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Exceptional set for sums of unlike powers of primes (II)

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Abstract
Let N be a sufficiently large integer. In this paper, it is proved that, with at most $O(N^{7/18+\varepsilon})$ exceptions, all even positive integers up to N can be represented in the form $p_1^2+p_2^2+p_3^3+p_4^3+p_5^4+p_6^4$, where $p_1, p_2, p_3, p_4, p_5, p_6$ are prime numbers, which constitutes an improvement over some previous work.

Keywords Waring–Goldbach problem · Circle method · Exceptional set

Mathematics Subject Classification 11P05 · 11P32 · 11P55

1 Introduction and main result

Let $N, k_1, k_2, \ldots, k_s$ be natural numbers such that $2 \le k_1 \le k_2 \le \cdots \le k_s$, $N > s$. Waring's problem of mixed powers concerns the representation of N as the form

$$N = x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s}.$$

Not very much is known about results of this kind. For historical literature the reader should consult section P12 of LeVeque's *Reviews in number theory* and the bibliography of Vaughan [9].

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In 1970, Vaughan [8] obtained the asymptotic formula for the number of representations of a number as the sum of two squares, two cubes and two biquadrates. He proved that, for any sufficiently large integer N, there holds

$$\sum_{x_1^2+x_2^2+x_3^3+x_4^3+x_5^4+x_6^4=N} 1 = \frac{\Gamma^2\left(\frac{3}{2}\right)\Gamma^2\left(\frac{4}{3}\right)\Gamma^2\left(\frac{5}{4}\right)}{\Gamma\left(\frac{13}{6}\right)}\mathfrak{S}_{2,3,4}(N)N^{\frac{7}{6}} + O\left(N^{\frac{7}{6}-\frac{1}{96}+\varepsilon}\right),$$

where the singular series is

$$\mathfrak{S}_{2,3,4}(N) = \sum_{q=1}^{\infty}\frac{1}{q^6}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\prod_{i=1}^{3}\left(\sum_{x_i=1}^{q} e\left(\frac{ax_i^{i+1}}{q}\right)\right)^2\right)e\left(-\frac{aN}{q}\right).$$

In view of Vaughan's result, it is reasonable to conjecture that, for every sufficiently large even integer N, the following Diophantine equation

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 \tag{1.1}$$

is solvable. Here and below the letter p, with or without subscript, always stands for a prime number. However, many authors approach this conjecture in different ways. For instance, in 2015, Lü [4] proved that for every sufficiently large even integer N, the following equation

$$N = x^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 \tag{1.2}$$

is solvable with x being an almost-prime $\mathcal{P}_6$ and the p_j $(j = 2, 3, 4, 5, 6)$ primes, where $\mathcal{P}_r$ denotes an almost-prime with at most r prime factors, counted according to multiplicity. Afterwards, Liu [3] enhanced the result of Lü [4] and showed that (1.2) is solvable with x being an almost-prime $\mathcal{P}_4$ and the p_j's primes. On the other hand, in 2019, Lü [5] proved that every sufficiently large even integer N can be represented as two squares of primes, two cubes of primes, two biquadrates of primes and 24 powers of 2, i.e.

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{24}}.$$

In 2018, Zhang and Li [11] establish the exceptional set of the problem (1.1). They proved that $E(N) \ll N^{13/16+\varepsilon}$, where $E(N)$ denotes the number of positive even integers n up to N, which cannot be represented as $p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4$.

In this paper, we shall continue to consider the exceptional set of the problem (1.1) and improve the previous result.

Theorem 1.1 *Let $E(N)$ denote the number of positive even integers n up to N, which cannot be represented as*

$$n = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4. \tag{1.3}$$

Then, for any $\varepsilon > 0$*, we have*

$$E(N) \ll N^{\frac{7}{18}+\varepsilon}.$$

We will establish Theorem 1.1 by using a pruning process into the Hardy–Littlewood circle method. In the treatment of the integrals over minor arcs, we will employ the methods, which is developed by Wooley in [10], combining with the new estimates for exponential sum over cubes of primes developed by Zhao [12]. For the treatment of the integrals on the major arcs, we shall prune the major arcs further and deal with them, respectively. The full details will be explained in the following relevant sections.

