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Solutions of KZ differential equations modulo *p*

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Abstract
We construct polynomial solutions of the KZ differential equations over a finite field $\mathbb{F}_p$ as analogs of hypergeometric solutions.

Keywords KZ differential equations · Multidimensional hypergeometric integrals · Polynomial solutions over finite fields

Mathematics Subject Classification 81R12 (11C08 · 14H52)

1 Introduction

The KZ equations were discovered by physicists Vadim Knizhnik and Alexander Zamolodchikov [6] to describe the differential equations for conformal blocks on sphere in the Wess–Zumino–Witten model of conformal field theory. As Gelfand said, the KZ equations are remarkable differential equations discovered by physicists, defined in terms of a Lie algebra and whose monodromy is described by the corresponding quantum group. It turned out that the KZ equations are realized as suitable Gauss–Manin connections and its solutions are represented by multidimensional hypergeometric integrals, see [1,3,8–11]. The fact that certain integrals of closed differential forms over cycles satisfy a linear differential equation follows by Stokes'

To Yu.I. Manin with admiration on the occasion of his 80th birthday.

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theorem from a suitable cohomological relation, in which the result of the application of the corresponding differential operator to the integrand of an integral equals the differential of a form of one degree less. Such cohomological relations for the KZ equations associated with Kac–Moody algebras were developed in [11].

The goal of this paper is to construct polynomial solutions of the KZ differential equations over a finite field $\mathbb{F}_p$ with p elements, where p is a prime number, as analogs of the hypergeometric solutions constructed in [11]. Our construction is based on the fact that all cohomological relations described in [11] are defined over $\mathbb{Z}$ and can be reduced modulo p. We learned how to construct polynomial solutions in this situation out of hypergeometric solutions from the remarkable paper by Manin [7], see a detailed exposition of Manin's idea in Section "Manin's Result: The Unity of Mathematics" in the book [2] by Clemens.

In the remainder of the introduction we consider the example of one-dimensional hypergeometric and p-hypergeometric integrals as an illustration of our constructions and results. The multidimensional case is considered in Sects. 2–4.

1.1 Case of field $\mathbb{C}$

Let $\kappa, m_1, \dots, m_n$ be nonzero complex numbers, $z = (z_1, \dots, z_n) \in \mathbb{C}^n, t \in \mathbb{C}$. Denote $|m| = m_1 + \cdots + m_n$. Consider the *master function*

$$\Phi(t, z_1, \dots, z_n) = \prod_{1 \leqslant a < b \leqslant n} (z_a - z_b)^{m_a m_b / 2\kappa} \prod_{a=1}^{n} (t - z_a)^{-m_a/\kappa}$$

and the n-vector

$$I^{(\gamma)}(z) = (I_1(z), \dots, I_n(z)), \tag{1.1}$$

where

$$I_j = \int \Phi(t, z_1, \dots, z_n) \frac{\mathrm{d}t}{t - z_j}, \qquad j = 1, \dots, n. \tag{1.2}$$

The integrals are over a closed (Pochhammer) curve γ in $\mathbb{C} - \{z_1, \dots, z_n\}$ on which one fixes a uni-valued branch of the master function to make the integral well defined. Starting from such a curve chosen for given $\{z_1, \dots, z_n\}$, the vector $I^{(\gamma)}(z)$ can be analytically continued as a multivalued holomorphic function of z to the complement in $\mathbb{C}^n$ to the union of the diagonal hyperplanes $z_i = z_j$.

Theorem 1.1 *The vector $I^{(\gamma)}(z)$ satisfies the algebraic equation*

$$m_1 I_1(z) + \cdots + m_n I_n(z) = 0 \tag{1.3}$$

and the differential KZ equations:

$$\frac{\partial I}{\partial z_i} = \frac{1}{\kappa}\sum_{j\neq i}\frac{\Omega_{i,j}}{z_i - z_j} I, \qquad i = 1, \dots, n, \tag{1.4}$$

where

$$\Omega_{i,j} = \begin{pmatrix} & \overset{i}{\vdots} & & \overset{j}{\vdots} & \\ i \cdots & \frac{(m_i-2)m_j}{2} & \cdots & m_j & \cdots \\ & \vdots & & \vdots & \\ j \cdots & m_i & \cdots & \frac{m_i(m_j-2)}{2} & \cdots \\ & \vdots & & \vdots & \end{pmatrix},$$

all other diagonal entries are $\frac{m_i m_j}{2}$ *and the remaining off-diagonal entries are all zero.*

Remark The vector $I^{(\gamma)}(z)$ depends on the choice of the curve γ. Different curves give different solutions of the same KZ equations and all solutions of Eqs. (1.3) and (1.4) are obtained in this way, if $\kappa, m_1, \dots, m_n$ are generic.

Remark The differential equations (1.4) are the KZ differential equations with parameter κ associated with the Lie algebra $\mathfrak{sl}_2$ and the singular weight subspace of weight $|m| - 2$ of the tensor product of $\mathfrak{sl}_2$-modules with highest weights $m_1, \dots, m_n$, see Sect. 2.

Remark The KZ equations define a flat connection over the complement in $\mathbb{C}^n$ to the union of all diagonal hyperplanes,

$$\left[\frac{\partial}{\partial z_i} - \frac{1}{\kappa}\sum_{j\neq i}\frac{\Omega_{i,j}}{z_i - z_j}, \ \frac{\partial}{\partial z_k} - \frac{1}{\kappa}\sum_{j\neq k}\frac{\Omega_{k,j}}{z_k - z_j}\right] = 0 \tag{1.5}$$

for all j, k.

Theorem 1.1 is a classical statement probably known in nineteenth century. Much more general algebraic and differential equations satisfied by analogous multidimensional hypergeometric integrals were considered in [11]. Theorem 1.1 is discussed as an example in [13, Sect. 1.1].

Below we give a proof of Theorem 1.1. A modification of this proof in Sect. 1.2 will produce for us polynomial solutions of Eqs. (1.3) and (1.4) modulo a prime p.

