The Ramanujan Journal (2021) 55:239–270
https://doi.org/10.1007/s11139-020-00259-w

Infinite series identities derived from the very well-poised Ω-sum

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Received: 30 June 2019 / Accepted: 6 February 2020 / Published online: 18 July 2020

Abstract
For the very well-poised Ω-series, a universal iteration pattern is established that yields numerous infinite series identities including several important ones discovered by Ramanujan (in Q J Pure Appl Math 45:350–372, 1914) and recently by Guillera.

Keywords Infinite series of Ramanujan-type · Apéry series · Formula of BBP-type · Bisection series · Dougall's formula for well-poised series

Mathematics Subject Classification Primary 33C20 · Secondary 65B10

1 Introduction and motivation

Let $\mathbb{N}$ be the set of natural numbers. The shifted factorial reads as $(x)_0 = 1$ and

$$(x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$

It can be expressed, for an integer n (even nonpositive), by the quotient

$$(x)_n = \Gamma(x+n)/\Gamma(x),$$

where the Γ-function is defined by the Euler integral

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\mathrm{d}u \quad \text{with} \quad \Re(x) > 0$$

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and admits the well-known reciprocal relations

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}\quad\text{and}\quad\Gamma(\tfrac{1}{2}+x)\Gamma(\tfrac{1}{2}-x)=\frac{\pi}{\cos\pi x}.$$

For the sake of brevity, we shall utilize, throughout the paper, the following multiparameter notations:

$$\begin{aligned}\left[\begin{matrix}\alpha, & \beta, & \cdots, & \gamma\\ A, & B, & \cdots, & C\end{matrix}\right]_n &= \frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n},\\ \Gamma\left[\begin{matrix}\alpha, & \beta, & \cdots, & \gamma\\ A, & B, & \cdots, & C\end{matrix}\right] &= \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.\end{aligned}$$

About one century ago, Ramanujan [58] discovered 17 remarkable infinite series for $1/\pi$. Three typical ones [58, Eqs. 28, 29 and 35] are reproduced as

$$\begin{aligned}\frac{4}{\pi}&=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac12, & \frac12, & \frac12\\ 1, & 1, & 1\end{matrix}\right]_k\frac{1+6k}{4^k},\\ \frac{8}{\pi}&=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac12, & \frac14, & \frac34\\ 1, & 1, & 1\end{matrix}\right]_k\frac{3+20k}{(-4)^k},\\ \frac{16}{\pi}&=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac12, & \frac12, & \frac12\\ 1, & 1, & 1\end{matrix}\right]_k\frac{5+42k}{64^k}.\end{aligned}$$

However their proofs were only obtained in the 1980s by the Borweins [23] and by the Chudnovskys [44] via modular equations. By using the powerful *WZ*-method, Guillera [48,49,51,53] proved recently several important formulae of Ramanujan-like for $1/\pi^2$ such as

$$\begin{aligned}\frac{8}{\pi^2}&=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}}\left[\begin{matrix}\frac12, & \frac12, & \frac12, & \frac12, & \frac12\\ 1, & 1, & 1, & 1, & 1\end{matrix}\right]_n\Big\{1+8n+20n^2\Big\},\\ \frac{32}{\pi^2}&=\sum_{n=0}^{\infty}\frac{1}{16^n}\left[\begin{matrix}\frac12, & \frac12, & \frac12, & \frac14, & \frac34\\ 1, & 1, & 1, & 1, & 1\end{matrix}\right]_n\Big\{3+34n+120n^2\Big\},\\ \frac{48}{\pi^2}&=\sum_{n=0}^{\infty}\Big(\frac{27}{64}\Big)^n\left[\begin{matrix}\frac12, & \frac12, & \frac12, & \frac13, & \frac23\\ 1, & 1, & 1, & 1, & 1\end{matrix}\right]_n\Big\{3+27n+74n^2\Big\},\\ \frac{128}{\pi^2}&=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\left[\begin{matrix}\frac12, & \frac12, & \frac12, & \frac12, & \frac12\\ 1, & 1, & 1, & 1, & 1\end{matrix}\right]_n\Big\{13+180n+820n^2\Big\}.\end{aligned}$$

More comprehensive investigation has been made by Chu and Zhang [40,42,67] by manipulating the classical hypergeometric series.

The objective of the present paper is to show that numerous infinite series identities can be proved in a unified manner from a transformation theorem about the following well-poised series

$$\Omega(a;b,c,d,e) := \sum_{k=0}^{\infty}(a+2k)\begin{bmatrix} b, & c, & d, & e \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{bmatrix}_k \tag{1}$$

provided that $\Re(1+2a-b-c-d-e) > 0$ for convergence. For the given nonnegative integers $\{\rho_a, \rho_b, \rho_c, \rho_d, \rho_e\}$, the corresponding recurrence relation expressing

$$\Omega(a;b,c,d,e) \quad \text{in terms of} \quad \Omega(a+\rho_a; b+\rho_b, c+\rho_c, d+\rho_d, e+\rho_e)$$

will be said of "iteration pattern" $[\rho_a; \rho_b, \rho_c, \rho_d, \rho_e]$. By exploring, in detail, the seven iteration patterns **[20101]**, **[20111]**, **[30112]**, **[30202]**, **[30122]**, **[31113]**, and **[31111]**, Chu and Zhang [42] derived numerous infinite series formulae for $\pi^{\pm 1}, \pi^{\pm 2}, \zeta(3)$ and the Catalan constant G, including several interesting identities discovered by Ramanujan [58] and recently by Guillera [47,48,51–54].

Instead of working on specific iteration patterns, we shall examine the universal iteration pattern for "$\Omega(a;b,c,d,e)$" and establish a general transformation. That contains the patterns mentioned above and those treated in [42] as particular cases. The proof is given in the next section.

Theorem A *For the five complex parameters $\{a,b,c,d,e\}$ satisfying the condition $\Re(1+2a-b-c-d-e) > 0$ and the iteration pattern $[\rho_a; \rho_b, \rho_c, \rho_d, \rho_e]$ subject to $\big|\Delta[\rho_a; \rho_b, \rho_c, \rho_d, \rho_e]\big| < 1$, the following transformation formula holds:*

$$\Omega(a;b,c,d,e) = \sum_{k=0}^{\infty} W_k\begin{bmatrix} a; & b, & c, & d, & e \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix} H_k\begin{bmatrix} a; & b, & c, & d, & e \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix}, \tag{2}$$

where

- $H_k[:::::]$ *is the quotient of Pochhammer symbols defined by* (16) *and* (18)*;*
- $W_k[:::::]$ *is the rational weight factor defined recursively by* (14) *and* (19)*;*
- $\Delta[\rho_a; \rho_b, \rho_c, \rho_d, \rho_e]$ *is the convergence rate defined by* (20)*.*

Remark When one of parameters $\{b,c,d,e\}$ is equal to a (for example $e = a$) in Theorem A, the sum on the left can be evaluated by Dougall's summation formula (cf. [8, §4.4]) on well-poised ${}_5F_4$-series as the Γ-function quotient

$$\Gamma\begin{bmatrix} 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d \\ a, 1+a-b-c, 1+a-b-d, 1+a-c-d \end{bmatrix}.$$

This is analytic in the whole complex space of dimension 4 except for the hyperplanes determined by

$$\{b-a \in \mathbb{N}\} \cup \{c-a \in \mathbb{N}\} \cup \{d-a \in \mathbb{N}\} \cup \{b+c+d-a \in \mathbb{N}\},$$

which is covered also by the convergence domain of the infinite series displayed on the right-hand side of Theorem A. Therefore in this case, the transformation in Theorem A becomes the following closed formula:

$$\sum_{k=0}^{\infty} W_k \begin{bmatrix} a; & b, & c, & d, & a \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix} H_k \begin{bmatrix} a; & b, & c, & d, & a \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix} = \Gamma \begin{bmatrix} 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d \\ a, 1+a-b-c, 1+a-b-d, 1+a-c-d \end{bmatrix}, \tag{3}$$

where the parameter restriction $\Re(1+a-b-c-d) > 0$ is removed by analytic continuation. This clarifies the concern raised recently by Guillera [56] and justifies the validity of those examples appearing in the compendium [42] even though their parameter settings fail to comply the condition $\Re(1+a-b-c-d) > 0$. The situation with one of parameters $\{b, c, d, e\}$ tending to $-\infty$ can be treated analogously.

The transformation formula in Theorem A will enable us, in Sect. 3, not only to review various infinite series identities of Ramanujan–Guillera type, but also to find several new formulae and to confirm a few challenging conjectured ones. As anticipation, we highlight nine examples of new summation formulae as follows:

- Example 6: Infinite series for π^{-1}

$$\frac{9\sqrt{3}}{2^{4/3}\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{3}, & \frac{2}{3}, & \frac{1}{6} \\ 1, & 1, & 1 \end{bmatrix}_k \frac{2+21k}{(-27)^k}.$$

- Example 7: Infinite series for π^{-1}

$$\frac{27\sqrt{3}}{2^{5/3}\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{3}, & \frac{2}{3}, & \frac{5}{6} \\ 1, & 1, & 1 \end{bmatrix}_k \frac{5+42k}{(-27)^k}.$$

- Example 18: Infinite series for π

$$\frac{15\pi}{8\sqrt{2}} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, & \frac{1}{4}, & \frac{1}{4} \\ \frac{7}{4}, & \frac{9}{8}, & \frac{13}{8} \end{bmatrix}_k \frac{4+13k+12k^2}{4^k}.$$

- Example 19: Infinite series for π

$$\frac{21\pi}{8\sqrt{2}} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, & \frac{3}{4}, & \frac{3}{4} \\ \frac{5}{4}, & \frac{11}{8}, & \frac{15}{8} \end{bmatrix}_k \frac{4+15k+12k^2}{4^k}.$$

- Example 24: Infinite series for π

$$\pi\sqrt{3}=\sum_{k=0}^{\infty}\begin{bmatrix}1, & \frac{3}{4}, & \frac{5}{4}\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\frac{5+8k}{9^k}.$$

- Example 34: Infinite series for π^2

$$\frac{\pi^2}{2}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ \frac{3}{2}, & \frac{5}{4}, & \frac{7}{4}\end{bmatrix}_k\frac{5+14k+10k^2}{(-4)^k\,(k+1)^2}.$$

- Example 38: Sun's conjecture [62, Eq. 1.1]

$$\frac{3\pi^2}{4}=\sum_{k=0}^{\infty}\Big(\frac{2}{27}\Big)^k\begin{bmatrix}1, & 1, & 1\\ \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k\Big\{7+10k\Big\}.$$

- Example 67: Sun's conjecture [62, Eq. 1.6]

$$\frac{21}{2}\zeta(3)=\sum_{k=0}^{\infty}\begin{bmatrix}1, & 1, & 1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k\frac{13+38k+28k^2}{(-27)^k}.$$

- Example 75: Guillera's conjecture [55, Eq. 42]

$$210\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-4^4}{5^5}\Big)^k\begin{bmatrix}1, & 1, & 1, & 1, & 1\\ \frac{3}{2}, & \frac{6}{5}, & \frac{7}{5}, & \frac{8}{5}, & \frac{9}{5}\end{bmatrix}_k\Big\{268+721k+483k^2\Big\}.$$

2 Proof of the main theorem

For the $\Omega(a; b, c, d, e)$ series, Chu and Zhang [42] derived several useful recurrence relations. By means of the iteration procedure, we shall utilize two of them (see Lemmas B and C in Appendix) to prove Theorem A.