Notation Throughout this paper, ε always denotes a sufficiently small positive constant, which may not be the same at different occurrences. As usual, we use $\varphi(n)$ to denote the Euler's function. $e(x) = e^{2\pi i x}$; $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$. N is a sufficiently large integer and $n \in (N/2, N]$, and thus $\log N \asymp \log n$. The letter c, with or without subscripts or superscripts, always denote a positive constant, which may not be the same at different occurrences.

2 Outline of the proof of Theorem 1.1

Let N be a sufficiently large positive integer. By a splitting argument, it is sufficient to consider the even integers $n \in (N/2, N]$. For the application of the Hardy–Littlewood method, we need to define the Farey dissection. For this object, we set

$$A = 100^{100}, \quad Q_0 = \log^A N, \quad Q_1 = N^{\frac{1}{6}}, \quad Q_2 = N^{\frac{5}{6}}, \quad \mathfrak{I}_0 = \left[-\frac{1}{Q_2}, 1 - \frac{1}{Q_2}\right].$$

By Dirichlet's lemma on rational approximation (for instance, see Lemma 12 on p. 104 of Pan and Pan [6]), each $\alpha \in [-1/Q_2, 1 - 1/Q_2]$ can be written as the form

$$\alpha = \frac{a}{q} + \lambda, \qquad |\lambda| \le \frac{1}{qQ_2} \tag{2.1}$$

for some integers $a,\ q$ with $1 \le a \le q \le Q_2$ and $(a,q) = 1$. Define

$$\mathfrak{M}(q,a) = \left[\frac{a}{q} - \frac{1}{qQ_2}, \frac{a}{q} + \frac{1}{qQ_2}\right], \qquad \mathfrak{M} = \bigcup_{1 \le q \le Q_1} \bigcup_{\substack{1 \le a \le q \\ (a,q)=1}} \mathfrak{M}(q,a),$$

$$\mathfrak{M}_0(q,a) = \left[\frac{a}{q} - \frac{Q_0^{100}}{qN}, \frac{a}{q} + \frac{Q_0^{100}}{qN}\right], \qquad \mathfrak{M}_0 = \bigcup_{1 \le q \le Q_0^{100}} \bigcup_{\substack{1 \le a \le q \\ (a,q)=1}} \mathfrak{M}_0(q,a),$$

$$\mathfrak{m}_1 = \mathfrak{I}_0 \backslash \mathfrak{M}, \qquad \mathfrak{m}_2 = \mathfrak{M} \backslash \mathfrak{M}_0.$$

Then we obtain the Farey dissection

$$\mathfrak{I}_0 = \mathfrak{M}_0 \cup \mathfrak{m}_1 \cup \mathfrak{m}_2. \tag{2.2}$$

For $k = 2, 3, 4$, we define

$$f_k(\alpha) = \sum_{X_k < p \le 2X_k} e(p^k\alpha),$$

where $X_k = (N/16)^{\frac{1}{k}}$. Let

$$\mathscr{R}(n) = \sum_{\substack{n=p_1^2+p_2^2+p_3^3+p_4^3+p_5^4+p_6^4 \\ X_2<p_1,p_2\le 2X_2 \\ X_3<p_3,p_4\le 2X_3 \\ X_4<p_5,p_6\le 2X_4}} 1.$$

From (2.2), one has

$$\begin{aligned}\mathscr{R}(n) &= \int_0^1 \Big(\prod_{k=2}^4 f_k^2(\alpha)\Big)e(-n\alpha)\mathrm{d}\alpha = \int_{-\frac{1}{Q_2}}^{1-\frac{1}{Q_2}} \Big(\prod_{k=2}^4 f_k^2(\alpha)\Big)e(-n\alpha)\mathrm{d}\alpha \\ &= \Big\{\int_{\mathfrak{M}_0} + \int_{\mathfrak{m}_1} + \int_{\mathfrak{m}_2}\Big\}\Big(\prod_{k=2}^4 f_k^2(\alpha)\Big)e(-n\alpha)\mathrm{d}\alpha.\end{aligned}$$

In order to prove Theorem 1.1, we need the two following propositions:

Proposition 2.1 *For $n \in [N/2, N]$, there holds*

$$\int_{\mathfrak{M}_0}\Big(\prod_{k=2}^4 f_k^2(\alpha)\Big)e(-n\alpha)\mathrm{d}\alpha = \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{5}{4})}{\Gamma(\frac{13}{6})}\mathfrak{S}(n)\frac{n^{\frac{7}{6}}}{\log^6 n} + O\Big(\frac{n^{\frac{7}{6}}}{\log^7 n}\Big), \tag{2.3}$$

where $\mathfrak{S}(n)$ is the singular series defined by

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty}\frac{1}{\varphi^6(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}\Big(\prod_{k=2}^4\Big(\sum_{\substack{r=1\\(r,q)=1}}^{q} e\Big(\frac{ar^k}{q}\Big)\Big)^2\Big)e\Big(-\frac{an}{q}\Big),$$

which is absolutely convergent and satisfies

$$0 < c^* \le \mathfrak{S}(n) \ll 1 \tag{2.4}$$

for any integer n satisfying $n \equiv 0 \,(\mathrm{mod} 2)$ and some fixed constant $c^* > 0$.

The proof of (2.3) in Proposition 2.1 follows from the well-known standard technique in the Hardy–Littlewood method. For more information, one can see pp. 90–99 of Hua [2], so we omit the details herein. For the property (2.4) of singular series, one can see Section 5 of Zhang and Li [11].

Proposition 2.2 *Let $\mathcal{Z}(N)$ denote the number of integers $n \in [N/2, N]$ satisfying $n \equiv 0 \,(\mathrm{mod} 2)$ such that*

$$\sum_{j=1}^{2}\left|\int_{\mathfrak{m}_j}\left(\prod_{k=2}^{4} f_k^2(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha\right| \gg \frac{n^{\frac{7}{6}}}{\log^7 n}.$$

Then we have

$$\mathcal{Z}(N) \ll N^{\frac{7}{18}+\varepsilon}.$$

The proof of Proposition 2.2 will be given in Sect. 4. The remaining part of this section is devoted to establishing Theorem 1.1 by using Propositions 2.1 and 2.2.

Proof of Theorem 1.1. From Proposition 2.2, we deduce that, with at most $O\big(N^{\frac{7}{18}+\varepsilon}\big)$ exceptions, all even integers $n \in [N/2, N]$ satisfy

$$\sum_{j=1}^{2}\left|\int_{\mathfrak{m}_j}\left(\prod_{k=2}^{4} f_k^2(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha\right| \ll \frac{n^{\frac{7}{6}}}{\log^7 n},$$

from which and Proposition 2.1, we conclude that, with at most $O\big(N^{\frac{7}{18}+\varepsilon}\big)$ exceptions, all even integers $n \in [N/2, N]$ can be represented in the form $p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4$, where $p_1, p_2, p_3, p_4, p_5, p_6$ are prime numbers. By a splitting argument, we get

$$E(N) \ll \sum_{0\le\ell\ll\log N} \mathcal{Z}\left(\frac{N}{2^\ell}\right) \ll \sum_{0\le\ell\ll\log N} \left(\frac{N}{2^\ell}\right)^{\frac{7}{18}+\varepsilon} \ll N^{\frac{7}{18}+\varepsilon}.$$

This completes the proof of Theorem 1.1. □

3 Some auxiliary Lemmas

Lemma 3.1 *Let $2 \le k_1 \le k_2 \le \cdots \le k_s$ be natural numbers such that*

$$\sum_{i=j+1}^{s} \frac{1}{k_i} \le \frac{1}{k_j}, \quad 1 \le j \le s-1.$$

Then we have

$$\int_0^1 \left|\prod_{i=1}^{s} f_{k_i}(\alpha)\right|^2 \mathrm{d}\alpha \ll N^{\frac{1}{k_1}+\cdots+\frac{1}{k_s}+\varepsilon}.$$

Proof See Lemma 1 of Brüdern [1]. □

Lemma 3.2 *Suppose that* α *is a real number, and that* $|\alpha - a/q| \leq q^{-2}$ *with* $(a, q) = 1$. *Let* $\beta = \alpha - a/q$. *Then we have*

$$f_k(\alpha) \ll d^{\delta_k}(q)(\log x)^c \left(X_k^{1/2}\sqrt{q(1+N|\beta|)} + X_k^{4/5} + \frac{X_k}{\sqrt{q(1+N|\beta|)}} \right),$$

where $\delta_k = \frac{1}{2} + \frac{\log k}{\log 2}$ *and c is a constant.*

Proof See Theorem 1.1 of Ren [7]. □

Lemma 3.3 *Suppose that* α *is a real number, and that there exist* $a \in \mathbb{Z}$ *and* $q \in \mathbb{N}$ *with*

$$(a, q) = 1, \qquad 1 \leq q \leq Q \qquad \text{and} \qquad |q\alpha - a| \leq Q^{-1}.$$