Proof of Theorem 1.1 Equations (1.3) and (1.4) are implied by the following cohomological identities. We have

$$\frac{-m_1}{\kappa}\Phi(t,z)\frac{\mathrm{d}t}{t - z_1} + \cdots + \frac{-m_1}{\kappa}\Phi(t,z)\frac{\mathrm{d}t}{t - z_n} = \mathrm{d}_t\Phi(t,z), \tag{1.6}$$

where d_t denotes the differential with respect to the variable t. This identity and Stokes' theorem imply equation (1.3).

Denote

$$V(t,x) = \left(\frac{dt}{t-z_1}, \dots, \frac{dt}{t-z_n}\right). \tag{1.7}$$

For any $i = 1, \dots, n$, let $W^i(t,z)$ be the vector of $\left(0, \dots, 0, \frac{-1}{t-z_i}, 0, \dots, 0\right)$ with nonzero element at the ith place. Then

$$\left(\frac{\partial I}{\partial z_i} - \frac{1}{\kappa}\sum_{j\neq i}\frac{\Omega_{i,j}}{z_i - z_j}\right)\Phi(t,z)V(t,x) = d_t(\Phi(t,z)W^i(t,z)). \tag{1.8}$$

The proof of this identity is straightforward. For much more general identities of this type, see [11, Lemmas 7.5.5 and 7.5.7], cf. identities in Sect. 2.4.

Identity (1.8) and Stokes' theorem imply the KZ equation (1.4). □

Example 1.1 Let $\kappa = 2$, $n = 3$, $m_1 = m_2 = m_3 = 1$. Then $I^{(\gamma)}(z) = (I_1(z), I_2(z), I_3(z))$, where

$$I_j(z) = \prod_{1\leqslant a<b\leqslant 3}\sqrt[4]{z_a - z_b}\int_{\gamma(z)}\frac{1}{\sqrt{(t-z_1)(t-z_2)(t-z_3)}}\frac{dt}{t-z_j}. \tag{1.9}$$

In this case, the curve $\gamma(z)$ may be thought of as a closed path on the elliptic curve

$$y^2 = (t-z_1)(t-z_2)(t-z_3).$$

Each of these integrals is an elliptic integral. Such an integral is a branch of analytic continuation of a suitable Euler hypergeometric function up to change of variables.

1.2 Case of field $\mathbb{F}_p$

Let $\kappa, m_1, \dots, m_n$ be positive integers. Let $p > 2$ be a prime number, $p \nmid \kappa$. The algebraic equation (1.3) and the differential KZ equations (1.4) are well defined when reduced modulo p. The reduction of the KZ equations satisfies the flatness condition (1.5). We construct solutions of Eqs. (1.3) and (1.4) with values in $(\mathbb{F}_p[z])^n$. Notice that the space of such solutions is a module over the ring $\mathbb{F}_p[z_1^p, \dots, z_n^p]$ since $\frac{\partial z_i^p}{\partial z_j} = 0$.

Choose positive integers M_a for $a = 1, \dots, n$ and $M_{a,b}$ for $1 \leqslant a < b \leqslant n$ such that

$$M_a \equiv -\frac{m_a}{\kappa}, \qquad M_{a,b} \equiv \frac{m_a m_b}{2\kappa} \qquad (\text{mod } p).$$

That means that we project $m_a, \kappa, 2$ to $\mathbb{F}_p$, calculate $-\frac{m_a}{\kappa}, \frac{m_a m_b}{2\kappa}$ in $\mathbb{F}_p$, and then choose positive integers $M_a, M_{a,b}$ satisfying these equations.

Fix an integer q. Consider the *master polynomial*

$$\Phi^{(p)}(t,z) = \prod_{1\leqslant a<b\leqslant n} (z_a - z_b)^{M_{a,b}} \prod_{a=1}^{n} (t - z_a)^{M_a},$$

and the Taylor expansion with respect to the variable t of the vector of polynomials

$$\Phi^{(p)}(t,z)\left(\frac{1}{t-z_1},\dots,\frac{1}{t-z_n}\right) = \sum_i \bar{I}^{(i)}(z,q)\,(t-q)^i,$$

where the $\bar{I}^{(i)}(z,q)$ are n-vectors of polynomials in z with integer coefficients. Let $I^{(i)}(z,q) \in (\mathbb{F}_p[z])^n$ be the canonical projection of $\bar{I}^{(i)}(z,q)$.

Theorem 1.2 *For any integer q and positive integer l, the vector of polynomials $I^{(lp-1)}(z,q)$ satisfies Eqs.* (1.3) *and* (1.4).

The parameters q and $lp-1$ are analogs of cycles γ in Sect. 1.1.

Proof To prove that $I^{(lp-1)}(z,q)$ satisfies (1.3) and (1.4), we consider the Taylor expansions at $t=q$ of both sides of Eqs. (1.6) and (1.8), divide them by dt, and then project the coefficients of $(t-q)^{lp-1}$ to $\mathbb{F}_p[z]$. The projections of the right-hand sides equal zero since $d(t^{lp})/\mathrm{d}t = lpt^{lp-1} \equiv 0 \pmod p$. □

Example 1.2 Let $\kappa = 2$, $m_1 = \dots = m_n = 1$, cf. Example 1.1. Given $p>2$ choose the master polynomial

$$\Phi^{(p)}(t,z) = \prod_{1\leqslant a<b\leqslant n} (z_a - z_b)^{\frac{(p+1)^2}{4}} \prod_{s=1}^{n} (t - z_s)^{\frac{p-1}{2}}. \tag{1.10}$$

Consider the Taylor expansion

$$\prod_{s=1}^{n} (t - z_s)^{\frac{p-1}{2}} \left(\frac{1}{t-z_1},\dots,\frac{1}{t-z_n}\right) = \sum_i \bar{c}^i(z) t^i, \tag{1.11}$$

where $\bar{c}^i = (\bar{c}^i_1,\dots,\bar{c}^i_n)$. Let c^i be the projection of $\bar{c}^i$ to $(\mathbb{F}_p[z])^n$. Then the vector of polynomials

$$\begin{aligned} I(z) &= (I_1(z),\dots,I_n(z)) \\ &= \prod_{1\leqslant a<b\leqslant n} (z_a - z_b)^{\frac{(p+1)^2}{4}} \left(c_1^{p-1}(z),\dots,c_n^{p-1}(z)\right) \end{aligned} \tag{1.12}$$

is a solution of the KZ differential equations over $\mathbb{F}_p[z]$ and $I_1(z)+\dots+I_n(z) = 0$.

