sum_{k=1}^{i_j} r_{j,k} x_{j,k}} = a_j e^{i \sum_{p \in I_{j-1}} r_{j,m_p} x_{j,m_p}} \\ &= a_j e^{i \sum_{p \in I_{j-1}} r_{j,m_p} (x_{p,m_p} + 2\pi n_{j,p})} = a_j e^{i\langle \mathbf{r}_j, \mathbf{x}_0 + \mathbf{p}_j \rangle}, \end{aligned}$$

where $\mathbf{p}_j = (2\pi n_{j,1}, 2\pi n_{j,2}, \ldots, 0, 0, \ldots)$. Moreover, by changing conveniently the vectors $\mathbf{p}_j$, we can take $\mathbf{x}_0 \in [0, 2\pi)^{\sharp G_\Lambda^*}$ without loss of generality.

Conversely, suppose the existence of $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*}$ satisfying $b_j = a_j e^{\langle \mathbf{r}_j, \mathbf{x}_0 + \mathbf{p}_j \rangle i}$ for some $\mathbf{p}_j = (2\pi n_{j,1}, 2\pi n_{j,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*}$, with $n_{j,k} \in \mathbb{Z}$. Let $r_{j,k} = \frac{p_{j,k}}{q_{j,k}}$ with $p_{j,k}$ and $q_{j,k}$ coprime integer numbers, and define

$q_{n,k} := \operatorname{lcm}(q_{1,k}, q_{2,k}, \ldots, q_{n,k})$ for each $k = 1, 2, \ldots$. Thus, for each integer number $n \geq 1$, take $\mathbf{x}_n = \mathbf{x}_0 + \mathbf{m}_n$, where $m_{n,k} = 2\pi p_{1,k} p_{2,k} \cdots p_{n,k} q_{n,k}$, $k = 1, 2, \ldots$. Therefore, it is satisfied $b_j = a_j e^{\langle \mathbf{r}_j, \mathbf{x}_n \rangle i}$ for each $j = 1, 2, \ldots, n$, which yields that $A_1 \overset{*}{\sim} A_2$.

We next study the case where the chosen basis is not the natural one. Fixed a set $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ of exponents, let G_Λ^* be the natural basis for Λ and G_Λ be an arbitrary basis for Λ. For each $j \geq 1$ let $\mathbf{r}_j$ and $\mathbf{s}_j$ be the vectors of rational components so that $\lambda_j = \langle \mathbf{r}_j, \mathbf{g} \rangle$ and $\lambda_j = \langle \mathbf{s}_j, \mathbf{h} \rangle$, with $\mathbf{g}$ and $\mathbf{h}$ the vectors associated with the basis G_Λ^* and G_Λ, respectively. Finally, for each $k \geq 1$, let $\mathbf{t}_k$ be the vector so that $h_k = \langle \mathbf{t}_k, \mathbf{g} \rangle$, i.e.

$$T = \begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,j} & \cdots \\ t_{2,1} & t_{2,2} & \cdots & t_{2,j} & \cdots \\ \vdots & \ldots & \ddots & \vdots & \cdots \\ t_{k,1} & t_{k,2} & \cdots & t_{k,j} & \cdots \\ \vdots & \ldots & \ddots & \vdots & \cdots \end{pmatrix} \tag{4}$$

is the change of basis matrix. Thus, for any $\mathbf{x}_0 \in \mathbb{R}^{\sharp G_\Lambda}$, we have

$$\langle \mathbf{r}_j, \mathbf{x}_0 \rangle = \langle \mathbf{s}_j, \mathbf{x}_1 \rangle,$$

where $\mathbf{x}_1$ is defined as $x_{1,k} = \langle \mathbf{t}_k, \mathbf{x}_0 \rangle$ for each $k \geq 1$. Indeed,

$$\begin{aligned} \langle \mathbf{s}_j, \mathbf{x}_1 \rangle &= \sum_k s_{j,k} x_{1,k} = \sum_k s_{j,k} \langle \mathbf{t}_k, \mathbf{x}_0 \rangle = \sum_k s_{j,k} \sum_m t_{k,m} x_{0,m} \\ &= \sum_m x_{0,m} \sum_k s_{j,k} t_{k,m} = \sum_m r_{j,m} x_{0,m} = \langle \mathbf{r}_j, \mathbf{x}_0 \rangle \end{aligned}$$

because $r_{j,m} = \sum_k s_{j,k} t_{k,m}$. Consequently, for each j and

$$\mathbf{p}_j = (2\pi n_{j,1}, 2\pi n_{j,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda} \text{with } n_{j,k} \in \mathbb{Z}, \tag{5}$$

we have

$$\langle \mathbf{r}_j, \mathbf{x}_0 + \mathbf{p}_j \rangle = \langle \mathbf{r}_j, \mathbf{x}_0 \rangle + \langle \mathbf{r}_j, \mathbf{p}_j \rangle = \langle \mathbf{s}_j, \mathbf{x}_1 \rangle + \langle \mathbf{s}_j, \mathbf{q}_j \rangle = \langle \mathbf{s}_j, \mathbf{x}_1 + \mathbf{q}_j \rangle,$$

where $\mathbf{q}_j$ is defined as $q_{1,k} = \langle \mathbf{t}_k, \mathbf{p}_j \rangle$ for each $k \geq 1$, i.e. $\mathbf{q}_j$ is obtained from $T \cdot \mathbf{p}_j^t$. In this way, we have proved the following result.