If $P^{\frac{1}{2}} \leq Q \leq P^{\frac{5}{2}}$, *then one has*

$$\sum_{P < p \leq 2P} e\big(p^3\alpha\big) \ll P^{1-\frac{1}{12}+\varepsilon} + \frac{q^{-\frac{1}{6}} P^{1+\varepsilon}}{\big(1 + P^3|\alpha - a/q|\big)^{1/2}}.$$

Proof See Lemma 8.5 of Zhao [12]. □

Lemma 3.4 *For* $\alpha \in \mathfrak{m}_1$, *we have*

$$f_3(\alpha) \ll N^{\frac{11}{36}+\varepsilon}.$$

Proof By Dirichlet's rational approximation (2.1), for $\alpha \in \mathfrak{m}_1$, we have $Q_1 \leq q \leq Q_2$. From Lemma 3.3 we obtain

$$f_3(\alpha) \ll X_3^{\frac{11}{12}+\varepsilon} + X_3^{1+\varepsilon} Q_1^{-\frac{1}{6}} \ll N^{\frac{11}{36}+\varepsilon}.$$

This completes the proof of Lemma 3.4. □

For $1 \leq a \leq q$ with $(a, q) = 1$, set

$$\mathcal{I}(q, a) = \left[\frac{a}{q} - \frac{1}{qQ_0}, \frac{a}{q} + \frac{1}{qQ_0}\right], \qquad \mathcal{I} = \bigcup_{1 \leq q \leq Q_0} \bigcup_{\substack{a=-q \\ (a,q)=1}}^{2q} \mathcal{I}(q, a). \tag{3.1}$$

For $\alpha \in \mathfrak{m}_2$, by Lemma 3.2, we have

$$f_3(\alpha) \ll \frac{N^{\frac{1}{3}} \log^c N}{q^{\frac{1}{2}-\varepsilon}\big(1 + N|\lambda|\big)^{1/2}} + N^{\frac{4}{15}+\varepsilon} = V_3(\alpha) + N^{\frac{4}{15}+\varepsilon}, \tag{3.2}$$

say. Then we obtain the following lemma.

Lemma 3.5 *We have*

$$\int_{\mathcal{I}}|V_3(\alpha)|^4\mathrm{d}\alpha=\sum_{1\le q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|V_3(\alpha)|^4\mathrm{d}\alpha\ll N^{\frac{1}{3}}\log^c N.$$

Proof We have

$$\begin{aligned}&\sum_{1\le q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|V_3(\alpha)|^4\mathrm{d}\alpha\\
&\ll\sum_{1\le q\le Q_0}q^{-2+\varepsilon}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{|\lambda|\le\frac{1}{Q_0}}\frac{N^{\frac{4}{3}}\log^c N}{(1+N|\lambda|)^2}\mathrm{d}\lambda\\
&\ll\sum_{1\le q\le Q_0}q^{-2+\varepsilon}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\left(\int_{|\lambda|\le\frac{1}{N}}N^{\frac{4}{3}}\log^c N\mathrm{d}\lambda+\int_{\frac{1}{N}\le|\lambda|\le\frac{1}{Q_0}}\frac{N^{\frac{4}{3}}\log^c N}{N^2\lambda^2}\mathrm{d}\lambda\right)\\
&\ll N^{\frac{1}{3}}\log^c N\sum_{1\le q\le Q_0}q^{-2+\varepsilon}\varphi(q)\ll N^{\frac{1}{3}}Q_0^{\varepsilon}\log^c N\ll N^{\frac{1}{3}}\log^c N.\end{aligned}$$

This completes the proof of Lemma 3.5. □

4 Proof of Proposition 2.2

In this section, we shall give the proof of Proposition 2.2. We denote by $\mathcal{Z}_j(N)$ the set of integers n satisfying $n\in[N/2,N]$ and $n\equiv 0\pmod 2$ for which the following estimate

$$\left|\int_{\mathfrak{m}_j}\left(\prod_{k=2}^{4}f_k^2(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha\right|\gg\frac{n^{\frac{7}{6}}}{\log^7 n}\tag{4.1}$$

holds. For convenience, we use $\mathcal{Z}_j$ to denote the cardinality of $\mathcal{Z}_j(N)$ for abbreviation. Also, we define the complex number $\xi_j(n)$ by taking $\xi_j(n)=0$ for $n\notin\mathcal{Z}_j(N)$, and when $n\in\mathcal{Z}_j(N)$ by means of the equation

$$\left|\int_{\mathfrak{m}_j}\left(\prod_{k=2}^{4}f_k^2(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha\right|=\xi_j(n)\int_{\mathfrak{m}_j}\left(\prod_{k=2}^{4}f_k^2(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha.\tag{4.2}$$