Example 1.3 Let $\kappa = 2$, $m_1 = \cdots = m_n = 1$, $p = 3$. We have

$$\begin{aligned}\Omega_{i,j} \quad & (I_1, \dots, I_n) \\ & = \frac{1}{2}(I_1, \dots, I_{i-1}, -I_i + 2I_j, I_{i+1}, \dots, I_{j-1}, 2I_i - I_j, I_{j+1}, \dots, I_n) \\ & \equiv (-I_1, \dots, -I_{i-1}, I_i + I_j, -I_{i+1}, \dots, -I_{j-1}, I_i + I_j, -I_{j+1}, \dots, -I_n)\end{aligned}$$

(mod 3). Equation (1.3) has the form $I_1(z) + \cdots + I_n(z) = 0$. We may choose the master polynomial

$$\Phi^{(p=3)}(t, z) = \prod_{1 \leqslant a < b \leqslant n} (z_a - z_b) \prod_{s=1}^{n} (t - z_s).$$

Choose a nonnegative integer l. Then the vector $I(z, q) := I^{(3l-1)}(z, q) = (I_1(z, q), \dots, I_n(z, q))$ of Theorem 1.2 has coordinates

$$I_j(z, q) = \left(\prod_{1 \leqslant a < b \leqslant n} (z_a - z_b) \right) \sum_{\substack{1 \leqslant i_1 < \cdots < i_{n-3-3l} \leqslant n, \\ j \notin \{i_1, \dots, i_{n-3-3l}\}}} \prod_{a=1}^{n-3-3l} (q - z_{i_a}) \tag{1.13}$$

and is a solution of (1.3) and (1.4) with values in $(\mathbb{F}_3[z])^n$ for any $q = 0, 1, 2$. Expanding these solutions into polynomials homogeneous in z we obtain solutions in homogeneous polynomials, which stabilize with respect to n as follows. The vector $I^{[r]}(z) = (I_1^{[r]}(z), \dots, I_n^{[r]}(z))$, with coordinates

$$I_j^{[r]}(z) = \left(\prod_{1 \leqslant a < b \leqslant n} (z_a - z_b) \right) \sum_{\substack{1 \leqslant i_1 < \cdots < i_r \leqslant n, \\ j \notin \{i_1, \dots, i_r\}}} \prod_{a=1}^{r} z_{i_a}, \tag{1.14}$$

is a solution of (1.3) and (1.4) with values in $(\mathbb{F}_3[z])^n$ if $r \equiv n \pmod 3$ and $r < n$. Thus, the vector $I^{[0]}(z)$, with coordinates

$$I_j^{[0]}(z) = \prod_{1 \leqslant a < b \leqslant n} (z_a - z_b), \tag{1.15}$$

is a solution with values in $(\mathbb{F}_3[z])^n$ for $n \equiv 0 \pmod 3$; the vector $I^{(1)}(z)$, with coordinates

$$I_j^{[1]}(z) = \left(\prod_{1 \leqslant a < b \leqslant n} (z_a - z_b) \right) \sum_{1 \leqslant i \leqslant n,\, i \neq j} z_i, \tag{1.16}$$

is a solution for $n \equiv 1 \pmod 3$ and so on. Note that the sum in (1.14) is the mth elementary symmetric function in $z_1, \dots, \widehat{z_j}, \dots, z_n$.

Solutions provided by Theorem 1.2 depend on parameters $q, lp-1$. In this example all solutions $I^{[r]}(z)$ can be obtained by putting $q = 0$ and varying $lp - 1$ only.

1.3 Relation of polynomial solutions to integrals over $\mathbb{F}_p$

For a polynomial $F(t) \in \mathbb{F}_p[t]$ define the integral

$$\int_{\mathbb{F}_p} F(t) := \sum_{t\in\mathbb{F}_p} F(t).$$

Recall that

$$\text{the sum } \sum_{t\in\mathbb{F}_p} t^i \text{ equals } -1 \text{ if } (p-1)\big| i \text{ and equals zero otherwise.} \tag{1.17}$$

Theorem 1.3 *Fix $x_1, \dots, x_n, q \in \mathbb{F}_p$. Consider the vector of polynomials*

$$F(t, x_1, \dots, x_n) := \Phi^{(p)}(t, x_1, \dots, x_n)\left(\frac{1}{t-x_1}, \dots, \frac{1}{t-x_n}\right) \in \mathbb{F}_p[t]$$

of Sect. 1.2. Assume that $\deg_t F(t, x_1, \dots, x_n) < 2p-2$. *Consider the polynomial solution $I^{(p-1)}(z_1, \dots, z_n, q)$ of Eqs. (1.3) and (1.4) defined in front of Theorem 1.2. Then*

$$I^{(p-1)}(x_1, \dots, x_n, q) = -\int_{\mathbb{F}_p} F(t, x_1, \dots, x_n). \tag{1.18}$$

This integral is a p-analog of the hypergeometric integral (1.2).

Proof Consider the Taylor expansion $F(t, x_1, \dots, x_n) = \sum_{i=0}^{2p-3} I^{(i)}(x_1, \dots, x_n, q)$ $(t-q)^i$. By formula (1.17), we have $\sum_{t\in\mathbb{F}_p} F(t, x_1, \dots, x_n) = -I^{(p-1)}(x_1, \dots, x_n, q)$. □

Example 1.4 Given $\kappa, n, m_1 = \dots = m_n = 1$, assume that $n \leqslant 2\kappa$ and $\kappa\big|(p-1)$. Then $F(t) = \prod_{a<b}(z_a - z_b)^{M_{a,b}} \prod_{s=1}^{n}(t-x_s)^{\frac{p-1}{\kappa}}\left(\frac{1}{t-x_1}, \dots, \frac{1}{t-x_n}\right)$ and $\deg_t F(t) < 2p-2$.