2.1 Iteration of Lemma B

Iterating λ-times of the recurrence relation displayed in Lemma B, we get the expression:

$$\Omega(a; b, c, d, e)=\Omega(a+\lambda; b+\lambda, c, d, e)\mathcal{A}_\lambda(a; b, c, d, e)+\mathcal{B}_\lambda(a; b, c, d, e), \tag{4}$$

where the $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$ coefficients are given by

$$\mathcal{A}_\lambda(a;b,c,d,e)=\left[\begin{matrix}b,1+a-c-d,1+a-c-e,1+a-d-e\\1+a-c,1+a-d,1+a-e,1+2a-b-c-d-e\end{matrix}\right]_\lambda, \tag{5}$$

$$\begin{aligned}\mathcal{B}_\lambda(a;b,c,d,e)=(a-b)\sum_{i=0}^{\lambda-1}&\frac{1+2a-c-d-e+2i}{1+2a-b-c-d-e}\\&\times\left[\begin{matrix}b,1+a-c-d,1+a-c-e,1+a-d-e\\1+a-c,1+a-d,1+a-e,2+2a-b-c-d-e\end{matrix}\right]_i.\end{aligned} \tag{6}$$

According to (4), by applying Lemma B successively

$$\begin{aligned}&bb\text{-times with respect to the parameter } b,\\&cc\text{-times with respect to the parameter } c,\\&dd\text{-times with respect to the parameter } d,\\&ee\text{-times with respect to the parameter } e;\end{aligned}$$

we derive the following transformation formula:

$$\begin{aligned}\Omega(a;b,c,d,e)=\mathbb{B}_{bb}+\mathbb{A}_{bb}\mathbb{B}_{cc}+\mathbb{A}_{bb}\mathbb{A}_{cc}\mathbb{B}_{dd}+\mathbb{A}_{bb}\mathbb{A}_{cc}\mathbb{A}_{dd}\mathbb{B}_{ee}+\mathbb{A}_{bb}\mathbb{A}_{cc}\mathbb{A}_{dd}\mathbb{A}_{ee}\\ \times\Omega(a+bb+cc+dd+ee;b+bb,c+cc,d+dd,e+ee),\end{aligned} \tag{7}$$

where for the sake of brevity, the notations are adopted:

$$\begin{aligned}\mathbb{A}_{bb}&:=\mathbb{A}_{bb}(a;b,c,d,e)=\mathcal{A}_{bb}(a;b,c,d,e),\\ \mathbb{A}_{cc}&:=\mathbb{A}_{cc}(a;b,c,d,e)=\mathcal{A}_{cc}(a+bb;c,b+bb,d,e),\\ \mathbb{A}_{dd}&:=\mathbb{A}_{dd}(a;b,c,d,e)=\mathcal{A}_{dd}(a+bb+cc;d,b+bb,c+cc,e),\\ \mathbb{A}_{ee}&:=\mathbb{A}_{ee}(a;b,c,d,e)=\mathcal{A}_{ee}(a+bb+cc+dd;e,b+bb,c+cc,d+dd);\\ \mathbb{B}_{bb}&:=\mathbb{B}_{bb}(a;b,c,d,e)=\mathcal{B}_{bb}(a;b,c,d,e),\\ \mathbb{B}_{cc}&:=\mathbb{B}_{cc}(a;b,c,d,e)=\mathcal{B}_{cc}(a+bb;c,b+bb,d,e),\\ \mathbb{B}_{dd}&:=\mathbb{B}_{dd}(a;b,c,d,e)=\mathcal{B}_{dd}(a+bb+cc;d,b+bb,c+cc,e),\\ \mathbb{B}_{ee}&:=\mathbb{B}_{ee}(a;b,c,d,e)=\mathcal{B}_{ee}(a+bb+cc+dd;e,b+bb,c+cc,d+dd).\end{aligned}$$

Furthermore, we can simplify the product

$$\begin{aligned}&\mathbb{A}_{bb}\mathbb{A}_{cc}\mathbb{A}_{dd}\mathbb{A}_{ee}\\&\quad=\frac{(b)_{bb}(c)_{cc}(d)_{dd}(e)_{ee}(1+a-b-c)_{dd+ee}(1+a-b-d)_{cc+ee}(1+a-b-e)_{cc+dd}}{(1+2a-b-c-d-e)_{bb+cc+dd+ee}}\\&\quad\times\frac{(1+a-c-d)_{bb+ee}(1+a-c-e)_{bb+dd}(1+a-d-e)_{bb+cc}}{(1+a-b)_{cc+dd+ee}(1+a-c)_{bb+dd+ee}(1+a-d)_{bb+cc+ee}(1+a-e)_{bb+cc+dd}}.\end{aligned}$$

2.2 Iteration of Lemma C

Instead, iterating μ-times of the recurrence relation displayed in Lemma C, we get the expression:

$$\Omega(a;b,c,d,e)=\Omega(a+\mu;b+\mu,c+\mu,d,e)\mathcal{C}_\mu(a;b,c,d,e)+\mathcal{D}_\mu(a;b,c,d,e), \tag{8}$$

where the $\mathcal{C}_\mu$ and $\mathcal{D}_\mu$ coefficients read as

$$\mathcal{C}_\mu(a;b,c,d,e)=\begin{bmatrix} b,c,1+a-d-e \\ 1+a-d,1+a-e,b+c-a\end{bmatrix}_\mu, \tag{9}$$

$$\mathcal{D}_\mu(a;b,c,d,e)=\frac{(a-b)(a-c)}{a-b-c}\sum_{j=0}^{\mu-1}\begin{bmatrix} b,c,1+a-d-e \\ 1+a-d,1+a-e,1+b+c-a\end{bmatrix}_j. \tag{10}$$

In view of (8), by applying Lemma C successively

bc-times with respect to the parameter pair (b,c),
bd-times with respect to the parameter pair (b,d),
be-times with respect to the parameter pair (b,e),
cd-times with respect to the parameter pair (c,d),
ce-times with respect to the parameter pair (c,e),
de-times with respect to the parameter pair (d,e);

we obtain another transformation formula

$$\begin{aligned}\Omega(a;b,c,d,e)&=\mathbb{D}_{bc}+\mathbb{C}_{bc}\mathbb{D}_{bd}+\mathbb{C}_{bc}\mathbb{C}_{bd}\mathbb{D}_{be}+\mathbb{C}_{bc}\mathbb{C}_{bd}\mathbb{C}_{be}\mathbb{D}_{cd}\\&+\mathbb{C}_{bc}\mathbb{C}_{bd}\mathbb{C}_{be}\mathbb{C}_{cd}\mathbb{D}_{ce}+\mathbb{C}_{bc}\mathbb{C}_{bd}\mathbb{C}_{be}\mathbb{C}_{cd}\mathbb{C}_{ce}\mathbb{D}_{de}\\&+\mathbb{C}_{bc}\mathbb{C}_{bd}\mathbb{C}_{be}\mathbb{C}_{cd}\mathbb{C}_{ce}\mathbb{C}_{de}\\&\times\Omega\big(a{+}bc{+}bd{+}be{+}cd{+}ce{+}de;b{+}bc{+}bd{+}be,\\&\quad c{+}bc{+}cd{+}ce,d{+}bd{+}cd{+}de,e{+}be{+}ce{+}de\big),\end{aligned} \tag{11}$$

where for the sake of consistence, we have utilized the notations:

$$\begin{aligned}\mathbb{C}_{bc}&=\mathbb{C}_{bc}(a;b,c,d,e):=\mathcal{C}_{bc}(a;b,c,d,e),\\ \mathbb{C}_{bd}&=\mathbb{C}_{bd}(a;b,c,d,e):=\mathcal{C}_{bd}(a+bc;b+bc,d,c+bc,e),\\ \mathbb{C}_{be}&=\mathbb{C}_{be}(a;b,c,d,e):=\mathcal{C}_{be}(a+bc+bd;b+bc+bd,e,c+bc,d+bd),\\ \mathbb{C}_{cd}&=\mathbb{C}_{cd}(a;b,c,d,e)\\ &\quad:=\mathcal{C}_{cd}(a+bc+bd+be;c+bc,d+bd,b+bc+bd+be,e+be),\\ \mathbb{C}_{ce}&=\mathbb{C}_{ce}(a;b,c,d,e)\end{aligned}$$

$$
\begin{aligned}
&\quad := \mathcal{C}_{ce}(a+bc+bd+be+cd;c+bc+cd,e+be,b+bc+bd+be,d+bd+cd),\\
\mathbb{C}_{de} &= \mathbb{C}_{de}(a;b,c,d,e)\\
&\quad := \mathcal{C}_{de}(a+bc+bd+be+cd+ce;d+bd+cd,e+be+ce,b+bc+bd+be,c+bc+cd+ce);\\
\mathbb{D}_{bc} &= \mathbb{D}_{bc}(a;b,c,d,e) := \mathcal{D}_{bc}(a;b,c,d,e),\\
\mathbb{D}_{bd} &= \mathbb{D}_{bd}(a;b,c,d,e) := \mathcal{D}_{bd}(a+bc;b+bc,d,c+bc,e),\\
\mathbb{D}_{be} &= \mathbb{D}_{be}(a;b,c,d,e) := \mathcal{D}_{be}(a+bc+bd;b+bc+bd,e,c+bc,d+bd),\\
\mathbb{D}_{cd} &= \mathbb{D}_{cd}(a;b,c,d,e)\\
&\quad := \mathcal{D}_{cd}(a+bc+bd+be;c+bc,d+bd,b+bc+bd+be,e+be),\\
\mathbb{D}_{ce} &= \mathbb{D}_{ce}(a;b,c,d,e)\\
&\quad := \mathcal{D}_{ce}(a+bc+bd+be+cd;c+bc+cd,e+be,b+bc+bd+be,d+bd+cd),\\
\mathbb{D}_{de} &= \mathbb{D}_{de}(a;b,c,d,e)\\
&\quad := \mathcal{D}_{de}(a+bc+bd+be+cd+ce;d+bd+cd,e+be+ce,b+bc+bd+be,c+bc+cd+ce).
\end{aligned}
$$

In addition, we can simplify the product explicitly

$$
\begin{aligned}
&\mathbb{C}_{bc}\mathbb{C}_{bd}\mathbb{C}_{be}\mathbb{C}_{cd}\mathbb{C}_{ce}\mathbb{C}_{de}\\
&= \frac{(b)_{bc+bd+be}(c)_{bc+cd+ce}(d)_{bd+cd+de}(e)_{be+ce+de}}{(1+a-b)_{cd+ce+de}(1+a-c)_{bd+be+de}(1+a-d)_{bc+be+ce}(1+a-e)_{bc+bd+cd}}\\
&\quad\times \frac{(-1)^{bc+bd+be+cd+ce+de}}{(b+c-a)_{bc-de}(b+d-a)_{bd-ce}(b+e-a)_{be-cd}(c+d-a)_{cd-be}(c+e-a)_{ce-bd}(d+e-a)_{de-bc}}.
\end{aligned}
$$

