The Ramanujan Journal (2020) 52:67–89
https://doi.org/10.1007/s11139-018-0127-3

Quadratic ideals and Rogers–Ramanujan recursions

Yuzhe Bai[1] · **Eugene Gorsky**[1,2] · **Oscar Kivinen**[1]

Received: 16 June 2018 / Accepted: 7 December 2018 / Published online: 22 May 2019

Abstract
We give an explicit recursive description of the Hilbert series and Gröbner bases for the family of quadratic ideals defining the jet schemes of a double point. We relate these recursions to the Rogers–Ramanujan identity and prove a conjecture of the second author, Oblomkov and Rasmussen.

Keywords Gröbner basis · Hilbert series · Jet scheme · Rogers–Ramanujan identity

Mathematics Subject Classification 13D02 · 13P10 · 05A19

1 Introduction

In this paper, we study a family of quadratic ideals defining the jet schemes for the double point $D = \operatorname{Spec} \mathbf{k}[x]/x^2$. Here $\mathbf{k}$ is a field of characteristic zero. Recall that the $(n-1)$-jet scheme of X is defined as the space of formal maps $\operatorname{Spec} \mathbf{k}[t]/t^n \to X$ [11]. In the case of the double point, such a formal map is defined by a polynomial

E.G. acknowledges the Russian Academic Excellence Project 5-100 for its support. The work of E.G. and O.K. was supported by the NSF Grants DMS-1700814 and DMS-1559338. The work of E.G. in Sect. 6 was supported by the RSF Grant 16-11-10160. O.K. was also supported by the Ville, Kalle, and Yrjö Väisälä foundation of the Finnish Academy of Science and Letters.

✉ Eugene Gorsky
egorskiy@math.ucdavis.edu

Yuzhe Bai
yzbai@ucdavis.edu

Oscar Kivinen
kivineo1@math.ucdavis.edu

1 Department of Mathematics, University of California, Davis, One Shields Avenue, Davis, CA 95616, USA

2 International Laboratory of Representation Theory and Mathematical Physics, National Research University Higher School of Economics, Moscow, Russia

$$x(t) = x_0 + x_1 t + \cdots + x_{n-1} t^{n-1},$$

such that $x(t)^2 \equiv 0 \mod t^n$. By expanding this equation, we get a system of equations

$$f_1 = x_0^2,\ f_2 = 2x_0 x_1, \ldots, f_n = \sum_{i=0}^{n-1} x_i x_{n-1-i}.$$

We denote the defining ideal of $\mathrm{Jet}^{n-1} D \subseteq \mathbb{A}^n$ by

$$I_n := \langle f_1, \ldots, f_n \rangle \subseteq R_n := \mathbf{k}[x_0, \ldots, x_{n-1}].$$

The ring R_n is $\mathbb{Z}^2_{\geq 0}$-graded by assigning the grading $(i, 1)$ to x_i. It is then clear that the ideal I_n is bihomogeneous. Let

$$H_n(q,t) = \sum_{i,j \geq 0} \dim_k (R_n/I_n)_{i,j} q^i t^j \in \mathbb{Z}[[q,t]]$$

denote the bigraded Hilbert series for R_n/I_n. Our first main result is the following.

Theorem 1.1 *The series $H_n(q,t)$ satisfies the recursion relation*

$$H_n(q,t) = \frac{H_{n-2}(q, qt) + t H_{n-3}(q, q^2 t)}{1 - q^{n-1} t}$$

with initial conditions

$$H_0(q,t) = 1, \quad H_1(q,t) = 1 + t, \quad H_2(q,t) = \frac{1}{1-qt} + t.$$

Using this recursion relation, we obtain explicit combinatorial formulas for $H_n(q,t)$:

Theorem 1.2 *The Hilbert series $H_n(q,t)$ is given by the following explicit formula:*

$$H_n(q,t) = \sum_{p=0}^{\infty} \frac{\binom{h(n,p)+1}{p}_q \cdot q^{p(p-1)} t^p}{(1 - q^{n-h(n,p)} t) \cdots (1 - q^{n-1} t)},$$

where $h(n,p) = \lfloor \frac{n-p}{2} \rfloor$.

In the limit $n \to \infty$, we reprove the theorem of Bruschek et al. [4], which relates the Hilbert series of the arc space for the double point to the Rogers–Ramanujan identity. In fact, we refine their result by considering an additional grading, see Eq. (7.1) . Similar results for $n = \infty$ were obtained by Feigin–Stoyanovsky [8,9], Lepowsky et al. [5,6], and the second author, Oblomkov and Rasmussen in [10].

Remark 1.3 We note that in the Lie theoretic literature the variables of R_n are often indexed $x_1, x_2, \ldots$ or $x_{-1}, x_{-2}, \ldots$ instead of our choice $x_0, x_1, \ldots$. We feel our normalization is more natural for purposes of commutative algebra.

Although our approach to the computation of the Hilbert series is inspired by [4], it is quite different. The key result in [4] shows that for $n = \infty$ the polynomials f_k form a Gröbner basis of the ideal I_∞. As shown below, the Gröbner basis of the ideal I_n for finite n is larger and has a very subtle recursive structure. We completely describe such a basis in Theorems 4.2 and 4.6. In particular, we prove the following.

Theorem 1.4 *Let $k > 2$. Then the reduced Gröbner basis for I_n contains $\binom{\lfloor \frac{n-k+1}{2} \rfloor}{k-2}$ polynomials of degree k.*

Our proof of Theorem 1.1 does not use Gröbner bases at all. First, by an explicit inductive argument in Theorem 2.2 we give a complete description of the first syzygy module for f_i. Then, we define a "shift operator" $S : R_n \to R_{n+1}$, which sends x_i to x_{i+1}, and identify $I_n \cap x_0 R_n$ and $I_n/(I_n \cap x_0 R_n)$ with the images of I_{n-3} and I_{n-2} under appropriate powers of S. This implies the recursion relation in Theorem 1.1.

Remark 1.5 The shift operator has a left inverse given by $x_i \mapsto x_{i-1}$ for $i \geq 1$. It can be extended to a derivation, and in the $n \to \infty$ limit this derivation has been successfully used in work of Capparelli–Lepowsky–Milas as well as Kanade [5,12] to prove results similar to ours, using a backward induction argument. This is in contrast to our forward inductions which work for all n. We note also that in [4] a forward induction argument is used in the $n \to \infty$ limit. The representation-theoretic origin of the shift operators lies in the lattice part of the affine Weyl group of type A_1, but we do not pursue this connection further.

We also observe a recursive structure in the minimal free resolution of R_n/I_n. In particular, we prove the following:

Theorem 1.6 *Let $b(i, n)$ denote the rank of the ith term in the minimal free resolution for R_n/I_n, in other words the ith Betti number. Then*

$$b(i, n) = b(i, n-1) + b(i-1, n-3) + b(i-2, n-3).$$

As a consequence, we can compute the projective dimension of R_n/I_n.

Corollary 1.7 *The projective dimension of R_n/I_n equals $\lceil \frac{2n}{3} \rceil$.*

Remark 1.8 It is easy to see that the reduced scheme $(\mathrm{Jet}^{n-1} D)^{\mathrm{red}}$ is a linear subspace given by the equations $x_0 = \ldots = x_{\lfloor \frac{n-1}{2} \rfloor} = 0$ and has dimension

$$\dim \mathrm{Jet}^{n-1} D = n - 1 - \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lceil \frac{n-1}{2} \right\rceil.$$

A more careful analysis of the gradings in Theorem 1.6 implies another formula for the series $H_n(q, t)$ which was first conjectured in [10].

Theorem 1.9 *The Hilbert series of R_n/I_n has the following form:*

$$H_n(q,t) = \frac{1}{\prod_{i=0}^{n-1}(1-q^i t)} \sum_{p=0}^{\infty} (-1)^p \prod_{k=0}^{p-1}(1-q^k t) \times \left(q^{\frac{5p^2-3p}{2}} t^{2p} \binom{n-2p+1}{p}_q - q^{\frac{5p^2+5p}{2}} t^{2p+2} \binom{n-2p-1}{p}_q \right).$$

The paper is organized as follows. In Sect. 2 we introduce the shift operator S, describe its properties and prove Theorem 2.2 which explicitly describes all syzygies between the f_i. In Sect. 3, we use the shift operator to find a recursive relation for the Hilbert series and to prove Theorem 1.1. In Sect. 4, we use the recursive structure to describe a Gröbner basis for I_n. In Sect. 5, we give a recursive description of the minimal free resolution of R_n/I_n and prove Theorem 1.6. In Sect. 6, we solve both of the above recursions explicitly (with the given initial conditions) and give two explicit combinatorial formulas for $H_n(q,t)$. Finally, in Sect. 7 we briefly discuss the limit of all these techniques at $n \to \infty$ and the connection to the Rogers–Ramanujan identity.