1.4 Relation of solutions to curves over $\mathbb{F}_p$

Example 1.5 Let $x_1, x_2, x_3 \in \mathbb{F}_p$. Let $\Gamma(x_1, x_2, x_3)$ be the projective closure of the affine curve

$$y^2 = (t-x_1)(t-x_2)(t-x_3) \tag{1.19}$$

over $\mathbb{F}_p$. For a rational function $h : \Gamma(x_1, x_2, x_3) \to \mathbb{F}_p$ define the integral

$$\int_{\Gamma(x_1,x_2,x_3)} h = \sum_{P\in\Gamma(x_1,x_2,x_3)}' h(P), \tag{1.20}$$

as the sum over all points $P \in \Gamma(x_1, x_2, x_3)$, where $h(P)$ is defined.

Theorem 1.4 *Let $p > 2$ be a prime. Let $\Big(c_1^{p-1}(x_1, x_2, x_3), c_2^{p-1}(x_1, x_2, x_3), c_3^{p-1}(x_1, x_2, x_3)\Big)$ be the vector of polynomials appearing in the solution (1.12) of the KZ equations of Example 1.2 for $n = 3$. Then*

$$\int_{\Gamma(x_1,x_2,x_3)} \frac{1}{t - x_j} = - c_j^{p-1}(x_1, x_2, x_3), \qquad j = 1, 2, 3. \tag{1.21}$$

Remark Theorems 1.2 and 1.4 say that the integrals $\int_{\Gamma(x_1,x_2,x_3)} \frac{1}{t-x_j}$ are polynomials in $x_1, x_2, x_3 \in \mathbb{F}_p$ and the triple of polynomials

$$\begin{aligned} &I(x_1, x_2, x_3)\\ &= \prod_{1\leqslant a<b\leqslant 3} (x_a - x_b)^{\frac{(p+1)^2}{4}} \left(\int_{\Gamma(x_1,x_2,x_3)} \frac{1}{t - x_1}, \int_{\Gamma(x_1,x_2,x_3)} \frac{1}{t - x_2}, \int_{\Gamma(x_1,x_2,x_3)} \frac{1}{t - x_3}\right) \end{aligned}$$

in these discrete variables satisfies the KZ differential equations! Cf. Example 1.1.

Proof of Theorem 1.4 The proof is analogous to the reasoning in [7, Sect. 2] and [2]. The value of $1/(t - x_j)$ at the infinite point of $\Gamma(x_1, x_2, x_3)$ equals zero. It is easy to see that

$$\begin{aligned} \int_{\Gamma(x_1,x_2,x_3)} \frac{1}{t - x_j} &= \sum_{t\in\mathbb{F}_p,\, t\neq x_j} \frac{1}{t - x_j} + \sum_{t\in\mathbb{F}_p} \frac{1}{t - x_j} \prod_{s=1}^{3} (t - x_s)^{\frac{p-1}{2}}\\ &= \sum_{t\in\mathbb{F}_p} (t - x_j)^{p-2} + \sum_{t\in\mathbb{F}_p} \sum_{i} c_j^i(x_1, x_2, x_3) t^i\\ &= -c_j^{p-1}(x_1, x_2, x_3), \end{aligned}$$

where the last equality is by formula (1.17). □

Remark In [7, Sect. 2] and in [2], an equation analogous to (1.21) is considered, where the left-hand side is the number of points on $\Gamma(x_1, x_2, x_3)$ over $\mathbb{F}_p$ and the right-hand side is the reduction modulo p of a solution of a second order Euler hypergeometric differential equation. Notice that the number of points on $\Gamma(x_1, x_2, x_3)$ is the discrete integral over $\Gamma(x_1, x_2, x_3)$ of the constant function $h = 1$. See details in Section "Manin's Result: The Unity of Mathematics" in [2].

Example 1.6 This example is a variant of Example 1.5.

Let $x_1, x_2, x_3, x_4 \in \mathbb{F}_p$. Let $\Gamma(x_1, x_2, x_3, x_4)$ be the projective closure of the affine curve

$$y^2 = (t - x_1)(t - x_2)(t - x_3)(t - x_4) \tag{1.22}$$

over $\mathbb{F}_p$.

Let $p > 3$ be a prime. Let

$$\Big(c_1^{p-1}(x_1, x_2, x_3, x_4), c_2^{p-1}(x_1, x_2, x_3, x_4), c_3^{p-1}(x_1, x_2, x_3, x_4), c_4^{p-1}(x_1, x_2, x_3, x_4)\Big)$$

be the vector of polynomials appearing in the solution (1.12) of the KZ equations of Example 1.2 for $n = 4$. Then

$$\int_{\Gamma(x_1,x_2,x_3,x_4)} \frac{1}{t - x_j} = -\, c_j^{p-1}(x_1, x_2, x_3, x_4), \qquad j = 1, 2, 3, 4. \tag{1.23}$$

Example 1.7 Let $\kappa = 3, n = 3, m_1 = m_2 = m_3 = 2$. Assume that $3\big|(p-1)$. Choose the master polynomial

$$\Phi^{(p)}(t, z) = \prod_{1 \leqslant a < b \leqslant 3} (z_a - z_b)^{\frac{p+2}{3}} \prod_{s=1}^{3} (t - z_s)^{2\frac{p-1}{3}}.$$

Consider the Taylor expansion

$$\prod_{s=1}^{3} (t - z_s)^{2\frac{p-1}{3}} \Big(\frac{1}{t - z_1}, \frac{1}{t - z_2}, \frac{1}{t - z_3}\Big) = \sum_i \bar{c}^i(z_1, z_2, z_3) t^i, \tag{1.24}$$

where $\bar{c}^i = (\bar{c}^i_1, \bar{c}^i_2, \bar{c}^i_3)$. Let c^i be the projection of $\bar{c}^i$ to $(\mathbb{F}_p[z])^3$. Then the vector

$$\begin{aligned} I(z) &= (I_1(z), I_2(z), I_3(z)) \\ &= \prod_{1 \leqslant a < b \leqslant 3} (z_a - z_b)^{\frac{p+2}{3}} \Big(c_1^{p-1}(z), c_2^{p-1}(z), c_3^{p-1}(z)\Big) \end{aligned} \tag{1.25}$$

is a solution of the corresponding KZ differential equations over $\mathbb{F}_p[z]$ and $I_1(z) + I_2(z) + I_3(z) = 0$.

For distinct $x_1, x_2, x_3 \in \mathbb{F}_p$ let $\Gamma(x_1, x_2, x_3)$ be the projective closure of the affine

$$y^3 = (t - x_1)(t - x_2)(t - x_3) \tag{1.26}$$

over $\mathbb{F}_p$. The curve has 3 points at infinity.