Corollary 1 *Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, consider $A_1(p)$ and $A_2(p)$ two exponential sums in the class $\mathcal{S}_\Lambda$, say $A_1(p) = \sum_{j \geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j \geq 1} b_j e^{\lambda_j p}$. Fixed a basis $G_\Lambda = \{g_1, g_2, \ldots, g_k, \ldots\}$ for Λ, for each $j = 1, 2, \ldots$ let $\mathbf{r}_j \in \mathbb{R}^{\sharp G_\Lambda}$ be the vector of rational components verifying (2). Then $A_1 \overset{*}{\sim} A_2$*

if and only if there exists $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in [0, 2\pi)^{\sharp G_\Lambda}$ *such that for each* $j = 1, 2, \ldots$ *it is satisfied*

$$b_j = a_j e^{\langle \mathbf{r}_j, \mathbf{x}_0 + \mathbf{q}_j \rangle i}$$

for some $\mathbf{q}_j \in \mathbb{R}^{\sharp G_\Lambda}$ *that is of the form* $\mathbf{q}_j = \mathbf{p}_j \cdot T^t$, *where* T^t *is the transpose of the change of basis matrix* (4) *and* $\mathbf{p}_j$ *is of the form* (5).

We next construct a generating expression for all exponential polynomials in a class $\mathcal{G} \in \mathcal{S}_\Lambda / \overset{*}{\sim}$. Let $2\pi\mathbb{Z}^m = \{(c_1, c_2, \ldots, c_m) \in \mathbb{R}^m : c_k = 2\pi n_k, \text{with } n_k \in \mathbb{Z},\ k = 1, 2, \ldots, m\}$. From Proposition 2, it is clear that the set of all exponential sums $A(p)$ in an equivalence class $\mathcal{G}$ in $\mathcal{S}_\Lambda / \overset{*}{\sim}$ can be determined by a function $E_\mathcal{G} : [0, 2\pi)^{\sharp G_\Lambda^*} \times \prod_{j \geq 1} 2\pi\mathbb{Z}^{\sharp G_\Lambda^*} \to \mathcal{S}_\Lambda$ of the form

$$E_\mathcal{G}(\mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots) := \sum_{j \geq 1} a_j e^{\langle \mathbf{r}_j, \mathbf{x} + \mathbf{p}_j \rangle i} e^{\lambda_j p}, \mathbf{x} \in [0, 2\pi)^{\sharp G_\Lambda^*},\ \mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G_\Lambda^*}, \quad (6)$$

where $a_1, a_2, \ldots, a_j, \ldots$ are the coefficients of an exponential sum in $\mathcal{G}$ and the $\mathbf{r}_j$'s are the vectors of rational components associated with the natural basis G_Λ^* for Λ. We recommend the reader compare the definition of the function $E_\mathcal{G}$ with that of [13, Expression (2.2)].

In particular, in this paper we are going to use Definition 2 for the case of exponential sums in $\mathcal{S}_\Lambda$ of a complex variable $s = \sigma + it$. Precisely, when the formal series in $\mathcal{S}_\Lambda$ are handled as exponential sums of a complex variable on which we fix a summation procedure, from equivalence class generating expression (6) we can consider an auxiliary function as follows (compare with [13, Definition 3]).

Definition 3 Fixed $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of frequencies or exponents, let $\mathcal{G}$ be an equivalence class in $\mathcal{S}_\Lambda / \overset{*}{\sim}$ and $a_1, a_2, \ldots, a_j, \ldots$ be the coefficients of an exponential sum in $\mathcal{G}$. For each $j = 1, 2, \ldots$ let $\mathbf{r}_j$ be the vector of rational components satisfying $\lambda_j = \langle \mathbf{r}_j, \mathbf{g} \rangle = \sum_{k=1}^{q_j} r_{j,k} g_k$, where $\mathbf{g} := (g_1, \ldots, g_k, \ldots)$ is the vector formed by the elements of the natural basis G_Λ^* for Λ. Suppose that some elements in $\mathcal{G}$, handled as exponential sums of a complex variable $s = \sigma + it$, are summable on at least a certain set $P \subset \mathbb{R}$ by some prefixed summation method. Then we define the auxiliary function $F_\mathcal{G} : P \times [0, 2\pi)^{\sharp G_\Lambda^*} \times \prod_{j \geq 1} 2\pi\mathbb{Z}^{\sharp G_\Lambda^*} \to \mathbb{C}$ associated with $\mathcal{G}$, relative to the basis G_Λ^*, as

$$F_\mathcal{G}(\sigma, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots) := \sum_{j \geq 1} a_j e^{\langle \mathbf{r}_j, \mathbf{x} + \mathbf{p}_j \rangle i} e^{\lambda_j \sigma}, \quad (7)$$

where $\sigma \in P,\ \mathbf{x} \in [0, 2\pi)^{\sharp G_\Lambda^*}$, $\mathbf{p}_k \in 2\pi\mathbb{Z}^{\sharp G_\Lambda^*}$, and the series in (7) is summed by the prefixed summation method, applied at $t = 0$ to the exponential sum obtained from the generating expression (6) with $p = \sigma + it$.

This auxiliary function can be immediately adapted to the case of almost periodic functions $AP(U, \mathbb{C})$ with the Bochner–Fejér summation method. In this case, the set P above is formed by the real projection of the strip of almost periodicity of the corresponding exponential sums (see Definition 5). In this theoretical framework, we will show later the strong link among the sets of values in the complex plane taken by a function in $AP(U, \mathbb{C})$, its Dirichlet series and its associated auxiliary function.

