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On modular equations of degree 25

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Abstract

On page 237–238 of his second notebook, Ramanujan recorded five modular equations of composite degree 25. Berndt proved all these using the method of parametrization. He also expressed that his proofs undoubtedly often stray from the path followed by Ramanujan. The purpose of this paper is to give direct proofs to four of the five modular equations using the identities known to Ramanujan.

Keywords Elliptic integrals · Modular equations · Theta functions

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1 Introduction

Let a be a complex number. In what follows, we employ the usual notation

$$(a)_0 = 1,$$
$$(a)_n = a(a+1)\cdots(a+n-1), \quad n \geq 1.$$

Gauss hypergeometric series ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1.$$

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The complete elliptic integral of first kind is denoted by $K(k)$ and is defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \quad |k| < 1.$$

k and $k' = \sqrt{1-k^2}$ are called modulus and complementary modulus of $K(k)$, respectively. This complete elliptic integral of first kind is related to the Gaussian hypergeometric series by the following equation

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

Set $\alpha = k^2$, $\beta = l_1^2$, $\gamma = l_2^2$ and $\delta = l_3^2$. Let $k' = \sqrt{1-k^2}$, $l_1' = \sqrt{1-l_1^2}$, $l_2' = \sqrt{1-l_2^2}$ and $l_3' = \sqrt{1-l_3^2}$. Suppose that the equality

$$n\frac{K(k')}{K(k)} = \frac{K(l_1')}{K(l_1)}$$

holds for some positive integer n. Any relation between α and β induced by the above is a called modular equation of degree n. We also say β has degree n over α. Also suppose that the equalities

$$m\,\frac{K(k')}{K(k)} = \frac{K(l_1')}{K(l)}, \quad n\frac{K(k')}{K(k)} = \frac{K(l_2')}{K(l_2)} \quad \text{and} \quad mn\,\frac{K(k')}{K(k)} = \frac{K(l_3')}{K(l_3)}$$

hold for positive integers m and n. Then any relation induced among α, β γ and δ by the above is called a modular equation of composite degree mn.
If β has degree n over α and if $\alpha = k^2$ and $\beta = l^2$, then the multiplier connecting α and β, denoted by m, is defined as

$$m = \frac{K(k)}{K(l)}.$$

On page 237–238 of his second notebook [5], Ramanujan recorded five modular equations of composite degree 25. In fact, these are the first set of modular equations of odd composite degree recorded by Ramanujan in his second note book. Modular equations of other composite degrees are recorded in Chapter 20 and in the unorganized portions. Following are the five modular equations of composite degree 25 recorded by Ramanujan:

Theorem 1.1 *Let α, β and γ be of first, fifth and twenty-fifth degrees respectively. Let m denote the multiplier connecting α and β and m' be the multiplier connecting β*

and γ. *Then*

$$\left(\frac{\gamma}{\alpha}\right)^{\frac{1}{8}}+\left(\frac{1-\gamma}{1-\alpha}\right)^{\frac{1}{8}}-\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{\frac{1}{8}}-2\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{\frac{1}{12}}=(mm')^{\frac{1}{2}}, \tag{1.1}$$

$$\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{8}}+\left(\frac{1-\alpha}{1-\gamma}\right)^{\frac{1}{8}}-\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{\frac{1}{8}}-2\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{\frac{1}{12}}=\frac{5}{(mm')^{\frac{1}{2}}}, \tag{1.2}$$

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{\frac{1}{8}}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{\frac{1}{8}}+\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{\frac{1}{8}}=\sqrt{\frac{m'}{m}}, \tag{1.3}$$

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{\frac{1}{4}}+\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{\frac{1}{4}}+\left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{\frac{1}{4}}$$
$$-2\left(\frac{\beta^2(1-\beta^2)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{\frac{1}{8}}\left\{1+\left(\frac{\beta^2}{\alpha\gamma}\right)^{\frac{1}{8}}+\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{\frac{1}{8}}\right\}=5\frac{m}{m'}, \tag{1.4}$$

and

$$\frac{1+4^{\frac{1}{3}}\left(\frac{\beta^{10}(1-\beta)^{10}}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{\frac{1}{24}}}{1+4^{\frac{1}{3}}\left(\frac{\alpha^5\gamma^5(1-\alpha)^5(1-\gamma)^5}{\beta^2(1-\beta)^2}\right)^{\frac{1}{24}}}=\frac{m}{m'}. \tag{1.5}$$