2 Ideals and syzygies

2.1 Ideals

Let $R_n = \mathbf{k}[x_0, \ldots, x_{n-1}]$ and $f_k = \sum_{i=0}^{k-1} x_i x_{k-1-i}$. Define $I_n \subseteq R_n$ to be the ideal generated by $f_1, \ldots, f_n$. Let F_n be the free R_n-module with the basis $e_1, \ldots, e_n$. Consider the map $\phi_n : F_n \to R_n$ given by the equation

$$\phi_n(\alpha_1, \ldots, \alpha_n) = f_1\alpha_1 + \ldots + f_n\alpha_n.$$

The R_n-module $\mathrm{Ker}(\phi_n)$ is called the first syzygy module of I_n.

Lemma 2.1 *One has*

$$\sum_{i=0}^{n} (n-3i) x_i f_{n+1-i} = 0. \tag{2.1}$$

Proof Indeed,

$$\sum_{i=0}^{n} (n-3i) x_i f_{n+1-i} = \sum_{i+k+l=n} (n-3i) x_i x_k x_l.$$

The coefficient at each monomial $x_i x_k x_l$ equals

$$(n-3i) + (n-3k) + (n-3l) = 3n - 3(i+k+l) = 3n - 3n = 0.$$

□

For $0 < k < n$, define

$$\mu_k := (-2kx_k, (-2k+3)x_{k-1}, \ldots, kx_0, 0, \ldots, 0) \in F_n.$$

By (2.1), we have $\phi_n(\mu_k) = 0$. Denote also $\nu_{ij} = f_i e_j - f_j e_i$ (for $i \neq j$). It is clear that $\phi_n(\nu_{ij}) = 0$. The main result of this section is the following.

Theorem 2.2 *The first syzygy module* $\mathrm{Ker}(\phi_n)$ *is generated by* μ_k *and* $\nu_{i,j}$ *over* R_n*.*

We prove Theorem 2.2 in Sect. 2.4.

Remark 2.3 For $n = \infty$ a similar result was obtained by Kanade [12], whose constructions are motivated by the affine Lie algebra $\widehat{\mathfrak{sl}_2}$. In particular, the ideals I_n in Kanade's picture, as well as in, e.g., Capparelli–Lepowsky–Milas [6], correspond to the principal subspaces of level 1 standard modules for $\widehat{\mathfrak{sl}_2}$. See also [8,10].

2.2 The shift operator

We define a ring homomorphism $S : R_n \to R_{n+1}$ by the equation $S(x_i) = x_{i+1}$. Note that S is injective and we can uniquely write any polynomial in R_n in the form

$$f = x_0 f' + S(f''),\ \ f' \in R_n,\ f'' \in R_{n-1}.$$

The following equation is clear from the definition and will be very useful:

$$f_n = 2x_0x_{n-1} + S(f_{n-2}). \tag{2.2}$$

By abuse of notation, denote also $S : F_n \to F_{n+2}$ the map which is given by

$$S(\alpha_1, \ldots, \alpha_n) = (0, 0, S(\alpha_1), \ldots, S(\alpha_n)). \tag{2.3}$$

Lemma 2.4 *Let* $\alpha \in F_n$*. Then* $\phi_{n+2}(S(\alpha))$ *is divisible by* x_0 *if and only if* $\phi_n(\alpha) = 0$*.*

Proof By (2.2) we have

$$\phi_{n+2}(S(\alpha)) = \sum_{i=1}^{n} S(\alpha_i) f_{i+2} \equiv S\left(\sum_{i=1}^{n} \alpha_i f_i\right) \mod x_0.$$

Therefore $\phi_{n+2}(S(\alpha))$ is divisible by x_0 if and only if $S(\sum \alpha_i f_i)$ is divisible by x_0. But since no shift contains x_0, this happens if and only if

$$S\left(\sum \alpha_i f_i\right) = 0 \Leftrightarrow \sum \alpha_i f_i = \phi_n(\alpha) = 0.$$

□

Since $\phi_n(\mu_k) = \phi_n(\nu_{ij}) = 0$, by Lemma 2.4 the images of $S(\mu_k)$ and $S(\nu_{ij})$ under ϕ_{n+2} are divisible by x_0. The following lemma describes these images explicitly.

Lemma 2.5 *One has* $\phi_{n+2}(S(\mu_k)) = (2k+6)x_{k+3}f_1+(2k+3)x_{k+2}f_2-(k+3)x_0 f_{k+4}$, $\phi_{n+2}(S(\nu_{ij})) = 2x_0x_{j+1}f_{i+2} - 2x_0x_{i+1}f_{j+2}$.

Proof By definition,

$$\begin{aligned} S(\mu_k) &= (0, 0, -2kx_{k+1}, (-2k+3)x_k, \dots, kx_1, 0, \dots, 0) \\ &= \mu_{k+3} + (2k+6)x_{k+3}e_1 + (2k+3)x_{k+2}e_2 - (k+3)x_0e_{k+4}, \end{aligned}$$

so

$$\phi_{n+2}(S(\mu_k)) = (2k+6)x_{k+3}f_1 + (2k+3)x_{k+2}f_2 - (k+3)x_0 f_{k+4}.$$

Also, $S(\nu_{ij}) = S(f_i)e_{j+2} - S(f_j)e_{i+2}$, so

$$\begin{aligned} \phi_{n+2}(S(\nu_{ij})) &= S(f_i)f_{j+2} - S(f_j)f_{i+2} \\ &= (f_{i+2} - 2x_0x_{i+1})f_{j+2} - (f_{j+2} - 2x_0x_{j+1})f_{i+2} \\ &= 2x_0x_{j+1}f_{i+2} - 2x_0x_{i+1}f_{j+2}. \end{aligned}$$

□

Corollary 2.6 *One has*

$$\begin{aligned} \phi_{n+2}(S(\mu_k)) &= (2k+3)x_{k+2}f_2 - (k+3)x_0S(f_{k+2}) \\ &= kx_{k+2}f_2 - (k+3)x_0S^2(f_k). \end{aligned}$$

Proof

$$\begin{aligned} \phi_{n+2}(S(\mu_k)) &= (2k+6)x_{k+3}f_1 + (2k+3)x_{k+2}f_2 - (k+3)x_0 f_{k+4} \\ &= (2k+6)x_{k+3}f_1 + (2k+3)x_{k+2}f_2 \\ &\quad - (k+3)(2x_0^2x_{k+3} + 2x_0x_1x_{k+2} + x_0S^2(f_k)) \\ &= (2k+3)x_{k+2}f_2 - (k+3)x_0S(f_{k+2}) \\ &= kx_{k+2}f_2 - (k+3)x_0S^2(f_k). \end{aligned}$$

□

Example 2.7 $\mu_1 = (-2x_1, x_0)$, so $S(\mu_1) = (0, 0, -2x_2, x_1)$, and

$$\begin{aligned} \phi_4(S(\mu_1)) &= -2x_2(2x_0x_2 + x_1^2) + x_1(2x_0x_3 + 2x_1x_2) \\ &= 2x_3x_0x_1 - 4x_0x_2^2 = x_3f_2 - 4x_0S^2(x_0^2). \end{aligned}$$

Lemma 2.8 *The polynomial* $x_1S(f_{n-2})$ *can be expressed via* $f_1, \dots, f_{n-1}$ *modulo* x_0.