Definition 4 Consider $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a countable set of distinct real numbers. We will say that a function $f : U \subset \mathbb{C} \to \mathbb{C}$ is in the class $\mathcal{D}_\Lambda$ if it is almost periodic (in the set $AP(U, \mathbb{C})$) and its associated Dirichlet series is of the form

$$\sum_{j\geq 1} a_j e^{\lambda_j s},\ a_j \in \mathbb{C},\ \lambda_j \in \Lambda, \tag{8}$$

where U is a strip of the type $\{s \in \mathbb{C} : \alpha < \operatorname{Re} s < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$.

Every almost periodic function in $AP(U, \mathbb{C})$ is determined by its Dirichlet series, which is of type (8). In fact it is convenient to remark that, even in the case that the sequence of the partial sums of its Dirichlet series does not converge uniformly, there exists a sequence of finite exponential sums, the Bochner–Fejér polynomials, of the type $P_k(s) = \sum_{j\geq 1} p_{j,k} a_j e^{\lambda_j s}$ where for each k only a finite number of the factors $p_{j,k}$ differ from zero, which converges uniformly to f in every reduced strip in U and converges formally to the Dirichlet series [3, Polynomial approximation theorem, pgs. 50,148].

Moreover, the equivalence relation of Definition 2 can be immediately adapted to the case of the functions (or classes of functions) which are identifiable by their also called Dirichlet series, in particular to the classes $\mathcal{D}_\Lambda$. More specifically, see [11, Sect. 4, Definition 5′ (mod.)] referred to the Besicovitch space which contains the classes of functions which are associated with Fourier or Dirichlet series and for which the extension of our equivalence relation makes sense.

3 The auxiliary functions

If we take Definition 3 as a reference to be applied to our particular case of almost periodic functions with the Bochner–Fejér summation method, notice that to every function $f \in \mathcal{D}_\Lambda$, with Λ an arbitrary set of exponents, we can associate an auxiliary function F_f of countably many real variables as follows.

Definition 5 Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, let $f(s) \in \mathcal{D}_\Lambda$ be an almost periodic function in the vertical strip $\{s \in \mathbb{C} : \alpha < \operatorname{Re} s < \beta\}$, $-\infty \leq \alpha < \beta \leq \infty$, whose Dirichlet series is given by $\sum_{j\geq 1} a_j e^{\lambda_j s}$. For each $j = 1, 2, \ldots$ let $\mathbf{r}_j$ be the vector of rational components satisfying the equality $\lambda_j = \langle \mathbf{r}_j, \mathbf{g}\rangle = \sum_{k=1}^{q_j} r_{j,k} g_k$, where $\mathbf{g} := (g_1, \ldots, g_k, \ldots)$ is the vector formed by the elements of the natural basis G_Λ^* for Λ. We define the auxiliary function $F_f : (\alpha, \beta) \times [0, 2\pi)^{\sharp G_\Lambda^*} \times \prod_{j\geq 1} 2\pi \mathbb{Z}^{\sharp G_\Lambda^*} \to \mathbb{C}$ associated with f, relative to the basis G_Λ^*, as

$$F_f(\sigma, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots) := \sum_{j\geq 1} a_j e^{\lambda_j \sigma} e^{\langle \mathbf{r}_j, \mathbf{x}+\mathbf{p}_j\rangle i}, \tag{9}$$

where $\sigma \in (\alpha, \beta)$, $\mathbf{x} \in [0, 2\pi)^{\sharp G^*_\Lambda}$, $\mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G^*_\Lambda}$ and the series in (9) is summed by Bochner–Fejér procedure, applied at $t = 0$ to the sum $\sum_{j\geq 1} a_j e^{\langle \mathbf{r}_j, \mathbf{x}+\mathbf{p}_j\rangle i} e^{\lambda_j s}$.

Given $f \in AP(U, \mathbb{C})$, it was proved in [11, Lemma 3] that every function in its equivalence class is also included in $AP(U, \mathbb{C})$. What is more, if $\sum_{j\geq 1} a_j e^{\lambda_j s}$ is the Dirichlet series of $f(s) \in AP(U, \mathbb{C})$, we first note that, for every choice of $\mathbf{x} \in [0, 2\pi)^{\sharp G_\Lambda}$ and $\mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G_\Lambda}$ with $j = 1, 2, \ldots$, the sum $\sum_{j\geq 1} a_j e^{\langle \mathbf{r}_j, \mathbf{x}+\mathbf{p}_j\rangle i} e^{\lambda_j s}$ represents the Dirichlet series of an almost periodic function. We second note that if the Dirichlet series of $f(s) \in AP(U, \mathbb{C})$ converges uniformly on $U = \{s \in \mathbb{C} : \alpha < \operatorname{Re} s < \beta\}$, then $f(s)$ coincides with its Dirichlet series and (9) can be viewed as summation by partial sums or ordinary summation.

In addition to this, we third note that the Dirichlet series $\sum_{j\geq 1} a_j e^{\lambda_j s}$, associated with a certain function $f \in \mathcal{D}_\Lambda$, arises from its auxiliary function F_f by a special choice of its variables, that is $F_f(\sigma, t\mathbf{g}, \mathbf{0}, \mathbf{0}, \ldots) = \sum_{j\geq 1} a_j e^{\lambda_j(\sigma+it)}$.

In this respect, under the assumption that the natural basis for the set of the exponents is also an integral basis, it is clear that the vectors $\mathbf{p}_j$ do not play any role and hence the auxiliary function F_f, associated with f, can be taken as $F_f(\sigma, \mathbf{x}) := \sum_{j\geq 1} a_j e^{\lambda_j \sigma} e^{\langle \mathbf{r}_j, \mathbf{x}\rangle i}$, $\sigma \in (\alpha, \beta)$, $\mathbf{x} \in [0, 2\pi)^{\sharp G^*_\Lambda}$.