Berndt [2] proved all these by the method of parametrization. While proving (1.4), Berndt [2, p. 294] remarked that "formula(1.4) appears to be more recondite than the preceding three formulas and it is not obvious how it can be deduced from them in any simple manner." Motivated by this remark, in this paper we give direct proofs of (1.1) to (1.4) using certain Ramanujan theta function identities which are easily deducible from the famous Ramanujan's ${}_1\psi_1$ summation formula. We are unable to prove (1.5) by the techniques used to prove (1.1)–(1.4).

In Sect. 2 of this paper, we recall certain facts and theta function identities which are required to prove (1.1)–(1.4). In Sect. 3, we prove (1.1)–(1.4).

2 Preliminary results

For complex numbers a and q with $|q|<1$, $(a;q)_\infty$ is defined as

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k).$$

Ramanujan's theta function $f(-a,-b)$ is defined as

$$f(-a,-b)=\sum_{n=-\infty}^{\infty}(-1)^n a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}=(a;ab)_\infty(b;ab)_\infty(ab,ab)_\infty,\quad |ab|<1. \tag{2.1}$$

Ramanujan also defines special cases of $f(-a,-b)$ by

$$\phi(q)=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}=(-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{2.2}$$

$$\psi(q)=f(q,q^3)=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.3}$$

and

$$f(-q)=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}}=(q;q)_\infty. \tag{2.4}$$

He also defines

$$\chi(-q)=(q;q^2)_\infty. \tag{2.5}$$

We use the following theorem due to Ramanujan [5, p. 211] [2, p. 124] in our proofs, to transform the theta function identities into modular equations.

Theorem 2.1 *If* $y=\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x).}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x).}$, $q=e^{-y}$, *and* $z=\phi^2(q)$, *where* $0<x<1$, *then*

$$f(-q)=\sqrt{z}2^{\frac{-1}{6}}(1-x)^{\frac{1}{6}}\left\{\frac{x}{q}\right\}^{\frac{1}{24}},$$

$$f(-q^2)=\sqrt{z}2^{\frac{-1}{3}}\left\{\frac{x(1-x)}{q}\right\}^{\frac{1}{12}},$$

and

$$f(-q^4)=\sqrt{z}4^{\frac{-1}{3}}(1-x)^{\frac{1}{24}}\left\{\frac{x}{q}\right\}^{\frac{1}{6}}.$$

We require the following two theorems to prove (1.1)–(1.4).

Theorem 2.2 *If* $P_{(m,n)}=\frac{1}{q^{\frac{n-m}{24}}}\frac{f(-q^m)}{f(-q^n)}$, *then*

$$P_{(1,5)}P_{(2,10)}+\frac{5}{P_{(1,5)}P_{(2,10)}}=\left(\frac{P_{(2,10)}}{P_{(1,5)}}\right)^3+\left(\frac{P_{(1,5)}}{P_{(2,10)}}\right)^3, \tag{2.6}$$

$$\left(P_{(1,2)}P_{(5,10)}\right)^2 + \frac{4}{\left(P_{(1,2)}P_{(5,10)}\right)^2} = \left(\frac{P_{(5,10)}}{P_{(1,2)}}\right)^3 - \left(\frac{P_{(1,2)}}{P_{(5,10)}}\right)^3, \tag{2.7}$$

$$P_{(2,5)}P_{(1,10)} - \frac{5}{P_{(2,5)}P_{(1,10)}} = \left(\frac{P_{(1,10)}}{P_{(2,5)}}\right)^2 - 4\left(\frac{P_{(2,5)}}{P_{(1,10)}}\right)^2, \tag{2.8}$$

and

$$\begin{aligned}\left(P_{(1,25)}P_{(2,50)}\right)^2 + 5\left(P_{(1,25)}P_{(2,50)}\right)^2 \\ = P_{(1,25)}^3 - 2P_{(1,25)}^2 P_{(2,50)} - 2P_{(1,25)}P_{(2,50)}^2 + P_{(2,50)}^3.\end{aligned} \tag{2.9}$$