Theorem 1.5 *Let p be a prime such that $3\big|(p-1)$. Let*

$$\Big(c_1^{p-1}(x_1,x_2,x_3),\, c_2^{p-1}(x_1,x_2,x_3),\, c_3^{p-1}(x_1,x_2,x_3)\Big)$$

be the vector of polynomials appearing in the solution (1.25) of the KZ equations. Then for $j=1,2,3$ we have

$$\int_{\Gamma(x_1,x_2,x_3)} \frac{1}{t-x_j} = -\,c_j^{p-1}(x_1,x_2,x_3). \tag{1.27}$$

Proof The value of $1/(t-x_j)$ at infinite points of Γ equals zero. It is easy to see that

$$\begin{aligned}\int_{\Gamma(x_1,x_2,x_3)} \frac{1}{t-x_j} &= \sum_{t\in\mathbb{F}_p,\, t\neq x_j} \frac{1}{t-x_j} + \sum_{t\in\mathbb{F}_p} \frac{1}{t-x_j}\prod_{s=1}^{3}(t-x_s)^{\frac{p-1}{3}} \\ &\quad + \sum_{t\in\mathbb{F}_p} \frac{1}{t-x_j}\prod_{s=1}^{3}(t-x_s)^{2\frac{p-1}{3}} \\ &= \sum_{t\in\mathbb{F}_p}(t-x_j)^{p-2} + \sum_{t\in\mathbb{F}_p}\sum_i c_j^i(x_1,x_2,x_3)t^i \\ &= -\,c_j^{p-1}(x_1,x_2,x_3). \end{aligned} \tag{1.28}$$

Notice that $\sum_{t\in\mathbb{F}_p}\frac{1}{t-x_j}\prod_{s=1}^{3}(t-x_s)^{\frac{p-1}{3}}=0$ since the polynomial under the sum is of degree $p-2$ which is less than $p-1$. The last equality in (1.28) is by formula (1.17). □

Example 1.8 Let $\kappa=3$, $n=3$, $m_1=m_2=1$, $m_3=2$. Assume that 3 divides $p-1$. Choose the master polynomial

$$\begin{aligned}\Phi^{(p)}(t,z) &= (z_1-z_2)^{\frac{5p+1}{6}}(z_1-z_3)^{\frac{2p+1}{3}}(z_2-z_3)^{\frac{2p+1}{3}} \\ &\quad\times(t-z_1)^{\frac{p-1}{3}}(t-z_2)^{\frac{p-1}{3}}(t-z_3)^{2\frac{p-1}{3}}.\end{aligned}$$

Consider the Taylor expansion

$$\begin{aligned}&(t-z_1)^{\frac{p-1}{3}}(t-z_2)^{\frac{p-1}{3}}(t-z_3)^{2\frac{p-1}{3}}\left(\frac{1}{t-z_1},\frac{1}{t-z_2},\frac{1}{t-z_3}\right) \\ &= \sum_i \bar b^i(z_1,z_2,z_3)t^i,\end{aligned} \tag{1.29}$$

where $\bar b^i=(\bar b_1^i,\bar b_2^i,\bar b_3^i)$. Let b^i be the projection of $\bar b^i$ to $(\mathbb{F}_p[z])^3$. Then the vector

$$I(z) = (z_1 - z_2)^{\frac{5p+1}{6}} (z_1 - z_3)^{\frac{2p+1}{3}} (z_2 - z_3)^{\frac{2p+1}{3}} \left(b_1^{p-1}(z), b_2^{p-1}(z), b_3^{p-1}(z) \right) \tag{1.30}$$

is a solution of the corresponding KZ differential equations over $\mathbb{F}_p[z]$ and $I_1(z) + I_2(z) + 2I_3(z) = 0$.

Similarly let $\kappa = 3$, $n = 3$, $m_1 = m_2 = 2$, $m_3 = 1$. Assume that 3 divides $p - 1$. Choose the master polynomial

$$\Phi^{(p)}(t, z) = (z_1 - z_2)^{\frac{p+2}{3}} (z_1 - z_3)^{\frac{2p+1}{3}} (z_2 - z_3)^{\frac{2p+1}{3}} \times (t - z_1)^{2\frac{p-1}{3}} (t - z_2)^{2\frac{p-1}{3}} (t - z_3)^{\frac{p-1}{3}}.$$

Consider the Taylor expansion

$$(t - z_1)^{2\frac{p-1}{3}} (t - z_2)^{2\frac{p-1}{3}} (t - z_3)^{\frac{p-1}{3}} \left(\frac{1}{t - z_1}, \frac{1}{t - z_2}, \frac{1}{t - z_3} \right) = \sum_i \bar{c}^i(z_1, z_2, z_3) t^i, \tag{1.31}$$

where $\bar{c}^i = (\bar{c}^i_1, \bar{c}^i_2, \bar{c}^i_3)$. Let c^i be the projection of $\bar{c}^i$ to $(\mathbb{F}_p[z])^3$. Then the vector

$$I(z) = (z_1 - z_2)^{\frac{p+2}{3}} (z_1 - z_3)^{\frac{2p+1}{3}} (z_2 - z_3)^{\frac{2p+1}{3}} \left(c_1^{p-1}(z), c_2^{p-1}(z), c_3^{p-1}(z) \right) \tag{1.32}$$

is a solution of the corresponding KZ differential equations over $\mathbb{F}_p[z]$ and $2I_1(z) + 2I_2(z) + I_3(z) = 0$.

For distinct $x_1, x_2, x_3 \in \mathbb{F}_p$ let $\Gamma(x_1, x_2, x_3)$ be the projective closure of the affine curve

$$y^3 = (t - x_1)(t - x_2)(t - x_3)^2 \tag{1.33}$$

over $\mathbb{F}_p$. The curve has genus 2 and one point at infinity.