In general, by taking into account Corollary 1, Definition 5 can be adapted to the case of an arbitrary basis for the set of exponents. For this purpose, given a basis G_Λ for Λ, let T be the change of basis matrix (4), with respect to the natural basis, and let

$$S_T = \{\mathbf{q} \in \mathbb{R}^{\sharp G_\Lambda} : \mathbf{q} = \mathbf{p} \cdot T^t, \text{ with } \mathbf{p} \text{ of the form (5)}\}.$$

Definition 6 Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, let $f(s) \in \mathcal{D}_\Lambda$ be an almost periodic function in $\{s \in \mathbb{C} : \alpha < \operatorname{Re} s < \beta\}$, $-\infty \leq \alpha < \beta \leq \infty$, whose Dirichlet series is given by $\sum_{j\geq 1} a_j e^{\lambda_j s}$. For each $j \geq 1$ let $\mathbf{s}_j$ be the vector of rational components satisfying the equality $\lambda_j = \langle \mathbf{s}_j, \mathbf{g}\rangle = \sum_{k=1}^{q_j} s_{j,k} g_k$, where $\mathbf{g} := (g_1, \ldots, g_k, \ldots)$ is the vector of the elements of an arbitrary basis G_Λ for Λ. Then we define the auxiliary function $F_f^{G_\Lambda} : (\alpha, \beta) \times [0, 2\pi)^{\sharp G_\Lambda} \times \prod_{j\geq 1} S_T \to \mathbb{C}$ associated with f, relative to the basis G_Λ, as

$$F_f^{G_\Lambda}(\sigma, \mathbf{x}, \mathbf{q}_1, \mathbf{q}_2, \ldots) := \sum_{j\geq 1} a_j e^{\lambda_j \sigma} e^{\langle \mathbf{s}_j, \mathbf{x}+\mathbf{q}_j\rangle i}, \tag{10}$$

where $\sigma \in (\alpha, \beta)$, $\mathbf{x} \in [0, 2\pi)^{\sharp G_\Lambda}$, $\mathbf{q}_j \in S_T$, and the series in (10) is summed by Bochner–Fejér procedure, applied at $t = 0$ to the sum $\sum_{j\geq 1} a_j e^{\langle \mathbf{r}_j, \mathbf{x}\rangle i} e^{\lambda_j s}$.

If we take the natural basis, it is obvious that the auxiliary functions F_f and $F_f^{G^*_\Lambda}$ of the respective Definitions 5 and 6 coincide.

We next show a characterization of the property of *-equivalence of functions in the classes $\mathcal{D}_\Lambda$ in terms of the auxiliary function relative to the natural basis.

Proposition 3 *Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, let f_1 and f_2 be two almost periodic functions in the class $\mathcal{D}_\Lambda$. Let $\mathbf{g} := (g_1, g_2, \ldots, g_k, \ldots)$ be the* *vector of the elements of the natural basis G_Λ^* for Λ. Then f_1 is *-equivalent to f_2 if and only if there exist some $\mathbf{y}\in\mathbb{R}^{\sharp G_\Lambda^*}$ and $\mathbf{p}_j\in 2\pi\mathbb{Z}^{\sharp G_\Lambda^*}$, $j=1,2,\ldots$, such that*

$$f_2(\sigma+it)=F_{f_1}(\sigma,\mathbf{y}+t\mathbf{g},\mathbf{p}_1,\mathbf{p}_2,\ldots)\quad \textit{for all}\quad \sigma+it\in U.$$

Proof Let $\sum_{j\geq 1}a_je^{\lambda_js}$ and $\sum_{j\geq 1}b_je^{\lambda_js}$ be the Dirichlet series associated with f_1 and f_2, respectively. Let U be an open vertical strip such that $f_2\in AP(U,\mathbb{C})$. If $f_1\overset{*}{\sim}f_2$, then Proposition 2 assures the existence of $\mathbf{x}_0\in[0,2\pi)^{\sharp G_\Lambda^*}$ such that for each $j=1,2,\ldots$ it is satisfied $b_j=a_je^{\langle\mathbf{r}_j,\mathbf{x}_0+\mathbf{p}'_j\rangle i}$ for some $\mathbf{p}'_j\in 2\pi\mathbb{Z}^{\sharp G_\Lambda^*}$. Now, let $P_k(s)=\sum_{j\geq 1}p_{j,k}b_je^{\lambda_js}$, $k=1,2,\ldots$, be the Bochner–Fejér polynomials associated with f_2 (recall that they converge uniformly to f_2 in every reduced strip in U, which yields $p_{j,k}\to 1$ as k goes to ∞). Thus, given $s=\sigma+it\in U$, we have

$$\begin{aligned}
f_2(\sigma+it)&=\lim_{k\to\infty}P_k(\sigma+it)=\lim_{k\to\infty}\sum_{j\geq 1}p_{j,k}b_je^{\lambda_j(\sigma+it)}\\
&=\lim_{k\to\infty}\sum_{j\geq 1}p_{j,k}a_je^{i\langle\mathbf{r}_j,\mathbf{x}_0+\mathbf{p}'_j\rangle}e^{\lambda_j\sigma}e^{i\lambda_jt}\\
&=\lim_{k\to\infty}\sum_{j\geq 1}p_{j,k}a_je^{\lambda_j\sigma}e^{i\langle\mathbf{r}_j,\mathbf{x}_0+\mathbf{p}'_j\rangle}e^{it\langle\mathbf{r}_j,\mathbf{g}\rangle}\\
&=\lim_{k\to\infty}\sum_{j\geq 1}p_{j,k}a_je^{\lambda_j\sigma}e^{i\langle\mathbf{r}_j,\mathbf{x}_0+\mathbf{p}'_j+t\mathbf{g}\rangle}\\
&=F_{f_1}(\sigma,\mathbf{y}_0+t\mathbf{g},\mathbf{p}_1,\mathbf{p}_2,\ldots),
\end{aligned}$$