2.3 Combined Iteration

Under the replacements

$$
\bigstar := \big\{a \to a+bb+cc+dd+ee,\ b \to b+bb,\ c \to c+cc,\ d \to d+dd,\ e \to e+ee\big\}
$$

the connection coefficients

$$
\big\{\mathbb{C}_{bc},\mathbb{C}_{bd},\mathbb{C}_{be},\mathbb{C}_{cd},\mathbb{C}_{ce},\mathbb{C}_{de};\mathbb{D}_{bc},\mathbb{D}_{bd},\mathbb{D}_{be},\mathbb{D}_{cd},\mathbb{D}_{ce},\mathbb{D}_{de}\big\}
$$

will be denoted by

$$
\big\{\mathbb{C}^{\star}_{bc},\mathbb{C}^{\star}_{bd},\mathbb{C}^{\star}_{be},\mathbb{C}^{\star}_{cd},\mathbb{C}^{\star}_{ce},\mathbb{C}^{\star}_{de};\mathbb{D}^{\star}_{bc},\mathbb{D}^{\star}_{bd},\mathbb{D}^{\star}_{be},\mathbb{D}^{\star}_{cd},\mathbb{D}^{\star}_{ce},\mathbb{D}^{\star}_{de}\big\}.
$$

By combining the two transformations (7) and (11), we get the recurrence relation

$$\Omega(a;b,c,d,e) = W\begin{bmatrix} a; & b, & c, & d, & e \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix} + H\begin{bmatrix} a; & b, & c, & d, & e \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix} \times \Omega(a+\rho_a; b+\rho_b, c+\rho_c, d+\rho_d, e+\rho_e), \tag{12}$$

where the W and H coefficients are given explicitly by

$$H\begin{bmatrix} a; & b, & c, & d, & e \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix} = \mathbb{A}_{bb}\mathbb{A}_{cc}\mathbb{A}_{dd}\mathbb{A}_{ee}\mathbb{C}^\star_{bc}\mathbb{C}^\star_{bd}\mathbb{C}^\star_{be}\mathbb{C}^\star_{cd}\mathbb{C}^\star_{ce}\mathbb{C}^\star_{de}, \tag{13}$$

$$\begin{aligned} W\begin{bmatrix} a; & b, & c, & d, & e \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix} &= \Big\{\mathbb{B}_{bb} + \mathbb{A}_{bb}\mathbb{B}_{cc} + \mathbb{A}_{bb}\mathbb{A}_{cc}\mathbb{B}_{dd} + \mathbb{A}_{bb}\mathbb{A}_{cc}\mathbb{A}_{dd}\mathbb{B}_{ee}\Big\} \\ &+ \mathbb{A}_{bb}\mathbb{A}_{cc}\mathbb{A}_{dd}\mathbb{A}_{ee} \\ &\times \Big\{\mathbb{D}^\star_{bc} + \mathbb{C}^\star_{bc}\mathbb{D}^\star_{bd} + \mathbb{C}^\star_{bc}\mathbb{C}^\star_{bd}\mathbb{D}^\star_{be} + \mathbb{C}^\star_{bc}\mathbb{C}^\star_{bd}\mathbb{C}^\star_{be}\mathbb{D}^\star_{cd} \\ &+ \mathbb{C}^\star_{bc}\mathbb{C}^\star_{bd}\mathbb{C}^\star_{be}\mathbb{C}^\star_{cd}\mathbb{D}^\star_{ce} + \mathbb{C}^\star_{bc}\mathbb{C}^\star_{bd}\mathbb{C}^\star_{be}\mathbb{C}^\star_{cd}\mathbb{C}^\star_{ce}\mathbb{D}^\star_{de}\Big\}; \end{aligned} \tag{14}$$

and the "iteration pattern" $[\rho_a; \rho_b, \rho_c, \rho_d, \rho_e]$ is defined by

$$\begin{aligned} \rho_a &= bb+cc+dd+ee+bc+bd+be+cd+ce+de, \\ \rho_b &= bb+bc+bd+be, \\ \rho_c &= bc+cc+cd+ce, \\ \rho_d &= bd+cd+dd+de, \\ \rho_e &= be+ce+de+ee; \end{aligned} \tag{15}$$

that satisfies the inequality

$$\rho_a \le \rho_b+\rho_c+\rho_d+\rho_e \le 2\rho_a.$$

By means of the induction principle on ρ_a, it can be shown that for any quintuple nonnegative integers $\{\rho_a; \rho_b, \rho_c, \rho_d, \rho_e\}$ satisfying the above inequality, they form an iteration pattern $[\rho_a; \rho_b, \rho_c, \rho_d, \rho_e]$; i.e., there exist nonnegative integers $\{bb, cc, dd, ee, bc, bd, be, cd, ce, de\}$ such that the linear system (15) holds.

In addition, we can check the following quotient expression of shifted factorials

$$\begin{aligned} H\begin{bmatrix} a; & b, & c, & d, & e \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix} &= \frac{(-1)^{\rho_a+\rho_b+\rho_c+\rho_d+\rho_e}(b)_{\rho_b}(c)_{\rho_c}(d)_{\rho_d}(e)_{\rho_e}}{(1+2a-b-c-d-e)_{2\rho_a-\rho_b-\rho_c-\rho_d-\rho_e}} \\ &\times \frac{(1+a-b-c)_{\rho_a-\rho_b-\rho_c}(1+a-b-d)_{\rho_a-\rho_b-\rho_d}(1+a-b-e)_{\rho_a-\rho_b-\rho_e}}{(1+a-b)_{\rho_a-\rho_b}(1+a-c)_{\rho_a-\rho_c}} \\ &\times \frac{(1+a-c-d)_{\rho_a-\rho_c-\rho_d}(1+a-c-e)_{\rho_a-\rho_c-\rho_e}(1+a-d-e)_{\rho_a-\rho_d-\rho_e}}{(1+a-d)_{\rho_a-\rho_d}(1+a-e)_{\rho_a-\rho_e}}. \end{aligned} \tag{16}$$

There are the following connections between iteration patterns and the corresponding transformations.

- For the fixed nonnegative integers $[\rho_a;\rho_b,\rho_c,\rho_d,\rho_e]$, they will become the iteration pattern of a transformation if and only if there exists a solution of the linear system (15) in nonnegative integers, which is, in turn, equivalent to the condition $\rho_a\le\rho_b+\rho_c+\rho_d+\rho_e\le 2\rho_a$.
- Different solutions of the system (15) determine formally different transformations (12). However, they are substantially equivalent because there is a unique difference equation of $\Omega(a;b,c,d,e)$ for the fixed pattern $[\rho_a;\rho_b,\rho_c,\rho_d,\rho_e]$.
- According to the explicit expression (16) of the function $H\begin{bmatrix}a; & b, & c, & d, & e\\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e\end{bmatrix}$, it becomes obvious that this function depends essentially upon the iteration pattern $[\rho_a;\rho_b,\rho_c,\rho_d,\rho_e]$, but not individual solutions.

By iterating n-times the relation displayed in (12), we derive the following partial sum expression

$$\begin{aligned}\Omega(a;b,c,d,e) &= \sum_{k=0}^{n-1} W_k\begin{bmatrix}a; & b, & c, & d, & e\\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e\end{bmatrix} H_k\begin{bmatrix}a; & b, & c, & d, & e\\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e\end{bmatrix}\\ &+\Omega(a+n\rho_a;b+n\rho_b,c+n\rho_c,d+n\rho_d,e+n\rho_e)H_n\begin{bmatrix}a; & b, & c, & d, & e\\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e\end{bmatrix},\end{aligned} \tag{17}$$

where

$$H_k\begin{bmatrix}a; & b, & c, & d, & e\\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e\end{bmatrix} = H\begin{bmatrix}a; & b, & c, & d, & e\\ k\rho_a; & k\rho_b, & k\rho_c, & k\rho_d, & k\rho_e\end{bmatrix}, \tag{18}$$

$$W_k\begin{bmatrix}a; & b, & c, & d, & e\\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e\end{bmatrix} = W\begin{bmatrix}a+k\rho_a; & b+k\rho_b, & c+k\rho_c, & d+k\rho_d, & e+k\rho_e\\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e\end{bmatrix}. \tag{19}$$

For an integer variable x, let the δ-function be given by

$$\delta(0)=1 \quad\text{and}\quad \delta(x)=x^x \quad\text{for}\quad x\neq 0.$$

Define further a quotient associated with an iteration pattern $[\rho_a;\rho_b,\rho_c,\rho_d,\rho_e]$ by

$$\begin{aligned}\Delta[\rho_a;\rho_b,\rho_c,\rho_d,\rho_e] &= (-1)^{\rho_a+\rho_b+\rho_c+\rho_d+\rho_e}\frac{\delta(\rho_b)\delta(\rho_c)\delta(\rho_d)\delta(\rho_e)}{\delta(2\rho_a-\rho_b-\rho_c-\rho_d-\rho_e)}\\ &\times\frac{\begin{Bmatrix}\delta(\rho_a-\rho_b-\rho_c)\delta(\rho_a-\rho_b-\rho_d)\delta(\rho_a-\rho_b-\rho_e)\\ \delta(\rho_a-\rho_c-\rho_d)\delta(\rho_a-\rho_c-\rho_e)\delta(\rho_a-\rho_d-\rho_e)\end{Bmatrix}}{\delta(\rho_a-\rho_b)\delta(\rho_a-\rho_c)\delta(\rho_a-\rho_d)\delta(\rho_a-\rho_e)}.\end{aligned} \tag{20}$$

By means of the Weierstrass M-test on uniformly convergent series, the rightmost term displayed in (17) is asymptotically determined by

$$H_n\begin{bmatrix} a; & b, & c, & d, & e \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix} \sim \mathcal{O}\Big(\Delta[\rho_a;\rho_b,\rho_c,\rho_d,\rho_e]\Big)^n \quad \text{as} \quad n\to\infty.$$

Consequently, Theorem A follows from the limiting case $n \to \infty$ of (17) under the convergence conditions $\Re(1+2a-b-c-d-e) > 0$ and $\big|\Delta[\rho_a;\rho_b,\rho_c,\rho_d,\rho_e]\big| < 1$. □

3 Classified infinite series identities

Further applications of Theorem A will be investigated, in this section, for detecting infinite series identities. This will be fulfilled almost automatically by an appropriately devised *Mathematica* package, which is described as follows.

- Fixing the parameters $\{a, b, c, d, e\}$, evaluate the corresponding $\Omega(a; b, c, d, e)$ in terms of the Γ-function quotient by Dougall's theorem on the well-poised ${}_5F_4$-series and the limiting relations appearing in Chu and Zhang [42, §2.5 and Corollary 6].
- Specifying the iteration pattern $[\rho_a;\rho_b,\rho_c,\rho_d,\rho_e]$, determine, by (20), the convergence rate $\Delta[\rho_a;\rho_b,\rho_c,\rho_d,\rho_e]$ for the corresponding infinite series.
- Factorizing the rational weight function $W_k\begin{bmatrix} a; & b, & c, & d, & e \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix}$, identify its largest irreducible polynomial factor and all the linear factors.
- Writing $H_k\begin{bmatrix} a; & b, & c, & d, & e \\ \rho_a; & \rho_b, & \rho_c, & \rho_d, & \rho_e \end{bmatrix}$ in terms of canonical shifted factorials of order k (but not its multiples), combine them with the linear factors extracted from the weight function W_k together.
- Display the resulting identity. In particular, when the Γ-function quotient is expressible in terms of $\pi^{\pm 1}$, $\pi^{\pm 2}$, $\zeta(3)$ and other known mathematical constants, write down the corresponding infinite series formulae.