Plainly, one has $|\xi_j(n)|=1$ whenever $\xi_j(n)$ is nonzero. Therefore, we obtain

$$\sum_{n\in\mathcal{Z}_j(N)}\xi_j(n)\int_{\mathfrak{m}_j}\left(\prod_{k=2}^{4}f_k^2(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha=\int_{\mathfrak{m}_j}\left(\prod_{k=2}^{4}f_k^2(\alpha)\right)\mathcal{K}_j(\alpha)\mathrm{d}\alpha,\tag{4.3}$$

where the exponential sum $\mathcal{K}_j(\alpha)$ is defined by

$$\mathcal{K}_j(\alpha)=\sum_{n\in\mathcal{Z}_j(N)}\xi_j(n)e(-n\alpha).$$

For $j=1,2$, set

$$I_j=\int_{\mathfrak{m}_j}\bigg(\prod_{k=2}^{4}f_k^2(\alpha)\bigg)\mathcal{K}_j(\alpha)\mathrm{d}\alpha.$$

From (4.1)–(4.3), we derive that

$$I_j\gg\sum_{n\in\mathcal{Z}_j(N)}\frac{n^{\frac{7}{6}}}{\log^7 n}\gg\frac{\mathcal{Z}_jN^{\frac{7}{6}}}{\log^7 N},\qquad j=1,2. \tag{4.4}$$

By Lemma 2.1 of Wooley [10] with $k=2$, we know that, for $j=1,2$, there holds

$$\int_0^1\big|f_2(\alpha)\mathcal{K}_j(\alpha)\big|^2\mathrm{d}\alpha\ll N^{\varepsilon}\big(\mathcal{Z}_jN^{\frac{1}{2}}+\mathcal{Z}_j^2\big). \tag{4.5}$$

It follows from Cauchy's inequality, Lemmas 3.1, 3.4 and (4.5) that

$$\begin{aligned}I_1&\ll\sup_{\alpha\in\mathfrak{m}_1}|f_3(\alpha)|^2\times\bigg(\int_0^1|f_2(\alpha)f_4^2(\alpha)|^2\mathrm{d}\alpha\bigg)^{\frac{1}{2}}\bigg(\int_0^1|f_2(\alpha)\mathcal{K}_1(\alpha)|^2\mathrm{d}\alpha\bigg)^{\frac{1}{2}}\\&\ll\big(N^{\frac{11}{36}+\varepsilon}\big)^2\cdot\big(N^{1+\varepsilon}\big)^{\frac{1}{2}}\cdot\Big(N^{\varepsilon}\big(\mathcal{Z}_1N^{\frac{1}{2}}+\mathcal{Z}_1^2\big)\Big)^{\frac{1}{2}}\\&\ll N^{\frac{10}{9}+\varepsilon}\Big(\mathcal{Z}_1^{\frac{1}{2}}N^{\frac{1}{4}}+\mathcal{Z}_1\Big)\ll\mathcal{Z}_1^{\frac{1}{2}}N^{\frac{49}{36}+\varepsilon}+\mathcal{Z}_1N^{\frac{10}{9}+\varepsilon}.\end{aligned} \tag{4.6}$$

Combining (4.4) and (4.6), we get

$$\mathcal{Z}_1N^{\frac{7}{6}}\log^{-7}N\ll I_1\ll\mathcal{Z}_1^{\frac{1}{2}}N^{\frac{49}{36}+\varepsilon}+\mathcal{Z}_1N^{\frac{10}{9}+\varepsilon},$$