where $\mathbf{y}_0\in\mathbb{R}^{\sharp G_\Lambda^*}$ and $\mathbf{p}_j\in 2\pi\mathbb{Z}^{\sharp G_\Lambda^*}$ are chosen so that $\mathbf{x}_0+t\mathbf{g}+\mathbf{p}'_j=\mathbf{y}_0+t\mathbf{g}+\mathbf{p}_j$, with $\mathbf{y}_0+t\mathbf{g}\in[0,2\pi)^{\sharp G_\Lambda^*}$.

Conversely, suppose the existence of $\mathbf{y}_0\in\mathbb{R}^{\sharp G_\Lambda}$ and $\mathbf{p}_j\in 2\pi\mathbb{Z}^{\sharp G_\Lambda^*}$, for $j=1,2,\ldots$, such that $f_2(\sigma+it)=F_{f_1}(\sigma,\mathbf{y}_0+t\mathbf{g},\mathbf{p}_1,\mathbf{p}_2,\ldots)$ for any $\sigma+it\in U$. Hence

$$\begin{aligned}
\lim_{k\to\infty}\sum_{j\geq 1}p_{j,k}b_je^{\lambda_j(\sigma+it)}&=F_{f_1}(\sigma,\mathbf{y}_0+t\mathbf{g},\mathbf{p}_1,\mathbf{p}_2,\ldots)\\
&=\sum_{j\geq 1}a_je^{i\langle\mathbf{r}_j,\mathbf{y}_0+\mathbf{p}_j\rangle}e^{\lambda_j(\sigma+it)}\ \forall\sigma+it\in U.
\end{aligned}$$

Now, by the uniqueness of the coefficients of an exponential sum in $\mathcal{D}_\Lambda$, it is clear that $b_j=a_je^{<\mathbf{r}_j,\mathbf{y}_0+\mathbf{p}_j>i}$ for each $j\geq 1$, which shows that $f_1\overset{*}{\sim}f_2$.

We next define the following set which will be widely used from now on.

Definition 7 Given $\Lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_j,\ldots\}$ a set of exponents, let $f(s)\in\mathcal{D}_\Lambda$ be an almost periodic function in an open vertical strip U, and $\sigma_0=\operatorname{Re}s_0$ with $s_0\in U$. We define $\operatorname{Img}\left(F_f^{G_\Lambda}(\sigma_0,\mathbf{x},\mathbf{q}_1,\mathbf{q}_2,\ldots)\right)$ to be the set of values in the complex plane taken on by the auxiliary function $F_f^{G_\Lambda}(\sigma, \mathbf{x}, \mathbf{q}_1, \mathbf{q}_2, \ldots)$, relative to a prefixed basis G_Λ, when $\sigma = \sigma_0$; that is $\operatorname{Img}\left(F_f^{G_\Lambda}(\sigma_0, \mathbf{x}, \mathbf{q}_1, \mathbf{q}_2, \ldots)\right) = \{s \in \mathbb{C} : \exists \mathbf{x} \in [0, 2\pi)^{\sharp G_\Lambda^*}$ and $\mathbf{q}_j \in S_T$ such that $s = F_f^{G_\Lambda}(\sigma_0, \mathbf{x}, \mathbf{q}_1, \mathbf{q}_2, \ldots)\}$.

We next prove that the sets of values taken on by the auxiliary function $F_f^{G_\Lambda}(\sigma, \mathbf{x}, \mathbf{q}_1, \mathbf{q}_2, \ldots)$ are independent of the basis G_Λ. The proof is similar to that of Corollary 1.

Lemma 1 *Given Λ a set of exponents and G_Λ an arbitrary basis for Λ, let $f(s) \in \mathcal{D}_\Lambda$ be an almost periodic function in an open vertical strip U, and $\sigma_0 = \operatorname{Re} s_0$ with $s_0 \in U$. Then*

$$\operatorname{Img}\left(F_f^{G_\Lambda}(\sigma_0, \mathbf{x}, \mathbf{q}_1, \mathbf{q}_2, \ldots)\right) = \operatorname{Img}\left(F_f^{G_\Lambda^*}(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right).$$

Proof Let $\sum_{j\geq 1} a_j e^{\lambda_j s}$ be the Dirichlet series associated with $f(s) \in \mathcal{D}_\Lambda$, and G_Λ^* and G_Λ be the natural and an arbitrary basis for Λ, respectively. For each $j \geq 1$ let $\mathbf{r}_j$ and $\mathbf{s}_j$ be the vector of integer components so that $\lambda_j = \langle \mathbf{r}_j, \mathbf{g}\rangle$ and $\lambda_j = \langle \mathbf{s}_j, \mathbf{h}\rangle$, with $\mathbf{g}$ and $\mathbf{h}$ the vectors associated with the basis G_Λ^* and G_Λ, respectively. Finally, for each integer $k \geq 1$, let $\mathbf{t}_k$ be the vector given by $h_k = \langle \mathbf{t}_k, \mathbf{g}\rangle$. Take $w_1 \in \operatorname{Img}\left(F_f^{G_\Lambda^*}(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right)$, then there exists $\mathbf{x}_1 \in [0, 2\pi)^{\sharp G_\Lambda}$ and $\mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G_\Lambda^*}$, $j = 1, 2, \ldots,$ such that $w_1 = F_f^{G_\Lambda^*}(\sigma_0, \mathbf{x}_1, \mathbf{p}_1, \mathbf{p}_2, \ldots)$. Hence