Theorem 1.6 *Let p be a prime such that 3 divides $p - 1$. Let*

$$\left(b_1^{p-1}(x_1, x_2, x_3), b_2^{p-1}(x_1, x_2, x_3), b_3^{p-1}(x_1, x_2, x_3) \right)$$

be the vector of polynomials appearing in the solution (1.30) of the KZ equations with $n = 3$, $\kappa = 3$, $m_1 = m_2 = 1$, $m_3 = 2$. Let

$$\left(c_1^{p-1}(x_1, x_2, x_3), c_2^{p-1}(x_1, x_2, x_3), c_3^{p-1}(x_1, x_2, x_3) \right)$$

be the vector of polynomials appearing in the solution (1.32) *of the KZ equations with* $n = 3$, $\kappa = 3$, $m_1 = m_2 = 2$, $m_3 = 1$. *Then for* $j = 1, 2, 3$ *we have*

$$\int_{\Gamma(x_1,x_2,x_3)} \frac{1}{t - x_j} = -\, b_j^{p-1}(x_1, x_2, x_3) - c_j^{p-1}(x_1, x_2, x_3). \tag{1.34}$$

Proof The value of $1/(t - x_j)$ at infinite points of Γ equals zero. It is easy to see that

$$\begin{aligned}
&\int_{\Gamma(x_1,x_2,x_3)} \frac{1}{t - x_j} \\
&= \sum_{t\in\mathbb{F}_p,\, t\neq x_j} \frac{1}{t - x_j} + \sum_{t\in\mathbb{F}_p} \frac{1}{t - x_j} (t - z_1)^{\frac{p-1}{3}} (t - z_2)^{1\frac{p-1}{3}} (t - z_3)^{2\frac{p-1}{3}} \\
&\quad + \sum_{t\in\mathbb{F}_p} \frac{1}{t - x_j} (t - z_1)^{2\frac{p-1}{3}} (t - z_2)^{2\frac{p-1}{3}} (t - z_3)^{4\frac{p-1}{3}} \\
&= \sum_{t\in\mathbb{F}_p} (t - x_j)^{p-2} + \sum_{t\in\mathbb{F}_p} \sum_i b_j^i(x_1, x_2, x_3) t^i \\
&\quad + \sum_{t\in\mathbb{F}_p} \frac{1}{t - x_j} (t - z_1)^{2\frac{p-1}{3}} (t - z_2)^{2\frac{p-1}{3}} (t - z_3)^{\frac{p-1}{3}} \\
&= -\, b_j^{p-1}(x_1, x_2, x_3) + \sum_{t\in\mathbb{F}_p} \sum_i c_j^i(x_1, x_2, x_3) t^i \\
&= -b_j^{p-1}(x_1, x_2, x_3) - c_j^{p-1}(x_1, x_2, x_3).
\end{aligned}$$

□

1.5 Resonances over $\mathbb{C}$ and $\mathbb{F}_p$

Under assumptions of Sect. 1.1 assume that

$$m_1 + \cdots + m_n = \kappa. \tag{1.35}$$

Then the vector $I^{(\gamma)}(z)$, defined in (1.1), in addition to the algebraic equation (1.3) and differential equations (1.4) satisfies the algebraic equation

$$z_1 m_1 I_1(z) + \cdots + z_n m_n I_n(z) = 0. \tag{1.36}$$

Equation (1.36) follows from the cohomological relation:

$$\mathrm{d}_t(t\Phi) = \Phi \mathrm{d}t - \Phi \sum_{j=1}^{n} \frac{m_j}{\kappa} \frac{t - z_j + z_j}{t - z_j} \mathrm{d}t$$
$$= \left(1 - \sum_{j=1}^{n} \frac{m_j}{\kappa}\right) \Phi \mathrm{d}t - \sum_{j=1}^{n} z_j \frac{m_j}{\kappa} \Phi \frac{\mathrm{d}t}{t - z_j}. \tag{1.37}$$

Relation (1.36) manifests resonances in conformal field theory, where solutions of KZ equations represent conformal blocks and conformal blocks satisfy algebraic equations analogous to (1.36), see [4,5], Sect. 3.6.2 in [13]. In conformal field theory the numbers $m_1, \ldots, m_n, \kappa$ are natural numbers. In that case the master function $\Phi(t, z)$ is an algebraic function and the hypergeometric integrals become integrals of algebraic forms over cycles lying on suitable algebraic varieties. The monodromy of the hypergeometric integrals $I^{(\gamma)}(z)$ in that case was studied in Sects. 13 and 14 of [12].

Relation (1.36) has an analog over $\mathbb{F}_p$.

Theorem 1.7 *Under assumptions of Theorem 1.2 let $I^{(lp-1)}(z, q) \in \mathbb{F}_p[z]^n$ be the polynomial solution of Eqs.* (1.3) *and* (1.4) *described in Theorem 1.2. Assume that*

$$M_1 + \cdots + M_n \equiv -1 \pmod{p}. \tag{1.38}$$

Then

$$z_1 M_1 I_1(z) + \cdots + z_n M_n I_n(z) = 0. \tag{1.39}$$

Proof The theorem follows from (1.37) similarly to the proof of Theorem 1.2. Namely, we consider the Taylor expansions at $t = q$ of both sides of Eq. (1.37), divide them by $\mathrm{d}t$, and then project the coefficients of $(t-q)^{lp-1}$ to $\mathbb{F}_p[z]$. The projection coming from $\mathrm{d}_t(t\Phi)$ equals zero since $\mathrm{d}(t^{lp})/\mathrm{d}t = lpt^{lp-1} \equiv 0 \pmod{p}$. The projection coming from $\left(1 - \sum_{j=1}^{n} \frac{m_j}{\kappa}\right)\Phi \mathrm{d}t$ equals zero by (1.38). The projection coming from $-\sum_{j=1}^{n} z_j \frac{m_j}{\kappa} \Phi \frac{\mathrm{d}t}{t-z_j}$ gives (1.39). □

Example 1.9 Let $\kappa = 2$, $m_1 = \cdots = m_n = 1$, $p = 3$, $M_1 = \cdots = M_n = 1$,

$$\Phi^{(p=3)}(t, z) = \prod_{1 \leqslant a < b \leqslant n} (z_a - z_b) \prod_{s=1}^{n} (t - z_s)$$

as in Example 1.3. Let $n \equiv 2 \pmod 3$, and then $M_1 + \cdots + M_n \equiv -1 \pmod 3$. Choose a positive integer r, such that $r \equiv n \pmod 3$ and $r < n$. Then the vector $I^{[r]}(z)$ given by (1.14) satisfies Eqs. (1.3), (1.4), and

$$z_1 I_1^{[r]}(z) + \cdots + z_n I_n^{[r]}(z) \equiv 0 \pmod{3}.$$

1.6 Exposition of material

In Sect. 2 we describe the hypergeometric solutions of the KZ equations associated with $\mathfrak{sl}_2$ and explain their reduction to polynomial solutions over $\mathbb{F}_p$. In Sect. 3 we describe the resonance relations for $\mathfrak{sl}_2$ conformal blocks and construct their reduction over $\mathbb{F}_p$. In Sect. 4 we explain how the results of Sects. 2 and 3 are extended to the KZ equations associated with simple Lie algebras.