Following this procedure, we have made a comprehensive exploration of infinite series identities. The results are displayed as examples in nine classes and classified according to the represented mathematical constants, that cover not only numerous known infinite series, but also several new important formulae including a few challenging conjectured series made experimentally by Guillera [55] and Sun [62].

In order to illustrate the approach, three examples are worked out in detail.

[Example 23] In Theorem A, let

$$\left\{ \begin{matrix} a=1, & b=\frac{1}{3}, & c=1, & d=\frac{2}{3}, & e=\frac{1}{2} \\ \rho_a=2, & \rho_b=0, & \rho_c=1, & \rho_d=1, & \rho_e=1 \end{matrix} \right\}.$$

According to Dougall's summation theorem, we can evaluate

$$\Omega\Big(1;\frac{1}{3},1,\frac{2}{3},\frac{1}{2}\Big)=\sum_{k=0}^{\infty}(1+2k)\begin{bmatrix}1,\frac{1}{2},\frac{1}{3},\frac{2}{3}\\ \\ 1,\frac{3}{2},\frac{5}{3},\frac{4}{3}\end{bmatrix}_k=\Gamma\begin{bmatrix}\frac{1}{2},\frac{3}{2},\frac{4}{3},\frac{5}{3}\\ \\ 1,2,\ \frac{5}{6},\frac{7}{6}\end{bmatrix}=\frac{2\pi}{3\sqrt{3}}.$$

By means of (16) and (18), we have

$$H_k\begin{bmatrix}1;\frac{1}{3},1,\frac{2}{3},\frac{1}{2}\\ 2;\ 0,1,1,1\end{bmatrix}=\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}1,\frac{2}{3},\frac{2}{3},\frac{7}{6}\\ \\ \frac{3}{2},\frac{4}{3},\frac{4}{3},\frac{5}{6}\end{bmatrix}_k.$$

With the aid of *Mathematica* commands implemented from (14) and (19), we can also determine

$$W_k\begin{bmatrix}1;\frac{1}{3},1,\frac{2}{3},\frac{1}{2}\\ 2;\ 0,1,1,1\end{bmatrix}=\frac{7+21k+15k^2}{5+6k}.$$

Then the corresponding formula in Theorem A

$$\Omega\Big(1;\frac{1}{3},1,\frac{2}{3},\frac{1}{2}\Big)=\sum_{k=0}^{\infty}W_k\begin{bmatrix}1;\frac{1}{3},1,\frac{2}{3},\frac{1}{2}\\ 2;\ 0,1,1,1\end{bmatrix}H_k\begin{bmatrix}1;\frac{1}{3},1,\frac{2}{3},\frac{1}{2}\\ 2;\ 0,1,1,1\end{bmatrix}$$

is evidently equivalent to the identity displayed in Example 23:

$$\frac{10\pi}{3\sqrt{3}}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}1,&\frac{2}{3},&\frac{2}{3},&\frac{7}{6}\\ \\ \frac{3}{2},&\frac{4}{3},&\frac{4}{3},&\frac{11}{6}\end{bmatrix}_k\Big\{7+21k+15k^2\Big\}.$$

□

[Example 45] In Theorem A, let

$$\begin{Bmatrix}a=2,&b=1,&c=1,&d=1,&e=\frac{3}{2}\\ \rho_a=3,&\rho_b=1,&\rho_c=1,&\rho_d=1,&\rho_e=1\end{Bmatrix}.$$

We can analogously evaluate

$$\Omega\Big(2;1,1,1,\frac{3}{2}\Big)=\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}=\frac{\pi^2}{6},$$

$$H_k\begin{bmatrix}2;1,1,1,\frac{3}{2}\\ 3;\ 1,1,1,1\end{bmatrix}=\Big(\frac{-1}{1024}\Big)^k\begin{bmatrix}1,1,1,\ \frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \\ \frac{3}{2},\frac{3}{2},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{5}{4}\end{bmatrix}_k,$$

$$W_k\begin{bmatrix}2;1,1,1,\frac{3}{2}\\ 3;\ 1,1,1,1\end{bmatrix}=\frac{(1+2k)^2(237+623k+410k^2)}{16(1+4k)(3+4k)^2}.$$

Then the corresponding formula in Theorem A

$$\Omega(2;\,1,\,1,\,1,\,\frac{3}{2}) = \sum_{k=0}^{\infty} W_k \begin{bmatrix} 2;\,1,\,1,\,1,\,\frac{3}{2} \\ 3;\,1,\,1,\,1,\,1 \end{bmatrix} H_k \begin{bmatrix} 2;\,1,\,1,\,1,\,\frac{3}{2} \\ 3;\,1,\,1,\,1,\,1 \end{bmatrix}$$

becomes, after some simplification, the identity displayed in Example 45:

$$24\pi^2 = \sum_{k=0}^{\infty} \Big(\frac{-1}{1024}\Big)^k \begin{bmatrix} 1, & 1, & 1, & \frac{1}{2} \\ \frac{5}{4}, & \frac{5}{4}, & \frac{7}{4}, & \frac{7}{4} \end{bmatrix}_k \Big\{237 + 623k + 410k^2\Big\}.$$

□

[Example 67]. In Theorem A, let

$$\left\{ \begin{matrix} a = 1, & b = \frac{1}{2}, & c = \frac{1}{2}, & d = \frac{1}{2}, & e = \frac{1}{2} \\ \rho_a = 3, & \rho_b = 0, & \rho_c = 1, & \rho_d = 1, & \rho_e = 1 \end{matrix} \right\}.$$

In this case, we have

$$\Omega\Big(1;\,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\Big) = \sum_{k=0}^{\infty} \frac{1}{(1+2k)^3} = \frac{7}{8}\zeta(3),$$

$$H_k \begin{bmatrix} 1;\,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 3;\;0,1,1,1 \end{bmatrix} = \Big(\frac{1}{729}\Big)^k \begin{bmatrix} 1, & 1, & 1, & 1, & 1 \\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{2}{3}, & \frac{4}{3} \end{bmatrix}_{2k},$$

$$W_k \begin{bmatrix} 1;\,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 3;\;0,1,1,1 \end{bmatrix} = \frac{P(k)}{36(1+3k)(2+3k)(3+4k)^3(5+6k)};$$

where $P(k)$ is a polynomial of degree 7 given by

$$P(k) = 10214+128199k+671402k^2+1904368k^3+3167648k^4+3098368k^5+1654272k^6+372736k^7.$$

Then the corresponding formula in Theorem A

$$\Omega\Big(1;\,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\Big) = \sum_{k=0}^{\infty} W_k \begin{bmatrix} 1;\,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 3;\;0,1,1,1 \end{bmatrix} H_k \begin{bmatrix} 1;\,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 3;\;0,1,1,1 \end{bmatrix}$$

results in the following series:

$$8505\zeta(3) = \sum_{k=0}^{\infty} \begin{bmatrix} 1, & 1, & 1, & 1, & 1 \\ \frac{5}{2}, & \frac{5}{2}, & \frac{5}{2}, & \frac{7}{3}, & \frac{8}{3} \end{bmatrix}_{2k} \frac{P(k)}{729^k}.$$

This series can be simplified futher. In fact, for the Λ-sequence defined by

$$\Lambda(k) := \begin{bmatrix} 1, & 1, & 1, & 1, & 1 \\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3} \end{bmatrix}_k \frac{13+38k+28k^2}{(-27)^k},$$

it is not hard to verify that the summand can be expressed as

$$\begin{bmatrix} 1, & 1, & 1, & 1, & 1 \\ \frac{5}{2}, & \frac{5}{2}, & \frac{5}{2}, & \frac{7}{3}, & \frac{8}{3} \end{bmatrix}_{2k} \frac{P(k)}{729^k} = 810\Big\{\Lambda(2k) + \Lambda(2k+1)\Big\}.$$

In view of the bisection series, we conclude that

$$\sum_{k\geq 0} \Lambda(k) = \sum_{k\geq 0} \Big\{\Lambda(2k) + \Lambda(2k+1)\Big\} = \frac{8505}{810}\zeta(3) = \frac{21}{2}\zeta(3),$$

which justifies the infinite series for $\zeta(3)$ displayed in Example 67:

$$\frac{21}{2}\zeta(3) = \sum_{k=0}^{\infty} \begin{bmatrix} 1, & 1, & 1, & 1, & 1 \\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3} \end{bmatrix}_k \frac{13+38k+28k^2}{(-27)^k}.$$

□

As shown above, all the series manipulations involved are entirely routine. Therefore for the other series in this section, their proofs will not be produced. In order to be concise, we shall indicate, in the header for each example, eventual references and how to derive the formula by $\Omega[\begin{smallmatrix} \rho_a \rho_b \rho_c \rho_d \rho_e \\ a\ b\ c\ d\ e \end{smallmatrix}]$, where $[a\,b\,c\,d\,e]$ and $[\rho_a \rho_b \rho_c \rho_d \rho_e]$ stand respectively for the parameter setting and the iteration pattern in the Ω-series.

3.1 Series for π^{-1}

Here we record 15 infinite series identities with four of them due to Ramanujan [58]. For different proofs of these identities and more formulae for $1/\pi$, refer to Andrews and Berndt [6, §15.6], Baruah and Berndt [16,18], Berndt et al. [20,21], Borwein et al. [24–27,30], Chan et al. [34–36], Guillera [50,51], Rogers [60] and Scarpello and Ritelli [61].