Proof We have $(n-3)x_0 f_{n-2} + (n-6)x_1 f_{n-3} + \cdots - 2(n-3)x_{n-2}f_0 = 0$, so

$$(n-3)x_1S(f_{n-2}) + (n-6)x_2S(f_{n-3}) + \cdots - 2(n-3)x_{n-1}S(f_0) = 0.$$

It remains to notice that $S(f_i) \equiv f_{i+2} \mod x_0$. □

Lemma 2.9 *Assume that* $\mathrm{Ker}(\phi_{n-2})$ *is generated by* μ_k *and* $\nu_{i,j}$ *and suppose that* $\phi_n(\alpha)$ *is divisible by* x_0. *Then* $\alpha_n = Ax_0 + Bx_1 + \sum_{i=3}^{n-1} \gamma_i f_i$ *for some* A, B, *and* γ_i.

Proof As stated above, we can write $\alpha_i = x_0\alpha_i' + S(\alpha_{i-2}'')$ for $i \geq 3$. Since f_1 and f_2 are divisible by x_0, we get

$$\phi_n(S(\alpha'')) = \sum_{i=3}^{n} S(\alpha_{i-2}'') f_i \equiv \sum_{i=1}^{n} \alpha_i f_i \equiv 0 \mod x_0.$$

By Lemma 2.4 we get $\phi_{n-2}(\alpha'') = 0$. By the assumption, we can write

$$\alpha'' = \sum_{k<n-2} \beta_k \mu_k + \sum_{i<j\leq n-2} \gamma_{i,j} \nu_{ij}.$$

Therefore

$$\alpha_{n-2}'' = \beta_{n-1} x_0 + \sum_{j\leq n-3} \gamma_{j,n-2} f_j,$$

and

$$\begin{aligned}\alpha_n &= x_0\alpha_n' + S(\alpha_{n-2}'') = x_0\alpha_n' + S(\beta_{n-1})x_1 \\ &\quad + \sum_{j\leq n-3} S(\gamma_{j,n-2})(f_{j+2} - 2x_0x_{j+1}).\end{aligned}$$

□

2.3 Examples

Before proving Theorem 2.2, we would like to present the proof for $n \leq 4$.

Example 2.10 For $n = 2$, we have $f_1 = x_0^2$ and $f_2 = 2x_0x_1$, so the module of syzygies is clearly generated by $(-2x_1, x_0) = \mu_1$.

Example 2.11 Let $n = 3$, suppose that $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0$. We can write $\alpha_3 = \alpha_3' x_0 + \alpha_3''$, where α_3'' does not contain x_0. Since f_1 and f_2 are divisible by x_0 and $f_3 = 2x_0x_2 + x_1^2$, we get $x_1^2\alpha_3'' = 0$, so $\alpha_3'' = 0$. Now $\alpha = \frac{1}{2}\alpha_3'\mu_2 + \gamma$, where γ is a syzygy between f_i with $\gamma_3 = 0$. By the previous example, γ is a multiple of μ_1, so the module of syzygies is actually generated by μ_1 and μ_2.

Example 2.12 Let $n = 4$, suppose that α is a syzygy. We can write $\alpha_3 = \alpha_3' x_0 + \alpha_3''$ and $\alpha_4 = \alpha_4' x_0 + \alpha_4''$ where α_i'' do not contain x_0. Similarly to the previous case, we obtain

$$\alpha_3'' x_1^2 + \alpha_4'' \cdot 2x_1x_2 = 0. \tag{2.4}$$

This means that there exists some β such that $\alpha_3'' = -2x_2\beta$ and $\alpha_4'' = x_1\beta$. Now

$$\alpha_1 x_0^2 + \alpha_2 \cdot 2x_0x_1 + (\alpha_3' x_0 - 2x_2\beta)(2x_0x_2 + x_1^2) + (\alpha_4' x_0 + x_1\beta)(2x_0x_3 + 2x_1x_2) = 0.$$

The terms without x_0 cancel, and the linear terms in x_0 are the following:

$$x_0(2\alpha_2 x_1 + \alpha_3' x_1^2 - 4x_2^2\beta + 2\alpha' 4x_1x_2 + 2\beta x_1x_3) = 0.$$

Note that all terms but $-4x_2^2\beta$ are divisible by x_1, so β is divisible by x_1, $\beta = mx_1$. Then

$$\alpha_4 = \alpha_4' x_0 + mx_1^2 = (\alpha_4' - 2x_2m)x_0 + mf_3.$$

By subtracting $m\nu_{3,4} + \frac{1}{3}(\alpha_4' - 2x_2m)\mu_3$ from α, we obtain a syzygy between f_1, f_2, f_3, and reduce to the previous case.

2.4 Syzygies

In this section, we prove Theorem 2.2 by induction on n. The base cases were covered in Sect. 2.3. Suppose that $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathrm{Ker}(\phi_n)$, i.e., is a linear relation between $f_1, \ldots, f_n$. As stated above, write $\alpha_i = \alpha_i' x_0 + S(\alpha_{i-2}'')$ for $i \geq 3$. Without loss of generality, we can assume that α_i' do not contain x_0 (otherwise we can subtract a multiple of $\nu_{1,i}$). Since

$$f_i = 2x_0x_{i-1} + S(f_{i-2}),$$

by collecting terms without x_0 we get $\sum_{i=3}^n S(\alpha_{i-2}'')S(f_{i-2}) = 0$. This means that $\phi_{n-2}(\alpha'') = 0$ and by the induction assumption we may then write

$$\alpha'' = \sum_{i=3}^{n-1} \beta_{i+1}\mu_{i-2} + \sum_{3\leq j<k\leq n,\, j\neq k} \beta_{j,k}\nu_{j-2,k-2}.$$

Because

$$S(\nu_{j-2,k-2}) = -S(f_{k-2})e_j + S(f_{j-2})e_k = \nu_{j,k} + 2x_0x_ke_j - 2x_0x_je_k,$$

without loss of generality we can assume $\alpha'' = S(\sum_{i=3}^{n-1} \beta_{i+1}\mu_{i-2})$. By Corollary 2.6 we get

$$\phi_n(S(\mu_{i-2})) = -(i+1)x_0S(f_i) + (2i-1)x_{i-1}f_2,$$

hence

$$\phi_n(\alpha) = \alpha_1 f_1 + (\alpha_2 + \sum_{i=3}^{n-1}(2i-1)S(\beta_{i+1})x_{i-1})f_2 + \sum_{i=3}^{n} x_0\alpha_i' f_i - \sum_{i=3}^{n-1}(i+1)S(\beta_{i+1})x_0 S(f_i) = 0.$$

By collecting the terms linear in x_0, we get

$$\left(\alpha_2 + \sum_{i=3}^{n-1}(2i-1)S(\beta_{i+1})x_{i-1}\right)2x_1 + \sum_{i=3}^{n}\alpha_i' S(f_{i-2}) - \sum_{i=3}^{n-1}(i+1)S(\beta_{i+1})S(f_i) = 0,$$

so

$$\sum_{i=3}^{n}\alpha_i' S(f_{i-2}) - \sum_{i=3}^{n-1}(i+1)S(\beta_{i+1})S(f_i)$$

is divisible by x_1, and

$$\sum_{i=3}^{n}\alpha_i''' f_{i-2} - \sum_{i=3}^{n-1}(i+1)\beta_{i+1} f_i$$

is divisible by x_0, where $\alpha_i' = S(\alpha_i''')$. By Lemma 2.9, this implies

$$\beta_n = Bx_0 + Cx_1 + \sum_{i=3}^{n-2}\gamma_i f_i$$

for some constants B, C. Now we can rewrite

$$\alpha_n = \alpha_n' x_0 + S(\beta_n x_0) = \alpha_n' x_0 + Bx_1^2 + Cx_1x_2 + \sum_{i=3}^{n-3}\gamma_i x_1(f_{i+2} - 2x_0x_{n-1}) + \gamma_{n-2}x_1 S(f_{n-2}).$$

Observe that $x_1^2 = f_3 - 2x_0x_2$, $x_1x_2 = \frac{1}{2}(f_3 - 2x_0x_3)$ and by Lemma 2.8 $x_1 S(f_{n-2})$ can be expressed via $f_1, \ldots, f_{n-1}$ modulo x_0. In other words,

$$\alpha_n = \delta x_0 + \sum_{i=3}^{n-1}\delta_i f_i$$

for some coefficients δ_i. Then $\alpha - \frac{1}{n-1}\delta\mu_{n-1} - \sum_{i=3}^{n-1}\delta_i\nu_{i,j}$ is a syzygy between $f_1, \ldots, f_{n-1}$, so by the induction assumption it can be expressed as an R_{n-1}-linear combination of the μ_i and $\nu_{i,j}$.