Ramanujan recorded identities (2.6), (2.8) and (2.9) on pages 325 and 327 of his second note book [5] and (2.7) on page 55 of his lost note book [6]. Berndt proved them in [3]. Recently Bhargava et al. [4] deduced Theorem 2.1 from the famous Ramanujan's ${}_1\psi_1$ summation formula. In his Ph.D thesis, Khaled Abed Azez Alloush [1] deduced the theta function identity

$$\begin{aligned}\frac{P_{(10,50)}P_{(5,25)}}{P_{(1,5)}P_{(2,10)}} - 1 &= \left(\frac{P_{(1,5)}P_{(5,25)}}{P_{(10,50)}P_{(2,10)}}\right)^2 + \left(\frac{P_{(10,50)}P_{(2,10)}}{P_{(1,5)}P_{(5,25)}}\right)^2 \\ &\quad - \left(\frac{P_{(1,5)}P_{(5,25)}}{P_{(10,50)}P_{(2,10)}} + \frac{P_{(10,50)}P_{(2,10)}}{P_{(1,5)}P_{(5,25)}}\right),\end{aligned} \tag{2.10}$$

by employing (2.6)–(2.8).

Theorem 2.3 *Let $P_{(m,n)}$ be as in the Theorem 2.1. Then*

$$P^6 - 2P^4Q - 5P^4Q^4 - 2P^3Q^3 + P^2Q^2 - 2PQ^4 + Q^6 = 0, \tag{2.11}$$

where $P = \frac{P_{(1,5)}}{P_{(5,25)}}$ *and* $Q = \frac{P_{(2,10)}}{P_{(10,50)}}$.

Proof Replacing q by q^5 in (2.6) and then multiplying the resulting identity with (2.6), we find that

$$P_{(1,5)}P_{(2,10)}P_{(5,25)}P_{(10,50)}+\frac{25}{P_{(1,5)}P_{(2,10)}P_{(5,25)}P_{(10,50)}}+5\Big(PQ+\frac{1}{PQ}\Big)$$
$$=\left(\frac{P_{(2,10)}P_{(10,50)}}{P_{(1,5)}P_{(5,25)}}\right)^3+\left(\frac{P_{(1,5)}P_{(5,25)}}{P_{(2,10)}P_{(10,50)}}\right)^3+\left(\frac{Q}{P}\right)^3+\left(\frac{Q}{P}\right)^3. \quad (2.12)$$

Replacing q by q^5 in (2.7) and then multiplying the resulting identity with (2.7), we find that

$$\Big(P_{(1,2)}P_{(5,10)}P_{(5,10)}P_{(25,50)}\Big)^2+\left(\frac{4}{P_{(1,2)}P_{(5,10)}P_{(5,10)}P_{(25,50)}}\right)^2$$
$$+4\left(\left(\frac{P_{(2,10)}P_{(10,50)}}{P_{(1,5)}P_{(5,25)}}\right)^2+\left(\frac{P_{(1,5)}P_{(5,25)}}{P_{(2,10)}P_{(10,50)}}\right)^2\right)$$
$$=\left(\frac{P_{(5,10)}P_{(25,50)}}{P_{(1,2)}P_{(5,10)}}\right)^3+\left(\frac{P_{(1,2)}P_{(5,10)}}{P_{(5,10)}P_{(25,50)}}\right)^3-\left(\frac{Q}{P}\right)^3-\left(\frac{P}{Q}\right)^3. \quad (2.13)$$

Replacing q by q^5 in (2.8) and multiplying the resulting identity with (2.8), we find that

$$P_{(1,5)}P_{(2,10)}P_{(5,25)}P_{(10,50)}+\frac{25}{P_{(1,5)}P_{(2,10)}P_{(5,25)}P_{(10,50)}}-5\Big(PQ+\frac{1}{PQ}\Big)$$
$$=\Big(P_{(1,2)}P_{(5,10)}P_{(5,10)}P_{(25,50)}\Big)^2+\left(\frac{4}{P_{(1,2)}P_{(5,10)}P_{(5,10)}P_{(25,50)}}\right)^2$$
$$-4\left(\left(\frac{P_{(5,10)}P_{(25,50)}}{P_{(1,2)}P_{(5,10)}}\right)^2+\left(\frac{P_{(1,2)}P_{(5,10)}}{P_{(5,10)}P_{(25,50)}}\right)^2\right). \quad (2.14)$$