Example 1 ($\Omega[\begin{smallmatrix} 2\,0\,1\,1\,1 \\ \frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\infty \end{smallmatrix}]$ Ramanujan [58, Eq. 28])

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} \Big(\frac{1}{4}\Big)^k \begin{bmatrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 1, & 1, & 1 \end{bmatrix}_k \Big\{1+6k\Big\}.$$

Example 2 ($\Omega\left[\begin{smallmatrix}2 & 0 & 1 & 1 & 1\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4}\end{smallmatrix}\right]$ Ramanujan [58, Eq. 35])

$$\frac{8}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ 1, & 1, & 1 \end{bmatrix}_k \Big\{3 + 20k\Big\}.$$

Example 3 ($\Omega\left[\begin{smallmatrix}3 & 1 & 1 & 1 & 1\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \infty\end{smallmatrix}\right]$ Guillera [51, Table 2:3])

$$\frac{2\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{8}\right)^k \begin{bmatrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 1, & 1, & 1 \end{bmatrix}_k \Big\{1 + 6k\Big\}.$$

Example 4 ($\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 1\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{6} & \frac{5}{6}\end{smallmatrix}\right]$ Trisection series: Ramanujan [58, Eq. 40])

$$\frac{2\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ 1, & 1, & 1 \end{bmatrix}_k \frac{1 + 8k}{9^k}.$$

Example 5 ($\Omega\left[\begin{smallmatrix}2 & 0 & 1 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{2}{3} & \frac{1}{3}\end{smallmatrix}\right]$)

$$\frac{15\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}, & \frac{1}{3}, & \frac{1}{3}, & \frac{2}{3}, & \frac{2}{3} \\ 1, & 1, & 1, & \frac{11}{12}, & \frac{17}{12} \end{bmatrix}_k \frac{8 + 75k + 135k^2}{16^k}.$$

Example 6 ($\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 1\\ \frac{5}{6} & \frac{5}{6} & \frac{5}{6} & \frac{5}{6} & \frac{2}{3}\end{smallmatrix}\right]$ Bisection series)

$$\frac{9\sqrt{3}}{2^{4/3}\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{3}, & \frac{2}{3}, & \frac{1}{6} \\ 1, & 1, & 1 \end{bmatrix}_k \frac{2 + 21k}{(-27)^k}.$$

Example 7 ($\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 1\\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3}\end{smallmatrix}\right]$ Bisection series)

$$\frac{27\sqrt{3}}{2^{5/3}\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{3}, & \frac{2}{3}, & \frac{5}{6} \\ 1, & 1, & 1 \end{bmatrix}_k \frac{5 + 42k}{(-27)^k}.$$

Example 8 ($\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 1\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{smallmatrix}\right]$ Bisection series)

$$\frac{3}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 1, & 1, & 1, & \frac{5}{6}, & \frac{7}{6} \end{bmatrix}_k \frac{1 + 10k + 28k^2}{(-27)^k}.$$

Example 9 $\left(\Omega\left[\begin{smallmatrix}3&0&1&1&2\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{4}&\frac{3}{4}&\frac{1}{2}\end{smallmatrix}\right]\right)$

$$\frac{64}{\pi}=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac{1}{2}, & \frac{1}{4}, & \frac{1}{4}, & \frac{3}{4}, & \frac{3}{4}\\ 1, & 1, & 1, & \frac{4}{3}, & \frac{5}{3}\end{matrix}\right]_k \frac{21+296k+992k^2+896k^3}{(-27)^k}.$$

Example 10 $\left(\Omega\left[\begin{smallmatrix}3&0&1&1&2\\ \frac{2}{3}&\frac{2}{3}&\frac{2}{3}&\frac{2}{3}&\infty\end{smallmatrix}\right]\right)$

$$\frac{243\sqrt{3}}{8\pi}=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac{2}{3}, & \frac{2}{3}, & \frac{2}{3}, & \frac{1}{6}, & \frac{1}{6}\\ 1, & 1, & 1, & \frac{3}{2}, & \frac{3}{2}\end{matrix}\right]_k \frac{17+279k+864k^2+756k^3}{(-27)^k}.$$

Example 11 $\left(\Omega\left[\begin{smallmatrix}3&0&1&1&2\\ \frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\infty\end{smallmatrix}\right]\right)$

$$\frac{243\sqrt{3}}{4\pi}=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & \frac{5}{6}, & \frac{5}{6}\\ 1, & 1, & 1, & \frac{3}{2}, & \frac{3}{2}\end{matrix}\right]_k \frac{35+504k+1620k^2+1512k^3}{(-27)^k}.$$

Example 12 $\left(\Omega\left[\begin{smallmatrix}3&0&1&1&2\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\infty&\frac{1}{2}\end{smallmatrix}\right]\right)$

$$\frac{32}{\pi}=\sum_{k=0}^{\infty}\left(\frac{4}{27}\right)^k\left[\begin{matrix}\frac{1}{2}, & \frac{1}{4}, & \frac{1}{4}, & \frac{3}{4}, & \frac{3}{4}\\ 1, & 1, & 1, & \frac{4}{3}, & \frac{5}{3}\end{matrix}\right]_k \left\{9+118k+400k^2+368k^3\right\}.$$

Example 13 $\left(\Omega\left[\begin{smallmatrix}3&0&2&0&2\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\infty&\frac{1}{2}\end{smallmatrix}\right]\right)$

$$\frac{16}{\pi}=\sum_{k=0}^{\infty}\left(\frac{16}{27}\right)^k\left[\begin{matrix}\frac{1}{2}, & \frac{1}{4}, & \frac{1}{4}, & \frac{3}{4}, & \frac{3}{4}\\ 1, & 1, & 1, & \frac{4}{3}, & \frac{5}{3}\end{matrix}\right]_k \left\{3+36k+108k^2+88k^3\right\}.$$

Example 14 $\left(\Omega\left[\begin{smallmatrix}3&0&1&2&2\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&\frac{2}{3}\end{smallmatrix}\right]\right)$

$$\frac{27\sqrt{3}}{\pi}=\sum_{k=0}^{\infty}\left(-\frac{16}{27}\right)^k\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3}, & \frac{1}{4}, & \frac{3}{4}\\ 1, & 1, & 1, & \frac{3}{2}\end{matrix}\right]_k \left\{20+167k+258k^2\right\}.$$

Example 15 $\left(\Omega\left[\begin{smallmatrix}3&1&1&1&1\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\infty\end{smallmatrix}\right]\right.$ Ramanujan [58, Eq. 29])

$$\frac{16}{\pi}=\sum_{k=0}^{\infty}\left(\frac{1}{64}\right)^k\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ 1, & 1, & 1\end{matrix}\right]_k \left\{5+42k\right\}.$$

3.2 Series for π

Example 16 $\left(\Omega\begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 1 & \frac{2}{3} & 1 & \frac{1}{3} & \infty \end{bmatrix}\right)$

$$\frac{16\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \begin{bmatrix} 1, & \frac{1}{3}, & \frac{1}{3} \\ \frac{5}{3}, & \frac{5}{3}, & \frac{7}{6} \end{bmatrix}_k \Big\{3 + 10k + 9k^2\Big\}.$$

Example 17 $\left(\Omega\begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 1 & \frac{1}{3} & 1 & \frac{2}{3} & \infty \end{bmatrix}\right)$

$$\frac{20\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \begin{bmatrix} 1, & \frac{2}{3}, & \frac{2}{3} \\ \frac{4}{3}, & \frac{4}{3}, & \frac{11}{6} \end{bmatrix}_k \Big\{3 + 11k + 9k^2\Big\}.$$

Example 18 $\left(\Omega\begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 1 & \frac{3}{4} & 1 & \frac{1}{4} & \infty \end{bmatrix}\right)$

$$\frac{15\pi}{8\sqrt{2}} = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \begin{bmatrix} 1, & \frac{1}{4}, & \frac{1}{4} \\ \frac{7}{4}, & \frac{9}{8}, & \frac{13}{8} \end{bmatrix}_k \Big\{4 + 13k + 12k^2\Big\}.$$

Example 19 $\left(\Omega\begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 1 & \frac{1}{4} & 1 & \frac{3}{4} & \infty \end{bmatrix}\right)$

$$\frac{21\pi}{8\sqrt{2}} = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \begin{bmatrix} 1, & \frac{3}{4}, & \frac{3}{4} \\ \frac{5}{4}, & \frac{11}{8}, & \frac{15}{8} \end{bmatrix}_k \Big\{4 + 15k + 12k^2\Big\}.$$

Example 20 $\left(\Omega\begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & \frac{1}{6} & \frac{5}{6} & \infty \end{bmatrix}\right)$

$$2\pi = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \begin{bmatrix} \frac{1}{6}, & \frac{5}{6} \\ 1, & \frac{3}{2} \end{bmatrix}_k \frac{31 + 108k + 108k^2}{(1+6k)(5+6k)}.$$

Example 21 $\left(\Omega\begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 1 & \frac{1}{3} & 1 & \frac{2}{3} & \infty \end{bmatrix}\right)$

$$\frac{40\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} 1, & \frac{1}{2} \\ \frac{7}{6}, & \frac{11}{6} \end{bmatrix}_k \Big\{9 + 10k\Big\}.$$

Example 22 $(\Omega\left[\begin{smallmatrix}2&0&1&1&1\\1&1&\frac{1}{2}&\frac{1}{4}&\frac{3}{4}\end{smallmatrix}\right])$

$$2\pi = \sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k \begin{bmatrix}\frac{1}{4}, & \frac{3}{4}\\ 1, & \frac{3}{2}\end{bmatrix}_k \frac{19+80k+80k^2}{(1+2k)(1+4k)(3+4k)}.$$

Example 23 $(\Omega\left[\begin{smallmatrix}2&0&1&1&1\\1&\frac{1}{3}&1&\frac{2}{3}&\frac{1}{2}\end{smallmatrix}\right])$

$$\frac{10\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k \begin{bmatrix}1, & \frac{2}{3}, & \frac{2}{3}, & \frac{7}{6}\\ \frac{3}{2}, & \frac{4}{3}, & \frac{4}{3}, & \frac{11}{6}\end{bmatrix}_k \Big\{7+21k+15k^2\Big\}.$$

Example 24 $(\Omega\left[\begin{smallmatrix}3&0&1&1&1\\1&\frac{1}{2}&1&\frac{1}{3}&\frac{2}{3}\end{smallmatrix}\right]$ Trisection series: Chu [41, Example 19])

$$\pi\sqrt{3} = \sum_{k=0}^{\infty} \begin{bmatrix}1, & \frac{3}{4}, & \frac{5}{4}\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k \frac{5+8k}{9^k}.$$

Example 25 $(\Omega\left[\begin{smallmatrix}3&0&1&1&2\\1&\frac{1}{2}&\frac{1}{4}&\frac{3}{4}&1\end{smallmatrix}\right])$

$$\frac{15\pi}{8} = \sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k \begin{bmatrix}1, & \frac{1}{2}\\ \frac{7}{6}, & \frac{11}{6}\end{bmatrix}_k \Big\{6+7k\Big\}.$$

Example 26 $(\Omega\left[\begin{smallmatrix}4&1&1&1&1\\1&1&\frac{1}{2}&\frac{1}{4}&\frac{3}{4}\end{smallmatrix}\right]$ Quartic section series: Zhang [67, Example 8])

$$\pi = \sum_{k=0}^{\infty}\Big(\frac{2}{27}\Big)^k \begin{bmatrix}1, & \frac{1}{2}\\ \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k \Big\{3+5k\Big\}.$$

Example 27 $(\Omega\left[\begin{smallmatrix}3&0&1&1&2\\1&\frac{2}{3}&\frac{1}{3}&\frac{1}{2}&1\end{smallmatrix}\right])$

$$\frac{56\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k \begin{bmatrix}1, & \frac{1}{3}, & \frac{1}{4}, & \frac{3}{4}\\ \frac{3}{2}, & \frac{10}{9}, & \frac{13}{9}, & \frac{16}{9}\end{bmatrix}_k \Big\{36+143k+129k^2\Big\}.$$

Example 28 $(\Omega\left[\begin{smallmatrix}3&1&1&1&1\\1&1&\frac{1}{3}&\frac{2}{3}&\infty\end{smallmatrix}\right])$

$$\frac{40\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix}1, & \frac{1}{3}, & \frac{2}{3}\\ \frac{3}{2}, & \frac{7}{6}, & \frac{11}{6}\end{bmatrix}_k \frac{8+27k+21k^2}{64^k}.$$

Example 29 $\left(\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 1\\ 1 & \infty & \frac{1}{2} & \frac{1}{2} & 1\end{smallmatrix}\right]\right)$

$$\frac{9\pi}{4}=\sum_{k=0}^{\infty}\left[\begin{matrix}1, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ \frac{5}{4}, & \frac{5}{4}, & \frac{7}{4}, & \frac{7}{4}\end{matrix}\right]_k \frac{7+42k+75k^2+42k^3}{64^k}.$$