Remark 2.13 The above proof shows that the syzygies $\nu_{1,k}$ and $\nu_{2,k}$ are not necessary, and can be expressed as linear combinations of other syzygies. Indeed, since the coefficients at e_k are divisible by x_0, one can subtract an appropriate multiple of μ_{k-1} and get a syzygy involving $e_1, \ldots, e_{k-1}$ only.

3 Hilbert series

In this section, we prove Theorem 3.5 by studying the relation between the ideals I_n and $x_0 R_n$.

Lemma 3.1 *One has*

$$R_n/(x_0 R_n + I_n) \simeq S(R_{n-2}/I_{n-2})[x_{n-1}]$$

as R_n-modules, the module structure on the right coming from $S : R_{n-1} \to R_n$.

Proof We have $x_0 R_n + I_n = \langle x_0, f_1, \ldots, f_n\rangle = \langle x_0, S(f_1), \ldots, S(f_{n-2})\rangle$, so

$$R_n/(x_0 R_n + I_n) = R_n/\langle x_0, S(f_1), \ldots, S(f_{n-2})\rangle = S(R_{n-2}/I_{n-2})[x_{n-1}].$$

□

Lemma 3.2 *The subspace $x_0 S^2(I_{n-3})[x_{n-1}]$ does not intersect the ideal $\langle f_1, f_2\rangle$ in R_n. Furthermore, $x_0 S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2\rangle$ is an ideal in R_n which is contained in $I_n \cap x_0 R_n$.*

Proof Given a non-zero polynomial $g \in I_{n-3}$, the iterated shift $S^2(g)$ does not contain x_0 or x_1, so that $x_0 S^2(g)$ is not contained in $\langle f_1, f_2\rangle$. Furthermore, I_{n-3} is stable under multiplication by $x_0, \ldots, x_{n-4}$, so $S^2(I_{n-3})$ is stable under multiplication by $x_2, \ldots, x_{n-2}$, and $x_0 S^2(I_{n-3})[x_{n-1}]$ is stable under multiplication by $x_2, \ldots, x_{n-1}$. Multiplication by x_0 or x_1 sends the latter subspace to $\langle f_1, f_2\rangle$, so $x_0 S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2\rangle$ is an ideal in R_n.

Finally, to prove that this ideal is contained in I_n, it is sufficient to prove that $x_0 S^2(f_k) \in I_n$ for $k \le n-3$. On the other hand, by Corollary 2.6:

$$x_0 S^2(f_k) = \frac{1}{k+3}\phi_n(S(\mu_k)) \mod \langle f_1, f_2\rangle.$$

□

Lemma 3.3 *One has*

$$I_n \cap x_0 R_n = x_0 S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2\rangle.$$

Proof By Lemma 3.2, the right-hand side is a submodule of the left-hand side, so it remains to prove the reverse inclusion. We have

$$f_i = 2x_0 x_{i-1} + S(f_{i-2}) = 2x_0 x_{i-1} + 2x_1 x_{i-2} + S^2(f_{i-4}).$$

Suppose that $\sum_{i=1}^{n} \alpha_i f_i \in I_n \cap x_0 R_n$. Then by Lemma 2.9,

$$\alpha_n = Ax_0 + Bx_1 + \sum_j \gamma_j f_j = A'x_0 + B'x_1 + \sum_j \gamma_j S^2(f_{j-4}).$$

Now by (2.1) and Corollary 2.6, $x_0 f_n$ and $x_1 f_n$ can be expressed as R_n-linear combinations of $f_1, \dots, f_{n-1}$ and elements of $x_0 S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2\rangle$, so $\sum_{i=1}^{n} \alpha_i f_i$ can be expressed as such a combination as well. Induction on n finishes the proof. □

Corollary 3.4 *One has*

$$x_0 R_n/(I_n \cap x_0 R_n) = x_0 S^2(R_{n-3}/I_{n-3})[x_{n-1}].$$

Proof We have

$$x_0 R_n/\langle f_1, f_2\rangle = x_0 R_n/(x_0^2, x_0 x_1) = x_0 \mathbf{k}[x_2, \dots, x_{n-1}] = x_0 S^2(R_{n-3})[x_{n-1}].$$

Therefore

$$\begin{aligned} x_0 R_n/(I_n \cap x_0 R_n) &= x_0 R_n/(x_0 S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2\rangle) \\ &= x_0 S^2(R_{n-3}/I_{n-3})[x_{n-1}]. \end{aligned}$$

□

Theorem 3.5 *Let $H_n(q,t)$ denote the bigraded Hilbert series of the quotient R_n/I_n. Then one has the following recursion relation:*

$$H_n(q,t) = \frac{H_{n-2}(q,qt) + tH_{n-3}(q,q^2t)}{1 - q^{n-1}t} \tag{3.1}$$

with initial conditions

$$H_0(q,t) = 1, \quad H_1(q,t) = 1 + t, \quad H_2(q,t) = \frac{1}{1-qt} + t.$$

Remark 3.6 This recursion is similar, but not identical to the various recursions considered by Andrews [1–3] in his proofs of the Rogers–Ramanujan identity. It is also similar to the recursions recently considered by Paramonov [13] in a different context. In the $n \to \infty$ limit, Capparelli–Lepowsky–Milas [5] use analogous exact sequences for principal level 1 subspaces of the standard modules of $\widehat{\mathfrak{sl}_2}$ to arrive at a similar formula.

Proof We have an exact sequence

$$0 \to x_0 R_n/(x_0 R_n \cap I_n) \to R_n/I_n \to R_n/(x_0 R_n + I_n) \to 0.$$

By Lemma 3.1, the Hilbert series of $R_n/(x_0 R_n + I_n)$ equals $\frac{H_{n-2}(q,qt)}{1-q^{n-1}t}$, and by Corollary 3.4 the Hilbert series of $x_0 R_n/(x_0 R_n \cap I_n)$ equals $\frac{tH_{n-3}(q,q^2t)}{1-q^{n-1}t}$. □

4 Gröbner bases

We will now compute Gröbner bases for the ideals I_n. Recall that a *Gröbner basis* for an ideal I is a subset $G = \{g_1, \ldots, g_s\} \subset I$ such that, for a chosen monomial ordering $<$,

$$\langle \mathrm{LT}_<(g_1), \ldots, \mathrm{LT}_<(g_s)\rangle = \mathrm{LT}_<(I),$$

where $\mathrm{LT}_<$ denotes leading term.

Let us order the monomials in R_n in grevlex order, that is

$$x^\alpha < x^\beta$$

if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and the rightmost entry of $\alpha - \beta$ is negative.

Remark 4.1 In fact, any order refining the reverse lexicographic order will work, but for definiteness and its popularity in computer algebra systems we shall fix grevlex order throughout.

Theorem 4.2 *Let*

$$G_1 = \{f_1\} \subseteq R_1, G_2 = \{f_1, f_2\} \subset R_2$$

and recursively define the sets $G_n, n \geq 3$ as follows:

$$G_n = x_0 S^2(G_{n-3}) \sqcup \{f_1, f_2\} \sqcup \widetilde{S}(G_{n-2}),$$

where $\widetilde{S}$ is a modified shift operator as explained below. Then G_n is a Gröbner basis for I_n.

Remark 4.3 The notation requires explanation. Note that any G_m is naturally a subset of $R_n, n \geq m$ so we can and will identify G_m inside a larger polynomial ring without explicit mention. Furthermore, we denote by $x_0 S^2(G_{n-3})$ the image of G_{n-3} under $S^2 : R_{n-2} \to R_n$ multiplied by x_0. The "operator" $\widetilde{S}$ is defined on elements $p \in I_{n-2}$ as follows: write $p = \sum_{i=1}^n \varphi_i f_i$, and let

$$\widetilde{S}(p) = \sum_{i=1}^n S(\varphi_i) f_{i+2}.$$

Note that by (2.2), we have $\widetilde{S}(p) = S(p) + \sum_{i=1}^{n} x_0 x_{i+2} S(\varphi_i) \in I_{n+2}$. In particular, if $p \neq 0$ and p is homogeneous then $\mathrm{LT}_<(\widetilde{S}(p)) = S(\mathrm{LT}_<(p))$. Therefore, the construction of $\widetilde{S}(p)$ requires a choice if φ_i, but the leading term of the result does not depend on this choice.