2 $\mathfrak{sl}_2$ KZ equations

In this section we describe solutions of the KZ equations associated with the Lie algebra $\mathfrak{sl}_2$. The solutions to the KZ equations over $\mathbb{C}$ in the form of multidimensional hypergeometric integrals are known since the end of 1980s. The polynomial solutions of the KZ equations over $\mathbb{F}_p$ in the form of $\mathbb{F}_p$-analogs of the multidimensional hypergeometric integrals are new.

2.1 $\mathfrak{sl}_2$ KZ equations

Let e, f, h be standard basis of the complex Lie algebra $\mathfrak{sl}_2$ with $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. The element

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h \ \in \ \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \tag{2.1}$$

is called the Casimir element. Given n, for $1 \leqslant i < j \leqslant n$ let $\Omega^{(i,j)} \in (U(\mathfrak{sl}_2))^{\otimes n}$ be the element equal to Ω in the ith and jth factors and to 1 in the other factors. For $i = 1, \dots, n$ and distinct $z_1, \dots, z_n \in \mathbb{C}$ introduce

$$H_i(z_1, \dots, z_n) = \sum_{j \neq i} \frac{\Omega^{(i,j)}}{z_i - z_j} \ \in \ (U(\mathfrak{sl}_2))^{\otimes n}, \tag{2.2}$$

the Gaudin Hamiltonians. For any $\kappa \in \mathbb{C}^\times$ and any i, k, we have

$$\left[\frac{\partial}{\partial z_i} - \frac{1}{\kappa} H_i(z_1, \dots, z_n), \frac{\partial}{\partial z_k} - \frac{1}{\kappa} H_k(z_1, \dots, z_n) \right] = 0, \tag{2.3}$$

and for any $x \in \mathfrak{sl}_2$ and i we have

$$[H_i(z_1, \dots, z_n), x \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes x] = 0. \tag{2.4}$$

Let $\otimes_{i=1}^n V_i$ be a tensor product of $\mathfrak{sl}_2$-modules. The system of differential equations

$$\frac{\partial I}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega^{(i,j)}}{z_i - z_j} I, \qquad i = 1, \dots, n, \tag{2.5}$$

on a $\otimes_{i=1}^n V_i$-valued function $I(z_1, \dots, z_n)$ is called the KZ equations.

2.2 Irreducible $\mathfrak{sl}_2$-modules

For a nonnegative integer i denote by L_i the irreducible $i+1$-dimensional module with basis $v_i, f v_i, \dots, f^i v_i$ and action $h.f^k v_i = (i-2k) f^k v_i$ for $k = 0, \dots, i$; $f.f^k v_i = f^{k+1} v_i$ for $k = 0, \dots, i-1$, $f.f^i v_i = 0$; $e.v_i = 0$, $e.f^k v_i = k(i-k+1) f^{k-1} v_i$ for $k = 1, \dots, i$.

For $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geqslant 0}$, denote $|m| = m_1 + \dots + m_n$ and $L^{\otimes m} = L_{m_1} \otimes \dots \otimes L_{m_n}$. For $J = (j_1, \dots, j_n) \in \mathbb{Z}^n_{\geqslant 0}$, with $j_s \leqslant m_s$ for $s = 1, \dots, n$, the vectors

$$f_J v_m := f^{j_1} v_{m_1} \otimes \dots \otimes f^{j_n} v_{m_n} \tag{2.6}$$

form a basis of $L^{\otimes m}$. We have

$$f.f_J v_m = \sum_{s=1}^n f_{J+1_s} v_m, \qquad h.f_J v_m = (|m| - 2|J|) f_J v_m,$$
$$e.f_J v_m = \sum_{s=1}^n j_s (m_s - j_s + 1) f_{J-1_s} v_m.$$

For $\lambda \in \mathbb{Z}$, introduce the weight subspace $L^{\otimes m}[\lambda] = \{ v \in L^{\otimes m} \mid h.v = \lambda v\}$ and the singular weight subspace $\mathrm{Sing} L^{\otimes m}[\lambda] = \{ v \in L^{\otimes m}[\lambda] \mid h.v = \lambda v,\ e.v = 0\}$. We have the weight decomposition $L^{\otimes m} = \oplus_{k=0}^{|m|} L^{\otimes m}[|m| - 2k]$. Denote

$$\mathcal{I}_k = \{J \in \mathbb{Z}^n_{\geqslant 0} \mid |J| = k,\ j_s \leqslant m_s,\ s = 1, \dots, n\}.$$

The vectors $(f_J v)_{J \in \mathcal{I}_k}$ form a basis of $L^{\otimes m}[|m| - 2k]$.

Remark The $\mathfrak{sl}_2$-action on the sum of singular weight subspaces $\mathrm{Sing} L^{\otimes m}[|m| - 2k]$ generates the entire $\mathfrak{sl}_2$-module $L^{\otimes m}$. If $I(z_1, \dots, z_n)$ is an $L^{\otimes m}$-valued solution of the KZ equations, then for any $x \in \mathfrak{sl}_2$ the function $x.I(z_1, \dots, z_n)$ is also a solution, see (2.4). These observations show that in order to construct all $L^{\otimes m}$-valued solutions of the KZ equations it is enough to construct all $\mathrm{Sing} L^{\otimes m}[|m| - 2k]$-valued solutions for all k and then generate the other solutions by the $\mathfrak{sl}_2$-action.