Example 30 $\left(\Omega\left[\begin{smallmatrix}3 & 1 & 1 & 1 & 3\\ 1 & \frac{1}{3} & 1 & \frac{2}{3} & \frac{1}{2}\end{smallmatrix}\right]\right)$

$$\frac{80\pi}{3\sqrt{3}}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\left[\begin{matrix}1, & \frac{1}{3}, & \frac{2}{3}\\ \frac{3}{2}, & \frac{7}{6}, & \frac{11}{6}\end{matrix}\right]_k \Big\{36+133k+111k^2\Big\}.$$

3.3 Series for π^2

Example 31 $\left(\Omega\left[\begin{smallmatrix}2 & 0 & 1 & 0 & 1\\ 1 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2}\end{smallmatrix}\right]\right.$ Bisection series: Guillera [53, §2.1: $a=\frac{1}{2}$])

$$\frac{\pi^2}{4}=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\left[\begin{matrix}1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{matrix}\right]_k \Big\{2+3k\Big\}.$$

Example 32 $\left(\Omega\left[\begin{smallmatrix}2 & 0 & 1 & 1 & 1\\ 1 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2}\end{smallmatrix}\right]\right.$ Chu [40, Theorem 3.1] with $\left[\begin{smallmatrix}1 & 0 & 1 & 2\\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2}\end{smallmatrix}\right]$)

$$\frac{3\pi^2}{8}=\sum_{k=0}^{\infty}\left[\begin{matrix}1, & 1, & \frac{1}{2}\\ \frac{3}{2}, & \frac{5}{4}, & \frac{7}{4}\end{matrix}\right]_k \frac{4+5k}{(-4)^k}.$$

Example 33 $\left(\Omega\left[\begin{smallmatrix}2 & 0 & 1 & 0 & 1\\ 2 & 1 & 1 & 1 & \infty\end{smallmatrix}\right]\right.$ Chu [40, Theorem 3.1] with $\left[\begin{smallmatrix}1 & 0 & 1 & 2\\ 1 & 1 & 2 & 2\end{smallmatrix}\right]$)

$$\frac{2\pi^2}{3}=\sum_{k=0}^{\infty}\left[\begin{matrix}1, & 1, & \frac{1}{2}\\ 2, & \frac{3}{2}, & \frac{3}{2}\end{matrix}\right]_k \frac{7+10k}{(-4)^k}.$$

Example 34 $\left(\Omega\left[\begin{smallmatrix}2 & 0 & 1 & 1 & 1\\ 2 & \frac{3}{2} & 1 & 1 & 1\end{smallmatrix}\right]\right)$

$$\frac{\pi^2}{2}=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ \frac{3}{2}, & \frac{5}{4}, & \frac{7}{4}\end{matrix}\right]_k \frac{5+14k+10k^2}{(-4)^k\,(k+1)^2}.$$

Example 35 $\left(\Omega\left[\begin{smallmatrix}2 & 0 & 1 & 0 & 1\\ \frac{3}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 1\end{smallmatrix}\right]\right.$ Guillera [53, §3.3: $a=\frac{1}{2}$])

$$\frac{8\pi^2}{3}=\sum_{k=0}^{\infty}\left[\begin{matrix}1, & 1, & 1, & \frac{3}{4}, & \frac{5}{4}\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{matrix}\right]_k \frac{25+77k+60k^2}{16^k}.$$

Example 36 $\left(\Omega\left[\begin{smallmatrix}3&0&1&1&2\\2&1&1&1&\frac{3}{2}\end{smallmatrix}\right]\right)$

$$\frac{\pi^2}{2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}1, & 1\\ \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k\frac{5+7k}{1+2k}.$$

Example 37 $\left(\Omega\left[\begin{smallmatrix}3&0&1&1&2\\\frac{3}{2}&1&\frac{1}{2}&\frac{1}{2}&1\end{smallmatrix}\right]\right)$

$$10\pi^2=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}\frac{1}{2}, & \frac{1}{2}\\ \frac{7}{6}, & \frac{11}{6}\end{bmatrix}_k\frac{99+314k+328k^2+112k^3}{(1+k)^2(1+2k)}.$$

Example 38 (Sun's conjecture [62, Eq. 1.1])

$$\frac{3\pi^2}{4}=\sum_{k=0}^{\infty}\Big(\frac{2}{27}\Big)^k\begin{bmatrix}1, & 1, & 1\\ \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k\Big\{7+10k\Big\}.$$

This is confirmed, via bisection series, by applying Theorem 3.1 of Chu [40] *under the parameter setting* $\big[1,1,2,3\mid\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2}\big]$.

Example 39 $\left(\Omega\left[\begin{smallmatrix}3&0&2&0&2\\\frac{3}{2}&\frac{1}{2}&1&\frac{1}{2}&1\end{smallmatrix}\right]\right)$

$$4\pi^2=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}1, & 1, & 1, & \frac{5}{6}, & \frac{7}{6}\\ \frac{3}{2}, & \frac{4}{3}, & \frac{4}{3}, & \frac{5}{3}, & \frac{5}{3}\end{bmatrix}_k\Big\{35+98k+69k^2\Big\}.$$

Example 40 $\left(\Omega\left[\begin{smallmatrix}3&0&2&0&2\\\frac{3}{2}&\frac{1}{2}&1&\infty&1\end{smallmatrix}\right]\right.$ Zhang [67, Example 1]: Sun's conjecture [62, Eq. 1.2])

$$\frac{3\pi^2}{2}=\sum_{k=0}^{\infty}\Big(\frac{16}{27}\Big)^k\begin{bmatrix}1, & 1, & 1\\ \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k\Big\{8+11k\Big\}.$$

Example 41 $\left(\Omega\left[\begin{smallmatrix}3&0&1&2&2\\2&1&1&1&\frac{3}{2}\end{smallmatrix}\right]\right)$

$$4\pi^2=\sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k\begin{bmatrix}1, & \frac{3}{4}, & \frac{5}{4}\\ \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k\frac{45+124k+86k^2}{(1+k)(1+2k)}.$$

Example 42 $\left(\Omega\left[\begin{smallmatrix}3&0&1&2&2\\1&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&1\end{smallmatrix}\right]\right)$

$$\frac{15\pi^2}{8}=\sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k\begin{bmatrix}1, & 1\\ \frac{7}{6}, & \frac{11}{6}\end{bmatrix}_k\frac{64+375k+646k^2+344k^3}{(1+2k)(1+4k)(3+4k)}.$$

Example 43 ($\Omega\left[\begin{smallmatrix}3111\,1\\2111\infty\end{smallmatrix}\right]$ Guillera [53, §2.2: $a=\frac{1}{2}$])

$$\frac{4\pi^2}{3}=\sum_{k=0}^{\infty}\Big(\frac{1}{64}\Big)^k\begin{bmatrix}1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\Big\{13+21k\Big\}.$$

This is rederived from Theorem 3.1 *of Chu* [40] *under the parameter setting* $\left[1122 \mid 1122\right]$.

Example 44 ($\Omega\left[\begin{smallmatrix}3011\,3\\2111\,\frac{3}{2}\end{smallmatrix}\right]$ Guillera [54, Eq. 18])

$$\frac{16\pi^2}{3}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}1, & 1, & 1, & \frac{5}{6}, & \frac{7}{6}\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\Big\{35+101k+74k^2\Big\}.$$

Example 45 ($\Omega\left[\begin{smallmatrix}3111\,1\\2111\,\frac{3}{2}\end{smallmatrix}\right]$)

$$24\pi^2=\sum_{k=0}^{\infty}\Big(\frac{-1}{1024}\Big)^k\begin{bmatrix}1, & 1, & 1, & \frac{1}{2}\\ \frac{5}{4}, & \frac{5}{4}, & \frac{7}{4}, & \frac{7}{4}\end{bmatrix}_k\Big\{237+623k+410k^2\Big\}.$$

3.4 Series for π^{-2}

Guillera [51–53] developed intensively the *computer algebra* approach via the *WZ*-method, reviewed systematically the formulae discovered by Ramanujan [58], found further π-formulae of Ramanujan-type and detected experimentally several beautiful and challenging series representations for $1/\pi^2$ (cf. [47–49]). More formulae of similar type can be found in Bailey et al. [15, §2.7], Baruah and Berndet [17], Chu [39] and Zudilin [69,70].

Example 46 ($\Omega\left[\begin{smallmatrix}2&0&1&1&1\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{smallmatrix}\right]$ Guillera [48, Eqs. 1–3])

$$\frac{8}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ 1, & 1, & 1, & 1, & 1\end{bmatrix}_k\Big\{1+8k+20k^2\Big\}.$$

Example 47 ($\Omega\left[\begin{smallmatrix}2&0&1&1&1\\\frac{3}{2}&\frac{1}{2}&\frac{3}{2}&\frac{1}{2}&\frac{1}{2}\end{smallmatrix}\right]$)

$$\frac{256}{\pi^2}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{3}{2}, & \frac{3}{2}\\ 1, & 2, & 2, & 2, & 2\end{bmatrix}_k\frac{27+94k+108k^2+40k^3}{(-4)^k}.$$

Example 48 ($\Omega\left[\begin{smallmatrix}2&0&1&0&1\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{smallmatrix}\right]$ Guillera [48, Eqs. 1–2])

$$\frac{32}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4}\\ 1, & 1, & 1, & 1, & 1\end{matrix}\right]_k\Big\{3+34k+120k^2\Big\}.$$

Example 49 ($\Omega\left[\begin{smallmatrix}2&0&1&0&1\\ \frac{3}{2}&\frac{3}{2}&\frac{1}{2}&\frac{3}{2}&\frac{1}{2}\end{smallmatrix}\right]$)

$$\frac{128}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & -\frac{1}{4}, & \frac{1}{4}\\ 1, & 1, & 1, & 2, & 2\end{matrix}\right]_k\Big\{13+118k+120k^2\Big\}.$$

Example 50 ($\Omega\left[\begin{smallmatrix}3&1&1&1&3\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&0\end{smallmatrix}\right]$ Guillera [54, Eq. 17])

$$\frac{48}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{3}, & \frac{2}{3}\\ 1, & 1, & 1, & 1, & 1\end{matrix}\right]_k\Big\{3+27k+74k^2\Big\}.$$

Example 51 ($\Omega\left[\begin{smallmatrix}3&1&1&1&3\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&1\end{smallmatrix}\right]$)

$$\frac{64}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{3}, & \frac{2}{3}\\ 1, & 1, & 2, & 2, & 2\end{matrix}\right]_k\Big\{6+69k+135k^2+74k^3\Big\}.$$

Example 52 ($\Omega\left[\begin{smallmatrix}3&1&1&1&1\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{smallmatrix}\right]$ Guillera [48, Eq. 1-1])

$$\frac{128}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{1024}\Big)^k\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ 1, & 1, & 1, & 1, & 1\end{matrix}\right]_k\Big\{13+180k+820k^2\Big\}.$$