Proof We will proceed by induction. The base cases $n = 1, 2$ are clear because the ideals are monomial. Consider now the ideal $\mathrm{LT}_<(I_n)$ generated by all the leading terms of elements of I_n. It is clear by Lemma 3.1 and the fact that S respects the reverse lexicographic order that if $g \in I_n$ is not divisible by x_0, its leading term is the image of a leading term in I_{n-2} under S. Since we assumed G_{n-2} to be a Gröbner basis, we must have $\mathrm{LT}_<(g)$ divisible by some monomial in $S(\mathrm{LT}_<(G_{n-2}))$.

Similarly, if g is divisible by x_0, we know by Lemma 3.2 and order preservation that its leading term is the image under $x_0 S^2$ of a leading term in I_{n-3} or divisible by f_1, f_2. By the induction assumption $\mathrm{LT}_<(g)$ is then divisible by an element of $x_0 S^2(\mathrm{LT}_<(G_{n-3})) \sqcup \{f_1, f_2\}$. In particular, $\mathrm{LT}_<(I_n) \subseteq \langle \mathrm{LT}_<(G_n)\rangle$. But the reverse inclusion is clear, so we have

$$\mathrm{LT}_<(I_n) = \langle \mathrm{LT}_<(G_n)\rangle$$

as desired, and G_n is a Gröbner basis for I_n. □

Example 4.4 We have

$$\begin{aligned}
G_3 &= \{f_1, f_2, f_3\},\\
G_4 &= \{f_1, f_2, f_3, f_4, x_0 x_2^2\},\\
G_5 &= \{f_1, f_2, f_3, f_4, f_5, x_0 x_2 x_3\},\\
G_6 &= \{f_1, \ldots, f_6, x_0 x_3^2 + 2x_0 x_2 x_4, 2x_1 x_3^2 + 3x_0 x_3 x_4 - x_0 x_2 x_5\}.
\end{aligned}$$

Note that the last polynomial in G_6 can be identified with $\widetilde{S}(x_0 x_2^2) \in \widetilde{S}(G_4)$. Indeed,

$$\begin{aligned}
4x_0 x_2^2 &= 2x_2(2x_0 x_2 + x_1^2) - x_1(2x_0 x_3 + 2x_1 x_2) + x_3(2x_0 x_1)\\
&= 2x_2 f_3 - x_1 f_4 + x_3 f_2,
\end{aligned}$$

so

$$\begin{aligned}
\widetilde{S}(4x_0 x_2^2) &= 2x_3 f_5 - x_2 f_6 + x_4 f_4\\
&= 2x_3(2x_0 x_4 + 2x_1 x_3 + x_2^2) - x_2(2x_0 x_5 + 2x_1 x_4 + 2x_2 x_3)\\
&\quad + x_4(2x_0 x_3 + 2x_1 x_2)\\
&= 4x_1 x_3^2 + 6x_0 x_3 x_4 - 2x_0 x_2 x_5.
\end{aligned}$$

Remark 4.5 The Gröbner basis constructed in Theorem 4.2 is far from being reduced. The following theorem describes the reduced basis implicitly.

Since all G_n contain $\{f_1, \ldots, f_n\}$ and none of their leading terms divides one another, we can throw away other polynomials in G_n in a controlled manner to obtain a minimal Gröbner basis. That is to say, if the leading terms of $G_n\backslash\{g\}$ still generate the leading ideal we are in business. Therefore after appropriate reduction [7, Proposition 6 on p. 92] we get a reduced Gröbner basis with the same leading terms.

Let us call a monomial $\prod x_i^{a_i}$ *admissible* if $a_i + a_{i+1} \leq 1$ for all i, that is, it is not divisible by x_i^2 or by $x_i x_{i+1}$.

Theorem 4.6 *Fix $k > 2$. The leading terms of (t-)degree k in a reduced Gröbner basis for I_n have the form $m(x)\,\mathrm{LT}_<(f_{n+k-2})$ where $m(x)$ is an admissible monomial of degree $k-2$ in variables $x_0, \ldots, x_{\lfloor \frac{n+k-7}{2} \rfloor}$. The number of degree k polynomials in the reduced Gröbner basis equals $\binom{\lfloor \frac{n-k+1}{2} \rfloor}{k-2}$.*

Remark 4.7 It is easy to see that there are no linear polynomials in the Gröbner basis (or in the ideal I_n), and $f_1, \ldots, f_n$ are the only quadratic polynomials in the reduced Gröbner basis.

Proof We prove the statement by induction in n. Suppose that it is true for G_{n-2} and G_{n-3}. By Theorem 4.2, the leading monomials in the degree k part of G_n consist of shifted degree k monomials in G_{n-2}, and twice shifted degree $(k-1)$ monomials in G_{n-3}, multiplied by x_0.

Consider first the case $k = 3$. We will prove that the leading terms in the reduced Gröbner basis have the form $x_j\,\mathrm{LT}_<(f_{n+1})$ for $j \leq \lfloor \frac{n-4}{2} \rfloor$. Indeed, in the first case, we get $S(x_j\,\mathrm{LT}_<(f_{(n-2)+1})) = x_{j+1}\,\mathrm{LT}_<(f_{n+1})$. In the second case, we have to consider the polynomials $x_0 S^2(f_i)$ for all $i \leq n-3$. Observe that for $i \leq n-4$ we get $\mathrm{LT}_<(x_0 S^2(f_i)) = x_0\,\mathrm{LT}_<(f_{i+4})$ and hence divisible by the leading term of f_{i+4} and can be eliminated. For $i = n-3$ we get $\mathrm{LT}_<(x_0 S^2(f_{n-3})) = x_0\,\mathrm{LT}_<(f_{n+1})$.

Assume now that $k > 3$. In the first case, we get

$$S(m(x)\,\mathrm{LT}_<(f_{(n-2)+k-2})) = S(m(x))\,\mathrm{LT}_<(f_{n+k-2}).$$

If $m(x)$ is an admissible monomial in x_j, $0 \leq j \leq \lfloor \frac{(n-2)+k-7}{2} \rfloor$ then $S(m(x))$ is an admissible monomial in x_j, $1 \leq j \leq \lfloor \frac{(n-2)+k-7}{2} \rfloor + 1 = \lfloor \frac{n+k-7}{2} \rfloor$.

In the second case, we get

$$x_0 S^2(m(x))\,\mathrm{LT}_<(f_{(n-3)+(k-1)-2})) = x_0 S^2(m(x))\,\mathrm{LT}_<(f_{n+k-2}).$$

Now $S^2(m(x))$ is an admissible monomial in x_j, $2 \leq j \leq \lfloor \frac{(n-3)+(k-1)-7}{2} \rfloor + 2 = \lfloor \frac{n+k-7}{2} \rfloor$, so $x_0 S^2(m(x))$ is also an admissible in a correct set of variables. In fact, all such monomials not divisible by x_0 appear from the first case, and the ones divisible by x_0 appear from the second case.

It is easy to see that none of these leading monomials are divisible by each other. Therefore after appropriate reduction [7] we get a reduced Gröbner basis with the same leading terms.

Finally, we can count monomials of given degree k. The number of admissible monomials of degree l in s variables equals $\binom{s-l+1}{l}$, so the number of polynomials in G_n of degree k equals

$$\binom{1 + \lfloor \frac{n+k-7}{2} \rfloor - (k-2) + 1}{k-2} = \binom{\lfloor \frac{n-k+1}{2} \rfloor}{k-2}.$$

□

Example 4.8 Let $n = 12$. The reduced Gröbner basis for I_{12} contains quadratic polynomials $f_1, \dots, f_{12}$. It also contains 5 cubic polynomials with leading terms

$$x_0x_6^2, x_1x_6^2, x_2x_6^2, x_3x_6^2, x_4x_6^2,$$

6 quartic polynomials with leading terms

$$x_0x_2x_6x_7, x_0x_3x_6x_7, x_0x_4x_6x_7, x_1x_3x_6x_7, x_1x_4x_6x_7, x_2x_4x_6x_7,$$

and 4 quintic polynomials with leading terms

$$x_0x_2x_4x_7^2, x_0x_2x_5x_7^2, x_0x_3x_5x_7^2, x_1x_3x_5x_7^2.$$

Observe that $\mathrm{LT}_<(f_{13}) = x_6^2$, $\mathrm{LT}_<(f_{14}) = x_6x_7$, and $\mathrm{LT}_<(f_{15}) = x_7^2$.