$$\begin{aligned} w_1 = F_f^{G_\Lambda^*}(\sigma_0, \mathbf{x}_1, \mathbf{p}_1, \mathbf{p}_2, \ldots) &= \sum_{j\geq 1} a_j e^{\lambda_j \sigma_0} e^{\langle \mathbf{r}_j, \mathbf{x}_1 + \mathbf{p}_j\rangle i} \\ &= \sum_{j\geq 1} a_j e^{\lambda_j \sigma_0} e^{\langle \mathbf{s}_j, \mathbf{x}_2 + \mathbf{q}_j\rangle i}, \end{aligned}$$

where $\mathbf{q}_j$ is defined as $q_{1,k} =< \mathbf{t}_k, \mathbf{p}_j >$ for each $k \geq 1$, and $\mathbf{x}_2$ is defined as $x_{2,k} =< \mathbf{t}_k, \mathbf{x}_1 >$ for each $k \geq 1$. Therefore, $w_1 = F_f^{G_\Lambda}(\sigma_0, \mathbf{x}_2, \mathbf{q}_1, \mathbf{q}_2, \ldots)$ and $w_1 \in \operatorname{Img}\left(F_f^{G_\Lambda}(\sigma_0, \mathbf{x}, \mathbf{q}_1, \mathbf{q}_2, \ldots)\right)$, which gives

$$\operatorname{Img}\left(F_f^{G_\Lambda^*}(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right) \subseteq \operatorname{Img}\left(F_f^{G_\Lambda}(\sigma_0, \mathbf{x}, \mathbf{q}_1, \mathbf{q}_2, \ldots)\right).$$

An analogous argument shows that the set $\operatorname{Img}\left(F_f^{G_\Lambda}(\sigma_0, \mathbf{x}, \mathbf{q}_1, \mathbf{q}_2, \ldots)\right)$ is included in the set $\operatorname{Img}\left(F_f^{G_\Lambda^*}(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right)$, which proves the result.

Consequently, from now on we will use the notation $\operatorname{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right)$ for the set of values taken on by the auxiliary function associated with a function $f(s) \in \mathcal{D}_\Lambda$. In this respect, without loss of generality, we can use the natural basis for the set of exponents Λ. Moreover, under the assumption of existence of an integral basis, we will use the notation $\operatorname{Img}\left(F_f(\sigma_0, \mathbf{x})\right)$ for the set above. In fact, in this case

$\text{Img}\left(F_f(\sigma_0, \mathbf{x})\right)$ is the same as that of [13, Definition 5] and all the results of [13] concerning integral basis are also valid for our case (see also [12, Remark 2]).

4 Main results

Given a function $f(s)$, take the notation

$$\text{Img}\,(f(\sigma_0 + it)) = \{w \in \mathbb{C} : \exists t \in \mathbb{R} \text{ such that } s = f(\sigma_0 + it)\}.$$

We next show the first important result in this paper concerning the connection between our equivalence relation and the set of values in the complex plane taken on by the auxiliary function (compare with [13, Lemma 9, Propositions 12 and 13]).

Proposition 4 *Given Λ a set of exponents, let $f(s) \in \mathcal{D}_\Lambda$ be an almost periodic function in an open vertical strip U, and $\sigma_0 = \text{Re}\, s_0$ with $s_0 \in U$.*

(i) *If $f_1 \overset{*}{\sim} f$, then $\text{Img}\,(f_1(\sigma_0 + it)) \subset \overline{\text{Img}\,(f(\sigma_0 + it))}$ and*

$$\overline{\text{Img}\,(f(\sigma_0 + it))} = \overline{\text{Img}\,(f_1(\sigma_0 + it))}.$$

(ii) $\text{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right) = \bigcup_{f_k \overset{*}{\sim} f} \text{Img}\,(f_k(\sigma_0 + it))$.

(iii) $\text{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right)$ *is a closed set.*

(iv) $\text{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right) = \overline{\text{Img}\,(f_1(\sigma_0 + it))}$ *for every $f_1 \overset{*}{\sim} f$.*

Proof (i) Note that [11, Theorem 4] shows that the functions in the same equivalence class are obtained as limit points of $\mathcal{T}_f = \{f_\tau(s) := f(s + i\tau) : \tau \in \mathbb{R}\}$, that is, every function $f_1 \overset{*}{\sim} f$ is the limit (in the sense of the uniform convergence on every reduced strip of U) of a sequence $\{f_{\tau_n}(s)\}$ with $f_{\tau_n}(s) := f(s + i\tau_n)$. Take $w_1 \in \text{Img}\,(f_1(\sigma_0 + it))$, then there exists $t_1 \in \mathbb{R}$ such that $w_1 = f_1(\sigma_0 + it_1)$. Now, given $\varepsilon > 0$ there exists $\tau > 0$ such that $|f_1(\sigma_0 + it_1) - f_\tau(\sigma_0 + it_1)| < \varepsilon$, which means that

$$|w_1 - f(\sigma_0 + i(t_1 + \tau))| < \varepsilon.$$

Now it is immediate that $w_1 \in \overline{\text{Img}\,(f(\sigma_0 + it))}$ and consequently

$$\text{Img}\,(f_1(\sigma_0 + it)) \subset \overline{\text{Img}\,(f(\sigma_0 + it))}.$$