3.5 Series for Catalan constant *G*

Example 53 ($\Omega\left[\begin{smallmatrix}2&0&1&1&1\\ 1&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\infty\end{smallmatrix}\right]$)

$$6G=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\left[\begin{matrix}1, & 1\\ \frac{5}{4}, & \frac{7}{4}\end{matrix}\right]_k\frac{5+6k}{1+2k}.$$

Example 54 ($\Omega\left[\begin{smallmatrix}2&0&1&0&1\\ 1&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\infty\end{smallmatrix}\right]$)

$$18G=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\left[\begin{matrix}1, & 1, & 1, & \frac{1}{2}\\ \frac{5}{4}, & \frac{5}{4}, & \frac{7}{4}, & \frac{7}{4}\end{matrix}\right]_k\Big\{19+56k+40k^2\Big\}.$$

Example 55 $\left(\Omega\left[\begin{smallmatrix}2 & 0 & 1 & 0 & 1\\ \frac{3}{2} & 1 & 1 & 1 & \infty\end{smallmatrix}\right]\right)$

$$6G=\sum_{k=0}^{\infty}\left(\frac{-1}{4}\right)^k\left[\begin{matrix}1, & \frac{1}{2}\\ \frac{5}{4}, & \frac{7}{4}\end{matrix}\right]_k\frac{17+94k+156k^2+80k^3}{(1+2k)(1+4k)(3+4k)}.$$

Example 56 $\left(\Omega\left[\begin{smallmatrix}3 & 1 & 1 & 1 & 3\\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \infty\end{smallmatrix}\right]\right.$ Bisection series: Guillera [53, §2.3: $a=\frac{1}{2}$])

$$2G=\sum_{k=0}^{\infty}\left(\frac{-1}{8}\right)^k\left[\begin{matrix}1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{matrix}\right]_k\Big\{2+3k\Big\}.$$

Example 57 $\left(\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 2\\ \frac{3}{2} & 1 & 1 & 1 & \infty\end{smallmatrix}\right]\right)$

$$30G=\sum_{k=0}^{\infty}\left(\frac{-1}{27}\right)^k\left[\begin{matrix}1, & 1\\ \frac{7}{6}, & \frac{11}{6}\end{matrix}\right]_k\frac{83+192k+112k^2}{(1+4k)(3+4k)}.$$

Example 58 $\left(\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 2\\ 1 & \frac{1}{2} & \frac{1}{2} & \infty & \frac{1}{2}\end{smallmatrix}\right]\right)$

$$30G=\sum_{k=0}^{\infty}\left(\frac{4}{27}\right)^k\left[\begin{matrix}1, & 1\\ \frac{7}{6}, & \frac{11}{6}\end{matrix}\right]_k\frac{79+428k+708k^2+368k^3}{(1+2k)(1+4k)(3+4k)}.$$

Example 59 $\left(\Omega\left[\begin{smallmatrix}3 & 0 & 2 & 0 & 2\\ \frac{3}{2} & 1 & 1 & 1 & \infty\end{smallmatrix}\right]\right)$

$$90G=\sum_{k=0}^{\infty}\left(\frac{-4}{27}\right)^k\left[\begin{matrix}1, & \frac{1}{2}\\ \frac{7}{6}, & \frac{11}{6}\end{matrix}\right]_k\frac{419+2610k+4404k^2+2232k^3}{(1+2k)(1+6k)(5+6k)}.$$

Example 60 $\left(\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 2 & 2\\ \frac{3}{2} & 1 & \infty & 1 & 1\end{smallmatrix}\right]\right)$

$$30G=\sum_{k=0}^{\infty}\left(\frac{16}{27}\right)^k\left[\begin{matrix}1, & 1\\ \frac{7}{6}, & \frac{11}{6}\end{matrix}\right]_k\frac{21+22k}{1+2k}.$$

3.6 Series for $\zeta(3)$

We give 15 spectacular series for $\zeta(3)$.

Example 61 ($\Omega\left[\begin{smallmatrix}2&0&1&0&1\\ \frac{3}{2}&1&1&1&\frac{1}{2}\end{smallmatrix}\right]$ Bisection series)

$$\frac{7}{4}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\begin{bmatrix}1, & 1\\ \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\frac{2+3k}{(1+k)(1+2k)}.$$

Example 62 ($\Omega\left[\begin{smallmatrix}2&0&1&1&1\\ \frac{3}{2}&\frac{1}{2}&1&1&1\end{smallmatrix}\right]$ Guillera [53, §3.1: $a=\frac{1}{2}$])

$$\frac{7}{2}\zeta(3)=\sum_{k=0}^{\infty}\begin{bmatrix}1, & 1, & 1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\frac{5+14k+10k^2}{(-4)^k}.$$

Example 63 ($\Omega\left[\begin{smallmatrix}2&0&1&1&1\\ 1&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{smallmatrix}\right]$)

$$\frac{21}{2}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}1, & 1\\ \frac{5}{4}, & \frac{7}{4}\end{bmatrix}_k\frac{13+32k+20k^2}{(1+k)(1+2k)^2}.$$

Example 64 ($\Omega\left[\begin{smallmatrix}2&0&1&0&1\\ 1&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{smallmatrix}\right]$)

$$\frac{63}{2}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\begin{bmatrix}1, & 1, & 1, & 1\\ \frac{5}{4}, & \frac{5}{4}, & \frac{7}{4}, & \frac{7}{4}\end{bmatrix}_k\frac{37+94k+60k^2}{1+2k}.$$

Example 65 ($\Omega\left[\begin{smallmatrix}2&0&1&0&1\\ 2&1&1&1&1\end{smallmatrix}\right]$)

$$16\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\begin{bmatrix}1, & 1\\ \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\frac{19+30k}{(1+k)(1+2k)}.$$

Example 66 ($\Omega\left[\begin{smallmatrix}3&0&1&1&2\\ 2&1&1&1&1\end{smallmatrix}\right]$ Amdelberhan [4] and Wilf [64, Eq. 8])

$$24\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}1, & 1\\ \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k\frac{29+80k+56k^2}{(1+k)(1+2k)^2}.$$

Example 67 ($\Omega\left[\begin{smallmatrix}3&0&1&1&1\\ 1&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{smallmatrix}\right]$ Bisection series: Sun's conjecture [62, Eq. 1.6])

$$\frac{21}{2}\zeta(3)=\sum_{k=0}^{\infty}\begin{bmatrix}1, & 1, & 1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k\frac{13+38k+28k^2}{(-27)^k}.$$

Example 68 $\left(\Omega\left[\begin{smallmatrix}3 & 0202\\ \frac{3}{2} & \frac{1}{2}111\end{smallmatrix}\right]\right)$

$$\frac{945}{2}\zeta(3)=\sum_{k=0}^{\infty}\left(\frac{4}{27}\right)^k\begin{bmatrix}1, & 1, & 1, & 1\\ \frac{5}{4}, & \frac{7}{4}, & \frac{7}{6}, & \frac{11}{6}\end{bmatrix}_k \times\frac{1078+7227k+17282k^2+17712k^3+6624k^4}{(1+2k)(1+3k)(2+3k)}.$$

Example 69 $\left(\Omega\left[\begin{smallmatrix}3 & 0122\\ \frac{3}{2} & 1\frac{1}{2}11\end{smallmatrix}\right]\right)$

$$\frac{105}{2}\zeta(3)=\sum_{k=0}^{\infty}\left(-\frac{16}{27}\right)^k\begin{bmatrix}1, & 1\\ \frac{7}{6}, & \frac{11}{6}\end{bmatrix}_k\frac{67+151k+86k^2}{(1+k)(1+2k)^2}.$$

Example 70 $\left(\Omega\left[\begin{smallmatrix}3 & 0122\\ \frac{3}{2} & \frac{1}{2}111\end{smallmatrix}\right]\right)$

$$63\zeta(3)=\sum_{k=0}^{\infty}\left(-\frac{16}{27}\right)^k\begin{bmatrix}1, & 1, & 1, & 1, & 1\\ \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3}, & \frac{5}{4}, & \frac{7}{4}\end{bmatrix}_k\left\{106+269k+172k^2\right\}.$$

Example 71 $\left(\Omega\left[\begin{smallmatrix}30111\\ 21111\end{smallmatrix}\right]\right)$

$$144\zeta(3)=\sum_{k=0}^{\infty}\left(\frac{1}{729}\right)^k\begin{bmatrix}1, & 1, & 1, & 1\\ \frac{4}{3}, & \frac{4}{3}, & \frac{5}{3}, & \frac{5}{3}\end{bmatrix}_k\frac{173+501k+364k^2}{1+2k}.$$

Example 72 $\left(\Omega\left[\begin{smallmatrix}31111\\ 21111\end{smallmatrix}\right]\right.$ Amdeberhan and Zeilberger [5])

$$64\zeta(3)=\sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k\begin{bmatrix}1, & 1, & 1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\left\{77+250k+205k^2\right\}.$$

Example 73 $\left(\Omega\left[\begin{smallmatrix}3 & 1 & 1 & 1 & 1\\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{smallmatrix}\right]\right)$

$$\frac{567}{2}\zeta(3)=\sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k\begin{bmatrix}1, & 1, & 1, & 1, & 1, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ \frac{5}{4}, & \frac{5}{4}, & \frac{5}{4}, & \frac{5}{4}, & \frac{7}{4}, & \frac{7}{4}, & \frac{7}{4}, & \frac{7}{4}\end{bmatrix}_k \times\left\{341+2844k+9122k^2+14236k^3+10888k^4+3280k^5\right\}.$$

Example 74 $\left(\Omega\left[\begin{smallmatrix}40112\\ 21111\end{smallmatrix}\right]\right)$

$$1728\zeta(3)=\sum_{k=0}^{\infty}\left(\frac{1}{4096}\right)^k\begin{bmatrix}1, & 1, & 1, & 1\\ \frac{5}{4}, & \frac{5}{4}, & \frac{7}{4}, & \frac{7}{4}\end{bmatrix}_k\frac{4154+26427k+62152k^2+64161k^3+24570k^4}{(1+2k)(1+3k)(2+3k)}.$$

Example 75 $\left(\Omega\left[\begin{smallmatrix}6\ 1\ 1\ 4\ 4\\ \frac32\ \frac12\ 1\ 1\ 1\end{smallmatrix}\right]\right.$ Bisection series: Guillera's conjecture [55, Eq. 42])

$$210\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-4^4}{5^5}\Big)^k\begin{bmatrix}1, & 1, & 1, & 1, & 1\\ \frac32, & \frac65, & \frac75, & \frac85, & \frac95\end{bmatrix}_k\Big\{268+721k+483k^2\Big\}.$$

3.7 Apéry-like series

They are infinite series expressions for the Riemann zeta series $\zeta(m)$ involving central binomial coefficients. The formulae of this kind first appear in the irrationality proof of $\zeta(3)$ given by Apéry [7]; see Van der Poorten [63] for a very readable version of this work. More formulae of similar type and extensions can be found in the books by Borwein and Borwein [23, §11] and Comtet [45, Page 89], as well as the papers by Almkvist et al. [3], Bailey et al. [9,11,14], Borwein et al. [28,29,31], Chu and Zheng [37,38,43], Elsner [46], Lehmer [57], Rivoal [59], Zagier [65] and Zucker [68].