Eliminating $\Big(P_{(1,2)}P_{(5,10)}P_{(5,10)}P_{(25,50)}\Big)^2+\left(\frac{4}{P_{(1,2)}P_{(5,10)}P_{(5,10)}P_{(25,50)}}\right)^2$ between (2.13) and (2.14), we obtain

$$4\left(\left(\frac{P_{(2,10)}P_{(10,50)}}{P_{(1,5)}P_{(5,25)}}\right)^2+\left(\frac{P_{(1,5)}P_{(5,25)}}{P_{(2,10)}P_{(10,50)}}\right)^2\right.$$
$$\left.+\left(\frac{P_{(5,10)}P_{(25,50)}}{P_{(1,2)}P_{(5,10)}}\right)^2+\left(\frac{P_{(1,2)}P_{(5,10)}}{P_{(5,10)}P_{(25,50)}}\right)^2\right)$$
$$=\left(\frac{P_{(5,10)}P_{(25,50)}}{P_{(1,2)}P_{(5,10)}}\right)^3+\left(\frac{P_{(1,2)}P_{(5,10)}}{P_{(5,10)}P_{(25,50)}}\right)^3-\left(\frac{Q}{P}\right)^3-\left(\frac{P}{Q}\right)^3$$
$$-P_{(1,5)}P_{(2,10)}P_{(5,25)}P_{(10,50)}-\frac{25}{P_{(1,5)}P_{(2,10)}P_{(5,25)}P_{(10,50)}}$$
$$+5\Big(PQ+\frac{1}{PQ}\Big). \quad (2.15)$$

Noticing that $\frac{P_{(5,10)}P_{(25,50)}}{P_{(1,2)}P_{(5,10)}}+\frac{P_{(1,2)}P_{(5,10)}}{P_{(5,10)}P_{(25,50)}}=\frac{P_{(2,10)}P_{(10,50)}}{P_{(1,5)}P_{(5,25)}}+\frac{P_{(1,5)}P_{(5,25)}}{P_{(2,10)}P_{(10,50)}}$ and eliminating $P_{(1,5)}P_{(2,10)}P_{(5,25)}P_{(10,50)}+\frac{25}{P_{(1,5)}P_{(2,10)}P_{(5,25)}P_{(10,50)}}$ between (2.15) and (2.12), we find that

$$5\Big(PQ+\frac{1}{PQ}\Big)=4\left\{\left(\frac{P_{(5,10)}P_{(25,50)}}{P_{(1,2)}P_{(5,10)}}\right)^2+\left(\frac{P_{(1,2)}P_{(5,10)}}{P_{(5,10)}P_{(25,50)}}\right)^2\right\}+\left(\frac{Q}{P}\right)^3+\left(\frac{P}{Q}\right)^3. \tag{2.16}$$

Set $x=\frac{P_{(5,10)}P_{(25,50)}}{P_{(1,2)}P_{(5,10)}}+\frac{P_{(1,2)}P_{(5,10)}}{P_{(5,10)}P_{(25,50)}}$. We can now write (2.16) and (2.10) as

$$4x^2-8+\left(\frac{Q}{P}\right)^3+\left(\frac{P}{Q}\right)^3-5\Big(PQ+\frac{1}{PQ}\Big)=0 \tag{2.17}$$

and

$$x^2-x-\frac{1}{PQ}-1=0, \tag{2.18}$$

respectively. Eliminating x between (2.17) and (2.18), we obtain

$$\Big(R-P\Big)\Big(P^6+P^2Q^2-2P^3Q^3-5P^4Q^4-2QP^4-2PQ^4+Q^6\Big)\times\Big(P^6+P^2Q^2-2P^3Q^3-5P^4Q^4+2QP^4+2PQ^4+Q^6\Big)=0. \tag{2.19}$$