which implies

$$\mathcal{Z}_1\ll N^{\frac{7}{18}+\varepsilon}. \tag{4.7}$$

Next, we give the upper bound for $\mathcal{Z}_2$. By (3.2), we obtain

$$\begin{aligned}I_2\ll&\int_{\mathfrak{m}_2}\big|f_2^2(\alpha)V_3^2(\alpha)f_4^2(\alpha)\mathcal{K}_2(\alpha)\big|\mathrm{d}\alpha\\&+N^{\frac{8}{15}+\varepsilon}\times\int_{\mathfrak{m}_2}\big|f_2^2(\alpha)f_4^2(\alpha)\mathcal{K}_2(\alpha)\big|\mathrm{d}\alpha\\=&\,I_{21}+I_{22},\end{aligned} \tag{4.8}$$

say. For $\alpha \in \mathfrak{m}_2$, we have either $Q_0^{100} < q < Q_1$ or $Q_0^{100} < N|q\alpha - a| < NQ_2^{-1} = Q_1$. Therefore, by Lemma 3.2, we get

$$\sup_{\alpha \in \mathfrak{m}_2} \left| f_4(\alpha) \right| \ll \frac{N^{\frac{1}{4}}}{\log^{40A} N}. \tag{4.9}$$

In view of the fact that $\mathfrak{m}_2 \subseteq \mathcal{I}$, where $\mathcal{I}$ is defined by (3.1), Cauchy's inequality, the trivial estimate $\mathcal{K}_2(\alpha) \ll \mathcal{Z}_2$ and Theorem 4 of Hua (See [2, p. 19]), we obtain

$$\begin{aligned} I_{21} &\ll \mathcal{Z}_2 \cdot \sup_{\alpha \in \mathfrak{m}_2} |f_4(\alpha)|^2 \times \left(\int_0^1 |f_2(\alpha)|^4 \mathrm{d}\alpha \right)^{\frac{1}{2}} \left(\int_{\mathcal{I}} |V_3(\alpha)|^4 \mathrm{d}\alpha \right)^{\frac{1}{2}} \\ &\ll \mathcal{Z}_2 \cdot \left(\frac{N^{\frac{1}{4}}}{\log^{40A} N} \right)^2 \cdot (N \log^c N)^{\frac{1}{2}} \cdot (N^{\frac{1}{3}} \log^c N)^{\frac{1}{2}} \ll \frac{\mathcal{Z}_2 N^{\frac{7}{6}}}{\log^{30A} N}. \end{aligned} \tag{4.10}$$

Moreover, it follows from Cauchy's inequality, (4.5) and Lemma 3.1 that

$$\begin{aligned} I_{22} &\ll N^{\frac{8}{15}+\varepsilon} \times \left(\int_0^1 |f_2(\alpha) f_4^2(\alpha)|^2 \mathrm{d}\alpha \right)^{\frac{1}{2}} \left(\int_0^1 |f_2(\alpha) \mathcal{K}_2(\alpha)|^2 \mathrm{d}\alpha \right)^{\frac{1}{2}} \\ &\ll N^{\frac{8}{15}+\varepsilon} \cdot \left(N^{1+\varepsilon}\right)^{\frac{1}{2}} \cdot \left(N^{\varepsilon} \left(\mathcal{Z}_2 N^{\frac{1}{2}} + \mathcal{Z}_2^2 \right) \right)^{\frac{1}{2}} \\ &\ll N^{\frac{31}{30}+\varepsilon} \left(\mathcal{Z}_2^{\frac{1}{2}} N^{\frac{1}{4}} + \mathcal{Z}_2 \right) \ll \mathcal{Z}_2^{\frac{1}{2}} N^{\frac{77}{60}+\varepsilon} + \mathcal{Z}_2 N^{\frac{31}{30}+\varepsilon}. \end{aligned} \tag{4.11}$$

Combining (4.4), (4.8), (4.10) and (4.11), we deduce that

$$\frac{\mathcal{Z}_2 N^{\frac{7}{6}}}{\log^7 N} \ll I_2 = I_{21} + I_{22} \ll \frac{\mathcal{Z}_2 N^{\frac{7}{6}}}{\log^{30A} N} + \mathcal{Z}_2^{\frac{1}{2}} N^{\frac{77}{60}+\varepsilon} + \mathcal{Z}_2 N^{\frac{31}{30}+\varepsilon},$$

which implies

$$\mathcal{Z}_2 \ll N^{\frac{7}{30}+\varepsilon}. \tag{4.12}$$

From (4.7) and (4.12), we have

$$\mathcal{Z}(N) \ll \mathcal{Z}_1 + \mathcal{Z}_2 \ll N^{\frac{7}{18}+\varepsilon}.$$

This completes the proof of Proposition 2.2.

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