Example 76 $\left(\Omega\left[\begin{smallmatrix}2011\ 1\\ 2111\infty\end{smallmatrix}\right]\right.$ Apéry [7]: see [46,63,68])

$$\frac{\pi^2}{18}=\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}.$$

Example 77 $\left(\Omega\left[\begin{smallmatrix}20111\\ 21111\end{smallmatrix}\right]\right.$ Apéry [7]: see [46,63,68])

$$\zeta(3)=\frac52\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}.$$

Example 78 $\left(\Omega\left[\begin{smallmatrix}20\ 10\ 1\\ 1\frac13 1\frac23\frac12\end{smallmatrix}\right]\right.$ Bisection series: Apéry [7]; see [46,57,63])

$$\frac{\pi}{3\sqrt3}=\sum_{k=1}^{\infty}\frac{1}{k\binom{2k}{k}}.$$

Example 79 $\left(\Omega\left[\begin{smallmatrix}201\ 0\ 1\\ 1\frac14 1\frac34\frac12\end{smallmatrix}\right]\right.$ Quartic section series: Elsner [46])

$$\frac{\pi}{2}=\sum_{k=0}^{\infty}\frac{2^k}{\binom{2k}{k}(2k+1)}.$$

Example 80 $\left(\Omega\left[\begin{smallmatrix}201\ 0\ 1\\ 11\frac14\frac12\frac34\end{smallmatrix}\right]\right.$ Bisection series: Lehmer [57])

$$\frac{\pi}{3}=\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{16^k(2k+1)}.$$

Example 81 ($\Omega\left[\begin{smallmatrix} 20 & 1 & 1 & 1 \\ 11 & \frac{1}{4} & \frac{3}{4} & \infty \end{smallmatrix}\right]$ Bisection series)

$$\frac{\pi}{2\sqrt{2}} = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{8^k(2k+1)}.$$

Example 82 ($\Omega\left[\begin{smallmatrix} 20 & 1 & 1 & 1 \\ 11 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{smallmatrix}\right]$)

$$\frac{\pi^2}{10} = \sum_{k=0}^{\infty} \Big(\frac{-1}{16}\Big)^k \frac{\binom{2k}{k}}{(2k+1)^2}.$$

3.8 BBP-type formulae

They are formulae for calculating π discovered firstly by Bailey et al. [12] in 1995, which may serve amazingly as digit-extraction algorithms for π. We refer to Adamchik and Wagon [1,2] for *Mathematica*-based approach and to Bailey et al. [9,10,12,13] for historical accounts. Further formulae of BBP-type can be found in Bellard [19], Borwein and Bailey [22, §3.6], Chan [32,33] and Zhang [66].

Example 83 ($\Omega\left[\begin{smallmatrix} 20 & 1 & 0 & 1 \\ 11 & \frac{1}{2} & \frac{3}{4} & \frac{1}{4} \end{smallmatrix}\right]$ Bailey et al. [12, Theorem 1])

$$\pi = \sum_{k=0}^{\infty} \Big(\frac{1}{16}\Big)^k \Big\{\frac{4}{1+8k} - \frac{2}{4+8k} - \frac{1}{5+8k} - \frac{1}{6+8k}\Big\}.$$

Example 84 ($\Omega\left[\begin{smallmatrix} 20 & 1 & 0 & 1 \\ 11 & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \end{smallmatrix}\right]$ Adamchik and Wagon [1])

$$2\pi = \sum_{k=0}^{\infty} \Big(\frac{1}{16}\Big)^k \Big\{\frac{8}{2+8k} + \frac{4}{3+8k} + \frac{4}{4+8k} - \frac{1}{7+8k}\Big\}.$$

Example 85 ($\Omega\left[\begin{smallmatrix} 20 & 1 & 0 & 1 \\ 11 & \frac{2}{3} & \frac{5}{6} & \frac{1}{6} \end{smallmatrix}\right]$)

$$\frac{4\cdot 2^{1/3}\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \Big(\frac{1}{16}\Big)^k \Big\{\frac{4}{1+12k} - \frac{4}{4+12k} + \frac{1}{7+12k} - \frac{1}{10+12k}\Big\}.$$

Example 86 ($\Omega\left[\begin{smallmatrix} 2 & 0 & 1 & 0 & 1 \\ \frac{5}{6} & \frac{5}{6} & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \end{smallmatrix}\right]$)

$$\frac{8\cdot 2^{2/3}\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \Big(\frac{1}{16}\Big)^k \Big\{\frac{4}{8+12k} + \frac{16}{2+12k} - \frac{4}{5+12k} - \frac{1}{11+12k}\Big\}.$$

Example 87 ($\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 1\\ 1 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{2}{3}\end{smallmatrix}\right]$ Bisection series)

$$\pi\sqrt{3}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\Big\{\frac{2}{1+2k}+\frac{3}{1+3k}+\frac{1}{2+3k}\Big\}.$$

Example 88 ($\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 1\\ 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{5}{6}\end{smallmatrix}\right]$ Bisection series)

$$\frac{9\pi}{2\sqrt{3}}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\Big\{\frac{9}{1+6k}-\frac{3}{3+6k}+\frac{1}{5+6k}\Big\}.$$

Example 89 ($\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 2\\ 1 & 1 & \frac{1}{3} & \frac{2}{3} & \infty\end{smallmatrix}\right]$)

$$2\sqrt{3}\pi=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\Big\{\frac{9}{1+6k}+\frac{3}{2+6k}+\frac{1}{4+6k}+\frac{1}{5+6k}\Big\}.$$

Example 90 ($\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 2\\ 1 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{2}\end{smallmatrix}\right]$)

$$\pi\sqrt{3}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\Big\{\frac{9}{1+6k}-\frac{3}{2+6k}-\frac{6}{3+6k}-\frac{1}{4+6k}+\frac{1}{5+6k}\Big\}.$$

3.9 Other infinte series

Example 91 ($\Omega\left[\begin{smallmatrix}2 & 0 & 1 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} & \infty\end{smallmatrix}\right]$)

$$6\sqrt{2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\left[\begin{matrix}\frac{3}{4}, & \frac{1}{8}, & \frac{5}{8}\\ 1, & \frac{7}{8}, & \frac{11}{8}\end{matrix}\right]_k\Big\{9+40k\Big\}.$$

Example 92 ($\Omega\left[\begin{smallmatrix}2 & 0 & 1 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & \frac{5}{6} & \frac{1}{6} & \infty\end{smallmatrix}\right]$)

$$\frac{64}{\sqrt{3}}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\left[\begin{matrix}\frac{5}{6}, & \frac{5}{12}, & \frac{11}{12}\\ 1, & \frac{2}{3}, & \frac{5}{3}\end{matrix}\right]_k\Big\{43+60k\Big\}.$$

Example 93 ($\Omega\left[\begin{smallmatrix}3 & 0 & 1 & 1 & 1\\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & \infty\end{smallmatrix}\right]$)

$$\frac{21}{\sqrt{2}}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}, & \frac{3}{4}\\ 1, & \frac{11}{12}, & \frac{19}{12}\end{matrix}\right]_k\Big\{15+28k\Big\}.$$

Example 94 $(\Omega\left[\begin{smallmatrix} 3 & 0 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & \infty \end{smallmatrix}\right])$

$$\frac{15}{\sqrt{2}} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \begin{bmatrix} \frac{1}{4}, & \frac{1}{4}, & \frac{3}{4} \\ 1, & \frac{13}{12}, & \frac{17}{12} \end{bmatrix}_k \Big\{11 + 124k + 224k^2\Big\}.$$

Example 95 $(\Omega\left[\begin{smallmatrix} 3 & 0 & 1 & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} & \infty \end{smallmatrix}\right])$

$$16\sqrt{2} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \begin{bmatrix} \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ 1, & \frac{4}{3}, & \frac{5}{3} \end{bmatrix}_k \Big\{23 + 110k + 112k^2\Big\}.$$

Example 96 $(\Omega\left[\begin{smallmatrix} 3 & 1 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{6} & \frac{5}{6} & \infty \end{smallmatrix}\right])$

$$\frac{256}{3\sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}, & \frac{1}{6}, & \frac{5}{6} \\ 1, & \frac{4}{3}, & \frac{5}{3} \end{bmatrix}_k \frac{49 + 240k + 252k^2}{64^k}.$$

Example 97 $(\Omega\left[\begin{smallmatrix} 3 & 1 & 1 & 1 & 3 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{smallmatrix}\right])$

$$8 = \sum_{k=0}^{\infty} \Big(\frac{27}{64}\Big)^k \begin{bmatrix} \frac{1}{2}, & \frac{1}{6}, & \frac{5}{6} \\ 1, & 1, & 1 \end{bmatrix}_k \frac{7 + 74k}{1 + 2k}.$$

Example 98 $(\Omega\left[\begin{smallmatrix} 6 & 1 & 1 & 4 & 4 \\ 2 & \frac{3}{2} & 1 & 1 & 1 \end{smallmatrix}\right]$ Bisection series)

$$0 = \sum_{k=0}^{\infty} (-1)^k \frac{2 + 213k + 652k^2 + 483k^3}{\binom{5k}{k}\,(5k+1)_4}.$$

The next example confirms a conjectured series about the Legendre symbol $\big(\frac{k}{3}\big)$.

Example 99 $(\Omega\left[\begin{smallmatrix} 2 & 0 & 1 & 1 & 1 \\ 1, & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{smallmatrix}\right]$ Sun's conjecture [62, Eq. 1.4])

$$\sum_{k=1}^{\infty} \frac{\big(\frac{k}{3}\big)}{k^2} = \sum_{k=0}^{\infty} \Big(\frac{-1}{4}\Big)^k \begin{bmatrix} 1, & 1, & 1 \\ \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3} \end{bmatrix}_k \frac{11 + 15k}{12}.$$

Finally, we point out that it is highly possible that most of the infinite series presented in this section can be verified automatically by symbolic languages, besides those examples attributed explicitly to the *WZ*-method, mainly done by Guillera [48–56].

Acknowledgements The author expresses his sincere gratitude to an anonymous referee for the careful reading, critical comments and helpful suggestions that make the manuscript improved during revision.

Appendix: Two useful recurrence relations

Chu and Zhang [42] examined $\Omega(a; b, c, d, e)$ mainly by utilizing the following two recurrence relations of iteration patterns **[11000]** and **[11100]**, respectively.

Lemma B (Chu and Zhang [42, Lemma 1]: $\Re(1 + 2a - b - c - d - e) > 0$).

$$\Omega(a; b, c, d, e) = \frac{(1 + 2a - c - d - e)(a - b)}{1 + 2a - b - c - d - e} + \Omega(1 + a; 1 + b, c, d, e) \times \frac{b(1 + a - c - d)(1 + a - c - e)(1 + a - d - e)}{(1 + a - c)(1 + a - d)(1 + a - e)(1 + 2a - b - c - d - e)}.$$

Lemma C (Chu and Zhang [42, Lemma 2]: $\Re(1 + 2a - b - c - d - e) > 0$).

$$\Omega(a; b, c, d, e) = \frac{(a - b)(a - c)}{a - b - c} + \Omega(1 + a; 1 + b, 1 + c, d, e) \times \frac{bc(1 + a - d - e)}{(1 + a - d)(1 + a - e)(b + c - a)}.$$

They have been employed further, in this paper, to establish the central transformation Theorem A of universal iteration pattern for $\Omega(a; b, c, d, e)$.

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