By definition, $R-P\neq 0$. Observe that

$$\begin{aligned}&P^6+P^2Q^2-2P^3Q^3-5P^4Q^4+2QP^4+2PQ^4+Q^6\\&=(P^3-Q^3)^2-5P^4Q^4+2QP^4+2PQ^4+P^2Q^2.\end{aligned} \tag{2.20}$$

By definition,

$$P=q^{\frac{2}{3}}\frac{f(-q)f(-q^{25})}{f^2-q^5}\quad\text{and}\quad Q=q^{\frac{4}{3}}\frac{f(-q^2)f(-q^{50})}{f^2(-q^{10})}.$$

$f(-q^n)$ is analytic in the disc $|q|<1$ and tends to 1 as $q\to 0^+$. Thus, $P\to 0$ and $Q\to 0$ as $q\to 0^+$. Thus in some neighborhood H of zero, $0<P<1$ and $0<Q<1$. Hence in H, $QP^4>P^4Q^4$, $PQ^4>P^4Q^4$ and $P^2Q^2>P^4Q^4$. Using these in (2.20) we can conclude that $-5P^4Q^4+2QP^4+2PQ^4+P^2Q^2>0$ in H. That is $P^6+P^2Q^2-2P^3Q^3-5P^4Q^4+2QP^4+2PQ^4+Q^6\neq 0$ in H. Applying now identity theorem for holomorphic functions to (2.19), we obtain (2.11). □

3 Proof of Theorem 1.1

In this section, we prove (1.1) to (1.4) of Theorem 1.1.

Proofs of (1.1) and (1.2) Let $P = P_{(1,25)}$, $Q = P_{(2,50)}$ and $R = P_{(4,100)}$. Then from (2.9)

$$(PQ)^2 + 5PQ - P^3 + 2P^2Q + 2PQ^2 - Q^3 = 0 \tag{3.1}$$

and

$$(RQ)^2 + 5RQ - R^3 + 2R^2Q + 2RQ^2 - Q^3 = 0. \tag{3.2}$$

Equation (3.2) is obtained by replacing q by q^2 in (2.9). Multiplying (3.1) by R and then subtracting the identity obtained by multiplying (3.7) with P, we arrive at

$$(R - P)(P^2R + PR^2 - Q^3 - 2PQR - PRQ^2) = 0. \tag{3.3}$$

Since $R - P \neq 0$, we obtain

$$P^2R + PR^2 - Q^3 - 2PQR - PRQ^2 = 0. \tag{3.4}$$

Transcribing (3.4) into modular equation by using Theorem 2.1, we obtain (1.1). Multiplying now (3.1) by R^2 and then subtracting the identity obtained by multiplying (3.2) with P^2, we arrive at

$$(R - P)(Q^3P + Q^3R - P^2R^2 - 2PRQ^2 - 5PQR) = 0. \tag{3.5}$$

Since $R - P \neq 0$, we obtain

$$Q^3P + Q^3R - P^2R^2 - 2PRQ^2 - 5PQR = 0. \tag{3.6}$$

Transcribing (3.6) into modular equation by using Theorem 2.1 we obtain (1.2). □

Proof of (1.3) If $P = \frac{P_{(1,5)}}{P_{(5,25)}}$, $Q = \frac{P_{(2,10)}}{P_{(10,50)}}$ and $R = \frac{P_{(4,20)}}{P_{(20,100)}}$, then

$$P^6 - 2P^4Q - 5P^4Q^4 - 2P^3Q^3 + P^2Q^2 - 2PQ^4 + Q^6 = 0.$$

The above is nothing but (2.11). Changing q to q^2 in the above, we find that

$$R^6 - 2R^4Q - 5R^4Q^4 - 2R^3Q^3 + R^2Q^2 - 2RQ^4 + Q^6 = 0. \tag{3.7}$$

Multiplying (2.11) by R^4 and then subtracting the identity obtained by multiplying (3.7) with P^4, we arrive at

$$(P - R)(R^2P^3 + P^2R^3 + QP^2R - Q^3P^2 + R^2QP - R^2Q^3)$$

$$\times(P^2R^2 + PQ^3 - PQR + Q^3R) = 0. \tag{3.8}$$

$P - R \neq 0$. Let

$$C(P, Q, R)\, D(P, Q, R) = 0,$$

where

$$C(P, Q, R) = R^2P^3 + P^2R^3 + QP^2R - Q^3P^2 + R^2QP - R^2Q^3. \tag{3.9}$$

and

$$D(P, Q, R) = P^2R^2 + PQ^3 - PQR + Q^3. \tag{3.10}$$

By definition

$$P = q^{\frac{2}{3}} - q^{\frac{5}{3}} - q^{\frac{8}{3}} + 3q^{\frac{17}{3}} - 2q^{\frac{20}{3}} - q^{\frac{23}{3}} + \cdots \tag{3.11}$$