Analogously, by symmetry we have $\text{Img}\,(f(\sigma_0 + it)) \subset \overline{\text{Img}\,(f_1(\sigma_0 + it))}$, which yields that

$$\overline{\text{Img}\,(f(\sigma_0 + it))} = \overline{\text{Img}\,(f_1(\sigma_0 + it))}.$$

(ii) Take $w_0 \in \bigcup_{f_k \overset{*}{\sim} f} \mathrm{Img}\,(f_k(\sigma_0 + it))$, then $w_0 \in \mathrm{Img}\,(f_k(\sigma_0 + it))$ for some $f_k \overset{*}{\sim} f$, which means that there exists $t_0 \in \mathbb{R}$ such that

$$w_0 = f_k(\sigma_0 + it_0).$$

Note that Proposition 3 assures the existence of a vector $\mathbf{y}_0 \in \mathbb{R}^{\sharp G^*_\Lambda}$ and $\mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G^*_\Lambda}$, $j = 1, 2, \ldots$, such that $w_0 = F_f(\sigma, \mathbf{y}_0 + t_0\mathbf{g}, \mathbf{p}_1, \mathbf{p}_2, \ldots)$. Hence $w_0 = F_f(\sigma_0, \mathbf{x}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots)$, with $\mathbf{x}_0 = \mathbf{y}_0 + t_0\mathbf{g} \in [0, 2\pi)^{\sharp G^*_\Lambda}$, which means that $w_0 \in \mathrm{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right)$. Conversely, if $w_0 \in \mathrm{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right)$, then $w_0 = F_f(\sigma_0, \mathbf{y}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots)$ for some $\mathbf{y}_0 \in [0, 2\pi)^{\sharp G_\Lambda}$ and $\mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G^*_\Lambda}$. Take $t_0 \in \mathbb{R}$. Since $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{g}$, with $\mathbf{x}_0 := \mathbf{y}_0 - t_0\mathbf{g}$, then

$$\begin{aligned} w_0 &= F_f(\sigma_0, \mathbf{x}_0 + t_0\mathbf{g}, \mathbf{p}_1, \mathbf{p}_2, \ldots) = \sum_{j\geq 1} a_j e^{\lambda_j \sigma_0} e^{\langle \mathbf{r}_j, \mathbf{x}_0 + t_0\mathbf{g} + \mathbf{p}_j\rangle i} \\ &= \sum_{j\geq 1} a_j e^{\lambda_j(\sigma_0 + it_0)} e^{\langle \mathbf{r}_j, \mathbf{x}_0 + \mathbf{p}_j\rangle i}. \end{aligned}$$

Hence $\sum_{j\geq 1} a_j e^{\langle \mathbf{r}_j, \mathbf{x}_0 + \mathbf{p}_j\rangle i} e^{\lambda_j s}$ is the Dirichlet series associated with an almost periodic function $h(s) \in AP(U, \mathbb{C})$ such that $h \overset{*}{\sim} f$ (see [11, Lemma 3]) and hence we have that $w_0 = h(\sigma_0 + it_0)$ (see also [13, Remark 11]), which shows that $w_0 \in \bigcup_{f_k \overset{*}{\sim} f} \mathrm{Img}\,(f_k(\sigma_0 + it))$.

(iii) Let $w_1, w_2, \ldots, w_j, \ldots$ be a sequence of points in $\mathrm{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right)$ tending to w_0. We next prove that $w_0 \in \mathrm{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right)$. Indeed, for each w_j, we deduce from (ii) the existence of $f_j \overset{*}{\sim} f$ such that $w_j \in \mathrm{Img}\left(f_j(\sigma_0 + it)\right)$. Now, since that $\{f_j(\sigma_0 + it)\}$ is a sequence in the same equivalence class, [11, Proposition 3] assures the existence of a subsequence $\{f_{j_k}\}$ which converges to a certain function $h \overset{*}{\sim} f$. Consequently, $\{w_{j_k}\}$ tends to $w_0 \in \mathrm{Img}\,(h(\sigma_0 + it))$. Finally, again by (ii), we conclude that

$$w_0 \in \mathrm{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right).$$

(iv) Let $\mathbf{g}$ be the vector associated with the natural basis G^*_Λ. Since the Fourier series of $f_{\sigma_0}(t) := f(\sigma_0 + it)$ can be obtained as $F_f(\sigma_0, t\mathbf{g}, \mathbf{0}, \mathbf{0}, \ldots)$, with $t \in \mathbb{R}$, it is clear that $\mathrm{Img}\,(f(\sigma_0 + it)) \subset \mathrm{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right)$. On the other hand, we deduce from (i) and (ii) that

$$\begin{aligned} &\mathrm{Img}\,(f(\sigma_0 + it)) \subset \mathrm{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right) \\ &= \bigcup_{f_k \overset{*}{\sim} f} \mathrm{Img}\,(f_k(\sigma_0 + it)) \subset \overline{\mathrm{Img}\,(f(\sigma_0 + it))}. \end{aligned}$$

Finally, by taking the closure and property (iii), we conclude that

$$\mathrm{Img}\left(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\right) = \overline{\mathrm{Img}\,(f(\sigma_0 + it))}.$$

Now, the result follows from property (i).

Remark 2 Given Λ a set of exponents, let $f(s) \in \mathcal{D}_\Lambda$ be an almost periodic function in an open vertical strip U, and $\sigma_0 = \operatorname{Re} s_0$ with $s_0 \in U$. As a consequence of Proposition 4, one gets that

$$\overline{\operatorname{Img}(f(\sigma_0 + it))} = \bigcup_{f_k \overset{*}{\sim} f} \operatorname{Img}(f_k(\sigma_0 + it)).$$

If we changed *-equivalence (that of Definition 2) by Bohr-equivalence (that of [13, Definition 2]) and Λ did not have an integral basis, this result would be false as [13, Remark 14] shows.