5 Minimal resolution

In this section, we describe the bigraded minimal free resolutions of I_n and R_n/I_n. We write them as follows:

$$0 \leftarrow I_n \leftarrow F(1,n) \leftarrow F(2,n) \leftarrow F(3,n) \cdots$$

and

$$0 \leftarrow R_n/I_n \leftarrow R_n = F(0,n) \leftarrow F(1,n) \leftarrow F(2,n) \leftarrow F(3,n) \cdots$$

Theorem 5.1 *Let $F(i,n)$ be the i-th term in the minimal free resolution for I_n. Then there is an injection $F(i,n-1) \hookrightarrow F(i,n)$, and*

$$F(i,n)/F(i,n-1) \simeq S(F(i-1,n-3)) \oplus x_0S(F(i-2,n-3))$$

as R_n-modules, and the shift of a free R_n-module is as in (2.3). Note that the gradings in the right-hand side are shifted by the bidegree of f_n (which equals $q^{n-1}t^2$).

Proof Observe that the ideal generated by $f_1, \dots, f_{n-1}$ in R_n is isomorphic to $I_{n-1}[x_{n-1}]$, so its minimal resolution over R_n is identical to the one for I_{n-1} over R_{n-1} tensored over R_n. Moreover, since $I_n = \langle f_1, \dots, f_n \rangle$, the minimal free R_n-resolution of $I_{n-1}[x_{n-1}]$ is naturally a subcomplex of the minimal free resolution for I_n. In other words, $F(i,n-1) \otimes_{R_{n-1}} R_n$ can be identified with a subspace in $F(i,n)$, which we will by abuse of notation also denote $F(i, n-1)$. We have a short exact sequence

$$0 \to F(i, n-1) \to F(i, n) \to F(i, n)/F(i, n-1) \to 0.$$

From the long exact sequence in cohomology, it is easy to see that $F(i, n)/F(i, n-1)$ is acyclic in positive degrees. Now $I_n = \langle f_1, \dots f_n \rangle$, so $F(1, n)/F(1, n-1) \cong R_n$ is generated by a single-vector $\tilde{f}_n$ corresponding to f_n. Furthermore, by Theorem 2.2 $F(2, n)$ has generators corresponding to $\mu_1, \dots, \mu_{n-1}$ and $\nu_{i,j}$ for $3 \le i < j \le n$, so $F(2, n)/F(2, n-1) \cong R_n^{n-2}$ is spanned by the basis elements corresponding to μ_{n-1} and $\nu_{i,n}$ for $3 \le i \le n-1$. The differential $d : F(2, n) \to F(1, n)$ descends to $\tilde{d} : F(2, n)/F(2, n-1) \to F(1, n)/F(1, n-1)$. It sends μ_{n-1} to $x_0 \tilde{f}_n$ and $\nu_{i,n}$ to $f_i \cdot \tilde{f}_n$.

Therefore, the quotient complex with terms $F(i, n)/F(i, n-1)$ is isomorphic to the minimal resolution of $R_n/\langle x_0, f_3, \dots, f_{n-1}\rangle = R_n/\langle x_0, S(f_1), \dots, S(f_{n-3})\rangle$. The latter is nothing but the (shifted) minimal resolution for I_{n-3} tensored with the two-term complex $R_n \xleftarrow{x_0} R_n$. □

Corollary 5.2 *Let $b(i, n)$ denote the rank of $F(i, n)$. Then*

$$b(i, n) = b(i, n-1) + b(i-1, n-3) + b(i-2, n-3). \tag{5.1}$$

Corollary 5.3 *Let $H_n(q, t)$ denote the Hilbert series for R_n/I_n, and let $\widetilde{H}_n(q, t) = H_n(q, t) \prod_{i=0}^{n-1} (1 - q^i t)$. Then $\widetilde{H}_n(q, t)$ satisfies the following recursion relation:*

$$\widetilde{H}_n(q, t) = \widetilde{H}_{n-1}(q, t) - q^{n-1} t^2 (1 - t^2) \widetilde{H}_{n-3}(q, qt). \tag{5.2}$$

Corollary 5.4 *The projective dimension of I_n equals $\lceil \frac{2n}{3} \rceil - 1$. The projective dimension of R_n/I_n equals $\lceil \frac{2n}{3} \rceil$.*

Proof By definition, the projective dimension $\mathrm{pd}(I_n)$ is equal to the length of the minimal free (or projective) resolution. By (5.1) we have $\mathrm{pd}(I_n) = \mathrm{pd}(I_{n-3}) + 2$. The minimal free resolutions for I_1, I_2, and I_3 are easy to compute:

$$I_1 \xleftarrow{\begin{pmatrix} f_1 \end{pmatrix}} R_1$$

$$I_2 \xleftarrow{\begin{pmatrix} f_1 & f_2 \end{pmatrix}} R_2^2 \xleftarrow{\begin{pmatrix} -2x_1 \\ x_0 \end{pmatrix}} R_2$$

$$I_3 \xleftarrow{\begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}} R_3^3 \xleftarrow{\begin{pmatrix} -2x_0 & -4x_2 \\ x_1 & -x_1 \\ 0 & 2x_0 \end{pmatrix}} R_3^2.$$

The minimal resolution of R_n/I_n is one step longer than the one for I_n. □

6 Combinatorial identities

We define

$$\binom{a}{b}_q = \frac{(1-q)\cdots(1-q^a)}{(1-q)\cdots(1-q^b)\cdot(1-q)\cdots(1-q^{a-b})}.$$

If $a < b$, we set $\binom{a}{b}_q = 0$. The following lemma is well known.

Lemma 6.1 *The following identities holds:*

$$\binom{a}{b}_q + q^{b+1}\binom{a}{b+1}_q = \binom{a+1}{b+1}_q = q^{a-b}\binom{a}{b}_q + \binom{a}{b+1}_q.$$

Proof One has

$$\binom{a}{b+1}_q = \frac{(1-q^{a-b})}{(1-q^{b+1})}\binom{a}{b}_q,$$

hence

$$\binom{a}{b}_q + q^{b+1}\binom{a}{b+1}_q = \binom{a}{b}_q\left(1 + q^{b+1}\frac{(1-q^{a-b})}{(1-q^{b+1})}\right)$$
$$= \binom{a}{b}_q\frac{(1-q^{a+1})}{(1-q^{b+1})} = \binom{a+1}{b+1}_q.$$

□

Theorem 6.2 *The Hilbert series $H_n(q,t)$ is given by the following explicit formula:*

$$H_n(q,t) = \sum_{p=0}^{\infty} \frac{\binom{h(n,p)+1}{p}_q \cdot q^{p(p-1)}t^p}{(1-q^{n-h(n,p)}t)\cdots(1-q^{n-1}t)}, \tag{6.1}$$

where $h(n,p) = \lfloor\frac{n-p}{2}\rfloor$.

Proof By Theorem 3.5 it is sufficient to prove that the right-hand side of (6.1) satisfies the recursion relation (3.1). Let us denote the p-th term in (6.1) by $H_{n,p}(q,t)$ so that $H_n(q,t) = \sum_p H_{n,p}(q,t)$. We have $h(n-2,p) = h(n-3,p-1) = h(n,p)-1$, so

$$H_{n-2,p}(q,qt) = \frac{\binom{h(n,p)}{p}_q \cdot q^{p(p-1)}t^p \cdot q^p}{(1-q^{n-h(n,p)}t)\cdots(1-q^{n-2}t)},$$
$$H_{n-3,p-1}(q,q^2t) = \frac{\binom{h(n,p)}{p-1}_q \cdot q^{(p-1)(p-2)}t^{p-1} \cdot q^{2p-2}}{(1-q^{n-h(n,p)}t)\cdots(1-q^{n-2}t)},$$

therefore

$$
\begin{aligned}
&H_{n-2,p}(q,qt)+tH_{n-3,p-1}(q,q^2t)\\
&\quad=\frac{q^{p(p-1)}t^p}{(1-q^{n-h(n,p)}t)\cdots(1-q^{n-2}t)}\left[q^p\binom{h(n,p)}{p}_q+\binom{h(n,p)}{p-1}_q\right]\\
&\quad=\frac{q^{p(p-1)}t^p}{(1-q^{n-h(n,p)}t)\cdots(1-q^{n-2}t)}\binom{h(n,p)+1}{p}_q\\
&\quad=(1-q^{n-1}t)H_{n,p}(q,t).
\end{aligned}
\tag{6.2}
$$

This proves (3.1), and the initial conditions are easy to check. □

The free resolution of I_n gives another formula for the Hilbert series of R_n/I_n.