Changing q to q^2 and changing q to q^4 in the above, we obtain

$$Q = q^{\frac{4}{3}} - q^{\frac{10}{3}} - q^{\frac{16}{3}} + 3q^{\frac{34}{3}} - 2q^{\frac{40}{3}} - q^{\frac{46}{3}} + \cdots \tag{3.12}$$

and

$$R = q^{\frac{8}{3}} - q^{\frac{20}{3}} - q^{\frac{32}{3}} + 3q^{\frac{68}{3}} - 2q^{\frac{80}{3}} - q^{\frac{92}{3}} + \cdots, \tag{3.13}$$

respectively Using these in (3.9) and (3.10), we obtain

$$C(P, Q, R) = 4q^{\frac{22}{3}} - 8q^{\frac{25}{3}} - 6q^{\frac{28}{3}} + 12q^{\frac{31}{3}} - 2q^{\frac{34}{3}} + \cdots \tag{3.14}$$

and

$$D(P, Q, R) = 4q^{\frac{83}{3}} - 40q^{\frac{86}{3}} - 28q^{\frac{89}{3}} - 20q^{\frac{95}{3}} + 300q^{\frac{98}{3}} + \cdots \quad . \tag{3.15}$$

As $q \to 0$, $q^{-\frac{22}{3}}C(P, Q, R) \not\to$ to 0, where as $q^{-\frac{22}{3}}D(P, Q, R) \to 0$. This implies that $C(P, Q, R) \neq 0$ in some neighborhood of 0^+. Therefore by identity theorem, $D(P, Q, R) = 0$ in some neighborhood of zero. By analytic continuation we can conclude that

$$P^2R^2 + PQ^3 - PQR + Q^3R = 0, \tag{3.16}$$

in the respective domain of q. Transcribing the above into modular equation by Theorem 2.1, we obtain (1.3). □

Proof of (1.4) Multiplying (2.6) with R^3 and then subtracting the identity obtained by multiplying (3.7) with P^3, we arrive at

$$(-R+P)(R^3P^5+R^4P^4-5P^3R^3Q^4-2P^3R^3Q+R^5P^3-R^2P^2Q^2 \\ +2P^2RQ^4-P^2Q^6+2R^2PQ^4-PRQ^6-R^2Q^6)=0. \tag{3.17}$$

$(P-R)\neq 0$. Therefore

$$R^3P^5+R^4P^4-5P^3R^3Q^4-2P^3R^3Q+R^5P^3-R^2P^2Q^2+2P^2RQ^4 \\ -P^2Q^6+2R^2PQ^4-PRQ^6-R^2Q^6=0. \tag{3.18}$$

Add and subtract $2P^3R^3Q$ and $R^2P^2Q^2$ to (3.18). Using (3.16) repeatedly in (3.18) (Grouping the terms $2P^3R^3Q+2P^2RQ^4+2R^2PQ^4-2R^2P^2Q^2$ and replacing $(PR)^4$ by $(PQR-PQ^3-Q^3R)^2$ and then again using (3.16) in the resulting identity), we arrive at

$$P^4R^2+P^2R^4+Q^6-5P^2R^2Q^4-2P^2R^2Q^2=0. \tag{3.19}$$

Finally, using $Q=\frac{Q^3}{P}+\frac{Q^3}{R}+PR$ (from 3.16) in the last term of (3.19), we arrive at

$$P^4R^2+P^2R^4+Q^6-2(P^3R^3+Q^3P^2R+Q^3PR^2)=5Q^4P^2R^2. \tag{3.20}$$

Transcribing the above theta function identity into a modular equation by using Theorem 2.1, we obtain (1.4). □

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