At this point we will demonstrate a result like Bohr's equivalence theorem [1, Sect. 8.11]. Given Λ an arbitrary set of exponents, let $f_1, f_2 \in \mathcal{D}_\Lambda$ be two *-equivalent almost periodic functions. We next show that, in every open half-plane or open vertical strip included in their region of almost periodicity, the functions f_1 and f_2 take the same set of values. In this sense, this result improves that of [13, Theorem 1] which was proved uniquely for almost periodic functions associated with sets of exponents which have an integral basis.

Theorem 1 *Fixed Λ a set of exponents, let $f_1, f_2 \in \mathcal{D}_\Lambda$ be two *-equivalent almost periodic functions in a strip $\{\sigma + it \in \mathbb{C} : \alpha < \sigma < \beta\}$, and consider E an open set of real numbers included in (α, β). Then*

$$\bigcup_{\sigma \in E} \operatorname{Img}(f_1(\sigma + it)) = \bigcup_{\sigma \in E} \operatorname{Img}(f_2(\sigma + it)).$$

That is, the functions f_1 and f_2 take the same set of values on the region $\{s = \sigma + it \in \mathbb{C} : \sigma \in E\}$.

Proof Without loss of generality, suppose that f_1 and f_2 are not constant functions (otherwise it is trivial). Take $w_0 \in \bigcup_{\sigma \in E} \operatorname{Img}(f_1(\sigma + it))$, then $w_0 \in \operatorname{Img}(f_1(\sigma_0 + it))$ for some $\sigma_0 \in E$ and hence $w_0 = f_1(\sigma_0 + it_0)$ for some $t_0 \in \mathbb{R}$. Furthermore, by Proposition 4, we get $w_0 \in \overline{\operatorname{Img}(f_1(\sigma_0 + it))} = \overline{\operatorname{Img}(f_2(\sigma_0 + it))}$, which yields the existence of a sequence $\{t_n\}$ of real numbers such that

$$w_0 = \lim_{n \to \infty} f_2(\sigma_0 + it_n).$$

Take $h_n(s) := f_2(s + it_n)$, $n \in \mathbb{N}$. By [11, Proposition 4], there exists a subsequence $\{h_{n_k}\}_k \subset \{h_n\}_n$ which converges uniformly on compact subsets to a function $h(s)$, with $h \overset{*}{\sim} f_2$. Observe that

$$\lim_{k \to \infty} h_{n_k}(\sigma_0) = h(\sigma_0) = w_0.$$

Therefore, by Hurwitz's theorem [2, Sect. 5.1.3], there is a positive integer k_0 such that for $k > k_0$ the functions $h^*_{n_k}(s) := h_{n_k}(s) - w_0$ have at least one zero in $D(\sigma_0, \varepsilon)$

for any $\varepsilon > 0$ sufficiently small. This means that for $k > k_0$ the functions $h_{n_k}(s) = f_2(s + it_{n_k})$, and hence the function $f_2(s)$, take the value w_0 on the region $\{s = \sigma + it : \sigma_0 - \varepsilon < \sigma < \sigma_0 + \varepsilon\}$ for any $\varepsilon > 0$ sufficiently small (recall that E is an open set). Consequently, $w_0 \in \bigcup_{\sigma \in E} \mathrm{Img}\,(f_2(\sigma + it))$. We analogously prove that $\bigcup_{\sigma \in E} \mathrm{Img}\,(f_2(\sigma + it)) \subset \bigcup_{\sigma \in E} \mathrm{Img}\,(f_1(\sigma + it))$.

It is worth noting that [13, Example 2] also shows that a converse to Theorem 1 cannot hold by fixing an open set E in (α, β). However, we are considering the existence of a certain converse statement of this present generalization (see the arXiv paper [9]).

Finally, from Remark 1 and Proposition 1, we know that Definition 2 and the definition of Bohr-equivalence are equivalent under the condition of existence of an integral basis for the set of exponents Λ, which means that all the results of this paper which can be formulated in terms of an integral basis are also valid under Bohr-equivalence. In fact, if two exponential sums or almost periodic functions are Bohr-equivalent, it is clear that they also are *-equivalent (according to Definition 2). Consequently, we can deduce immediately from our main result (Theorem 1) that two Bohr-equivalent almost periodic functions take the same values on every open vertical strip included in their strip of almost periodicity U, i.e. [13, Theorem 18] is also true if the condition of existence of an integral basis is omitted. This constitutes an extension of Bohr's equivalence theorem.

Corollary 2 *Fixed Λ a set of exponents, let $f_1, f_2 \in \mathcal{D}_\Lambda$ be two Bohr-equivalent almost periodic functions in a strip $\{\sigma + it \in \mathbb{C} : \alpha < \sigma < \beta\}$, and consider E an open set of real numbers included in (α, β). Then*

$$\bigcup_{\sigma \in E} \mathrm{Img}\,(f_1(\sigma + it)) = \bigcup_{\sigma \in E} \mathrm{Img}\,(f_2(\sigma + it))\,.$$

That is, the functions f_1 and f_2 take the same set of values on the region $\{s = \sigma + it \in \mathbb{C} : \sigma \in E\}$.

For the case of Bohr's equivalence relation, we would like to point out that the existence of a converse statement of the above result will be possible only assuming the existence of an integral basis (see [9]).

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