Proposition 6.3 *Let $b(i,n)$, as stated above, denote the rank of i-th module in the free resolution of R_n/I_n. Then*

$$
b(i,n)=\sum_p\left[\binom{n-2p+1}{p}\binom{p}{i-p}+\binom{n-2p-1}{p}\binom{p}{i-p-1}\right].
$$

Remark 6.4 The terms in the first sum are non-zero if $p\le(n+1)/3$ and $i/2\le p\le i$. The terms in the second sum are non-zero if $p\le(n-1)/3$ and $(i-1)/2\le p\le(i-1)$.

Proof Let

$$
A(n,p,i)=\binom{n-2p+1}{p}\binom{p}{i-p},\ B(n,p,i)=\binom{n-2p-1}{p}\binom{p}{i-p-1}.
$$

Then

$$
\begin{aligned}
&A(n-1,p,i)+A(n-3,p-1,i-1)+A(n-3,p-1,i-2)\\
&\quad=\binom{n-2p}{p}\binom{p}{i-p}+\binom{n-2p}{p-1}\binom{p-1}{i-p}+\binom{n-2p}{p-1}\binom{p-1}{i-p-1}\\
&\quad=\binom{n-2p}{p}\binom{p}{i-p}+\binom{n-2p}{p-1}\binom{p}{i-p}\\
&\quad=\binom{n-2p+1}{p}\binom{p}{i-p}=A(n,p,i).
\end{aligned}
$$

Similarly, $B(n-1,p,i)+B(n-3,p-1,i-1)+B(n-3,p-1,i-2)=B(n,p,i)$, so the right-hand side satisfies the recursion relation (5.1). It remains to check the base cases:

$$
\begin{aligned}
f(0,n)&=1=\binom{n-1}{0},\\
f(1,n)&=n=\binom{n-1}{1}+\binom{n-3}{0},
\end{aligned}
$$

$$f(2,n) = (n-1) + \binom{n-2}{2} = \binom{n-1}{1} + \binom{n-3}{1} + \binom{n-3}{2}.$$

By Corollary 5.4 $b(i,n) = 0$ for $i > 2$ and $n \leq 3$. □

We have the following (q,t)-analog of Proposition 6.3.

Proposition 6.5 *Let* $\widehat{b}(i,n)$ *denote the bigraded Hilbert polynomial for the generating set in* $F(i,n)$*. Then*

$$\widehat{b}(i,n) = \sum_{p>0} q^{\frac{5p^2-3p+(i-p)(i-p-1)}{2}} t^{2p+(i-p)} \binom{n-2p+1}{p}_q \binom{p}{i-p}_q + q^{\frac{5p^2+5p+(i-p)(i-p-1)}{2}} t^{2p+2+(i-p)} \binom{n-2p-1}{p}_q \binom{p}{i-p-1}_q. \quad (6.3)$$

Proof The proof is completely analogous to the proof of Proposition 6.3, but we include it here for completeness. By Theorem 5.1 we have a recursion relation

$$\widehat{b}(i,n) = \widehat{b}(i,n-1) + q^{n-1}t^2\widehat{b}(i-1,n-3)(q,qt) + q^{n-1}t^3\widehat{b}(i-2,n-3)(q,qt). \quad (6.4)$$

We need to prove that the right-hand side of (6.3) satisfies (6.4). Let

$$\widehat{A}(n,p,i) = q^{\frac{5p^2-3p+(i-p)(i-p-1)}{2}} t^{2p+(i-p)} \binom{n-2p+1}{p}_q \binom{p}{i-p}_q.$$

Then

$$\widehat{A}(n-3,p-1,i-1)(q,qt) = q^{\frac{5p^2-9p+4+(i-p)(i-p+1)}{2}} t^{2p-2+(i-p)} \times \binom{n-2p}{p-1}_q \binom{p-1}{i-p}_q,$$

$$\widehat{A}(n-3,p-1,i-2)(q,qt) = q^{\frac{5p^2-9p+4+(i-p)(i-p-1)}{2}} t^{2p-2+(i-p-1)} \times \binom{n-2p}{p-1}_q \binom{p-1}{i-p-1}_q,$$

so

$$\widehat{A}(n-3,p-1,i-1)(q,qt) + t\widehat{A}(n-3,p-1,i-2)(q,qt) = q^{\frac{5p^2-9p+4+(i-p)(i-p-1)}{2}} t^{2p-2+(i-p)} \binom{n-2p}{p-1}_q \binom{p}{i-p}_q.$$

Now

$$\begin{aligned}
&\widehat{A}(n-1,p,i)+q^{n-1}t^2\widehat{A}(n-3,p-1,i-1)(q,qt)\\
&\quad+q^{n-1}t^3\widehat{A}(n-3,p-1,i-2)(q,qt)\\
&\qquad=q^{\frac{5p^2-3p+(i-p)(i-p-1)}{2}}t^{2p+(i-p)}\\
&\qquad\times\left[\binom{n-2p}{p}_q\binom{p}{i-p}_q+q^{n-3p+1}\binom{n-2p}{p-1}_q\binom{p}{i-p}_q\right]\\
&\qquad=q^{\frac{5p^2-3p+(i-p)(i-p-1)}{2}}t^{2p+(i-p)}\binom{n-2p+1}{p}_q\binom{p}{i-p}_q=\widehat{A}(n,p,i).
\end{aligned}$$

A similar recursion holds for $\widehat{B}(n,p,i)$. It remains to check the initial conditions:

$$\begin{aligned}
\widehat{b}(0,n)&=1,\\
\widehat{b}(1,n)&=(t^2+qt^2+\ldots+q^{n-1}t^2)=qt^2\binom{n-1}{1}_q+t^2\binom{n-3}{0},\\
\widehat{b}(2,n)&=qt^3[n-1]_q+q^5t^4\binom{n-2}{2}_q\\
&=qt^3\binom{n-1}{1}_q+q^5t^4\binom{n-3}{1}+q^7t^4\binom{n-3}{2}_q.
\end{aligned}$$

□

The following result was conjectured by the second author, Oblomkov and Rasmussen in [10, Conjecture 4.1].

Theorem 6.6 *The Hilbert series of R_n/I_n has the following form:*

$$\begin{aligned}
H_n(q,t)&=\frac{1}{\prod_{i=0}^{n-1}(1-q^it)}\sum_{p=0}^{\infty}(-1)^p\prod_{k=0}^{p-1}(1-q^kt)\\
&\times\left(q^{\frac{5p^2-3p}{2}}t^{2p}\binom{n-2p+1}{p}_q-q^{\frac{5p^2+5p}{2}}t^{2p+2}\binom{n-2p-1}{p}_q\right).
\end{aligned}\tag{6.5}$$

Proof It is clear that $H_n(q,t)=\frac{1}{\prod_{i=0}^{n-1}(1-q^it)}\sum_{i=0}^{\infty}(-1)^i\widehat{b}(i,n)$. The latter can be computed by (6.3), and it remains to use the identity

$$\prod_{k=0}^{p-1}(1-q^kt)=\sum_{j=0}^{p}(-1)^jq^{j(j-1)/2}t^j\binom{p}{j}.$$

□

7 Limit at $n \to \infty$

In the limit $n \to \infty$ both formulas for the Hilbert series simplify. Indeed, for fixed p we have

$$\lim_{n\to\infty} \binom{n}{p}_q = \frac{1}{(1-q)\cdots(1-q^p)},$$

so we can take the limit of all the above results.

Proposition 7.1 *The limit of the Hilbert series $H_n(q,t)$ has the following form:*

$$H_\infty(q,t) = \sum_{p=0}^{\infty} \frac{q^{p(p-1)} t^p}{(1-q)(1-q^2)\cdots(1-q^p)}. \tag{7.1}$$

Proposition 7.2 *The limit of the bigraded rank of the ith syzygy module $F(i,n)$ equals*

$$\widehat{b}(i,\infty) = \sum_{p>0} \left(q^{\frac{5p^2-3p+(i-p)(i-p-1)}{2}} t^{2p+(i-p)} \binom{p}{i-p}_q \frac{1}{(1-q)\cdots(1-q^p)} \right.$$
$$+ q^{\frac{5p^2+5p+(i-p)(i-p-1)}{2}} t^{2p+2+(i-p)} \binom{p}{i-p-1}_q$$
$$\left. \times \frac{1}{(1-q)\cdots(1-q^p)} \right). \tag{7.2}$$

Proposition 7.3 *The limit of the Hilbert series $H_n(q,t)$ has the following form:*

$$H_n(q,t) = \frac{1}{\prod_{i=0}^{\infty}(1-q^i t)} \sum_{p=0}^{\infty} (-1)^p \prod_{k=0}^{p-1} \frac{1-q^k t}{1-q^{k+1}}$$
$$\times \left(q^{\frac{5p^2-3p}{2}} t^{2p} - q^{\frac{5p^2+5p}{2}} t^{2p+2} \right). \tag{7.3}$$

The equality between the right-hand sides of (7.3) and (7.1) was proved in [9, Theorem 3.3.2(b)]. At $t = 1$ and $t = q$, one recovers more familiar Rogers–Ramanujan identities.

The following proposition concerning Gröbner bases in the limit was proved first in [4], but we give an alternative proof here. In fact, [4] use a slightly different basis of Bell polynomials. In [14, Section 17], a vertex-algebraic proof of essentially the same fact was also obtained. Yet another proof can be obtained by taking the limit in Theorem 4.6, as follows.

Proposition 7.4 *For $n \to \infty$, the polynomials f_i form a Gröbner basis for the ideal I_∞.*

Before embarking on the proof, we record the following lemmas concerning Gröbner bases here for the convenience of the reader.

Lemma 7.5 ([7] Proposition 8 on p. 106). *Given* $(g_1, \ldots, g_s) \in F_s$, *the* S*-pairs*

$$S_{ij} := \frac{\mathrm{lcm}(\mathrm{LT}_<(g_i), \mathrm{LT}_<(g_j))}{\mathrm{LT}_<(g_i)} e_i - \frac{\mathrm{lcm}(\mathrm{LT}_<(g_i), \mathrm{LT}_<(g_j))}{\mathrm{LT}_<(g_j)} e_j \tag{7.4}$$

form a homogeneous basis for the syzygies on $\{\mathrm{LT}_<(g_1), \ldots, \mathrm{LT}_<(g_s)\}$.

Lemma 7.6 ([7] Proposition 9 on p. 107) *Let* $I = \langle g_1, \ldots, g_s \rangle$. *Then* $G = \{g_1, \ldots, g_s\}$ *is a Gröbner basis for* I *if and only if every element of a homogeneous basis for the syzygies on* $\mathrm{LT}_<(G)$ *reduces to zero modulo* G.

Lemma 7.7 ([7] Proposition 4 on p.103) $G = \{g_1, \ldots, g_s\} \subset R_n$, *and suppose* $g_i, g_j \in G$ *have relatively prime leading monomials. Then the* S*-polynomial*

$$\begin{aligned} S(g_i, g_j) := \phi_n(S_{ij}) = \frac{\mathrm{lcm}(\mathrm{LT}_<(g_i), \mathrm{LT}_<(g_j))}{\mathrm{LT}_<(g_i)} g_j \\ - \frac{\mathrm{lcm}(\mathrm{LT}_<(g_i), \mathrm{LT}_<(g_j))}{\mathrm{LT}_<(g_j)} g_j \end{aligned} \tag{7.5}$$

reduces to zero modulo G.

Proof of Proposition 7.4 Consider $S(f_i, f_j)$. By Lemma 7.7, $\gcd(\mathrm{LT}_<(f_i), \mathrm{LT}_<(f_j)) = 1$ implies that $S(f_i, f_j)$ reduces to zero modulo $\{f_k\}_{k=1}^{\infty}$. Write $i = 2q+r$, where $r = 0, 1$. Then $\mathrm{LT}_<(f_i) = x_q^2$ if i is even and $\mathrm{LT}_<(f_i) = 2x_q x_{q+1}$ if i is odd. So the only case we need to consider is $j = i + 1$. In this case, we have

$$\mathrm{lcm}(\mathrm{LT}_<(f_i), \mathrm{LT}_<(f_{i+1})) = \begin{cases} 2x_q^2 x_{q+1}, & i \text{ even} \\ 2x_q x_{q+1}^2, & i \text{ odd}. \end{cases}$$

Additionally

$$S(f_i, f_{i+1}) = \begin{cases} 2x_{q+1} f_i - x_q f_{i+1}, & i \text{ even} \\ x_q f_i - 2x_{q+1} f_{i+1}, & i \text{ odd}. \end{cases}$$

But from (2.1) it follows that these S-pairs appear in the relations $\phi_n(\mu_{n-1}) = 0$ for $n \gg 0$. Since $n = \infty$, we always have these relations in I_∞. Additionally, moving the S-pair to the right-hand side we reduce $S(f_i, f_{i+1}) \equiv 0$ modulo $\{f_k\}_{k=1}^{\infty}$. In particular, Lemma 7.6 implies that $\{f_k\}_{k=1}^{\infty}$ is a Gröbner basis for I_∞. □

Acknowledgements E.G. would like to thank Boris Feigin, Mikhail Bershtein, James Lepowsky, Kirill Paramonov, and Anne Schilling for useful discussions. O.K. thanks Eric Babson and Jésus de Loera for discussions in the initial stages of the project.

References

1. Andrews, G.E.: On the proofs of the Rogers–Ramanujan identities. In: q-series and partitions (Minneapolis, MN, 1988),volume 18 of IMA Vol. Math. Appl., pp. 1–14. Springer, New York (1989)
2. Andrews, G.E.: The theory of partitions. Cambridge Mathematical Library. Cambridge University Press, Cambridge (1998). Reprint of the 1976 original
3. Andrews, G.E., Baxter, R.J.: A motivated proof of the Rogers–Ramanujan identities. Am. Math. Mon. **96**(5), 401–409 (1989)
4. Bruschek, C., Mourtada, H., Schepers, J.: Arc spaces and the Rogers–Ramanujan identities. Ramanujan J. **30**(1), 9–38 (2013)
5. Calinescu, C., Lepowsky, J., Milas, A.: Vertex-algebraic structure of the principal subspaces of certain $A_1^{(1)}$-modules. I. Level one case. Int. J. Math. **19**(1), 71–92 (2008)
6. Capparelli, S., Lepowsky, J., Milas, A.: The Rogers–Ramanujan recursion and intertwining operators. Commun. Contemp. Math. **5**(6), 947–966 (2003)
7. Cox, D., Little, J., O'Shea, D.: Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra. Undergraduate Texts in Mathematics, third edition. Springer, New York (2006)
8. Feigin, B.: Abelianization of the BGG resolution of representations of the Virasoro algebra. Funct. Anal. Appl. **45**(4), 297–304 (2011)
9. Feigin, B., Stoyanovsky, A.: Functional models of the representations of current algebras, and semi-infinite Schubert cells. Funct. Anal. Appl. **28**(1), 55–72 (1994)
10. Gorsky, E., Oblomkov, A., Rasmussen, J.: On stable Khovanov homology of torus knots. Exp. Math. **22**(3), 265–281 (2013)
11. Ishii, S.: Jet schemes, arc spaces and the Nash problem. C. R. Math. Acad. Sci. Soc. R. Can. **29**(1), 1–21 (2007)
12. Kanade, S.: On a Koszul complex related to the principal subspace of the basic vacuum module for $A_1^{(1)}$. J. Pure Appl. Algebra **222**(2), 323–339 (2018)
13. Paramonov, K.: Cores with distinct parts and bigraded Fibonacci numbers. Discret. Math. **341**(4), 875–888 (2018)
14. van Ekeren, J., Heluani, R.: Chiral Homology of elliptic curves and Zhu's algebra. arXiv preprint arXiv:1804.00017 (2018)