2.3 Solutions of KZ equations with values in $\mathrm{Sing} L^{\otimes m}[|m| - 2k]$ over $\mathbb{C}$

Given $k, n \in \mathbb{Z}_{>0}$, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$, $\kappa \in \mathbb{C}^\times$, denote $t = (t_1, \dots, t_k)$, $z = (z_1, \dots, z_n)$, define the *master function*

$$\begin{aligned}\Phi_{k,n,m}(t,z) &:= \Phi_{k,n,m}(t_1,\dots,t_k,z_1,\dots,z_n,\kappa)\\ &= \prod_{i<j}(z_i-z_j)^{m_im_j/2\kappa}\prod_{1\leqslant i\leqslant j\leqslant k}(t_i-t_j)^{2/\kappa}\prod_{l=1}^{n}\prod_{i=1}^{k}(t_i-z_l)^{-m_l/\kappa}.\end{aligned}\tag{2.7}$$

For any function or differential form $F(t_1,\dots,t_k)$, denote

$$\mathrm{Sym}_t[F(t_1,\dots,t_k)] = \sum_{\sigma\in S_k} F(t_{\sigma_1},\dots,t_{\sigma_k}),$$
$$\mathrm{Ant}_t[F(t_1,\dots,t_k)] = \sum_{\sigma\in S_k}(-1)^{|\sigma|} F(t_{\sigma_1},\dots,t_{\sigma_k}).$$

For $J=(j_1,\dots,j_n)\in\mathcal{I}_k$ define the *weight function*

$$W_J(t,z) = \frac{1}{j_1!\dots j_n!}\,\mathrm{Sym}_t\left[\prod_{s=1}^{n}\prod_{i=1}^{j_s}\frac{1}{t_{j_1+\dots+j_{s-1}+i}-z_s}\right].\tag{2.8}$$

For example,

$$W_{(1,0,\dots,0)} = \frac{1}{t_1-z_1},\qquad W_{(2,0,\dots,0)} = \frac{1}{t_1-z_1}\frac{1}{t_2-z_1},$$
$$W_{(1,1,0,\dots,0)} = \frac{1}{t_1-z_1}\frac{1}{t_2-z_2}+\frac{1}{t_2-z_1}\frac{1}{t_1-z_2}.$$

The function

$$W_{k,n,m}(t,z) = \sum_{J\in\mathcal{I}_k} W_J(t,z)\,f_J v_m\tag{2.9}$$

is the $L^{\otimes m}[|m|-2k]$-valued *vector weight function*.

Consider the $L^{\otimes m}[|m|-2k]$-valued function

$$I^{(\gamma)}(z_1,\dots,z_n) = \int_{\gamma(z)}\Phi_{k,n,m}(t,z,\kappa)W_{k,n,m}(t,z)\mathrm{d}t_1\wedge\dots\wedge\mathrm{d}t_k,\tag{2.10}$$

where $\gamma(z)$ in $\{z\}\times\mathbb{C}_t^k$ is a horizontal family of k-dimensional cycles of the twisted homology defined by the multivalued function $\Phi_{k,n,m}(t,z,m)$, see [11–13]. The cycles $\gamma(z)$ are multidimensional analogs of Pochhammer double loops.

Theorem 2.1 *The function $I^{(\gamma)}(z)$ takes values in* $\mathrm{Sing}L^{\otimes m}[|m|-2k]$ *and satisfies the KZ equations.*

This theorem and its generalizations can be found, for example, in [1,3,9–11].

The solutions in Theorem 2.1 are called the *multidimensional hypergeometric solutions* of the KZ equations. The coordinate functions

$$I_J^{(\gamma)}(z_1,\dots,z_n)=\int_\gamma \Phi_{k,n,m}(t,z)W_J(t,z)\mathrm{d}t_1\wedge\dots\wedge\mathrm{d}t_k,\qquad J\in\mathcal{I}_k,\quad (2.11)$$

are called the *multidimensional hypergeometric functions* associated with the master function $\Phi_{k,n,m}$.

The fact that solutions in Theorem 2.1 take values in $\mathrm{Sing}L^{\otimes m}[|m|-2k]$ may be reformulated as follows. For any $J\in\mathcal{I}_{k-1}$, we have

$$\sum_{s=1}^{n}(j_s+1)(m_s-j_s)I_{J+\mathbf{1}_s}^{(\gamma)}(z)=0,\qquad (2.12)$$

where we set $I_{J+\mathbf{1}_s}^{(\gamma)}(z)=0$ if $J+\mathbf{1}_s\notin\mathcal{I}_k$.

The pair consisting of the KZ equations (1.4) and hypergeometric solutions (1.2) is identified with the pair consisting of the KZ equations (2.5) and hypergeometric solutions (2.10) with values in $\mathrm{Sing}L^{\otimes m}[|m|-2]$. In this case the system of equations in (2.12) is identified with Eq. (1.3).

2.4 Proof of Theorem 2.1

We sketch the proof following [11]. The reason to present a proof is to show later in Sect. 2.5 how a modification of this reasoning leads to a construction of polynomial solutions of the KZ equations over $\mathbb{F}_p$.

The proof of Theorem 2.1 is a generalization of the proof of Theorem 1.1 and is based on cohomological relations.

It is convenient to reformulate the definition of the hypergeometric integral (2.10). Given $k,n\in\mathbb{Z}_{>0}$ and a multi-index $J=(j_1,\dots,j_n)$ with $|J|\leqslant k$, introduce a differential form

$$\eta_J=\frac{1}{j_1!\cdots j_n!}\mathrm{Ant}_t\Bigg[\frac{\mathrm{d}(t_1-z_1)}{t_1-z_1}\wedge\dots\wedge\frac{\mathrm{d}(t_{j_1}-z_1)}{t_{j_1}-z_1}\wedge\frac{\mathrm{d}(t_{j_1+1}-z_2)}{t_{j_1+1}-z_2}\wedge\dots$$
$$\wedge\frac{\mathrm{d}(t_{j_1+\dots+j_{n-1}+1}-z_n)}{t_{j_1+\dots+j_{n-1}+1}-z_n}\wedge\dots\wedge\frac{\mathrm{d}(t_{j_1+\dots+j_n}-z_n)}{t_{j_1+\dots+j_n}-z_n}\Bigg],$$

which is a logarithmic differential form on $\mathbb{C}^n\times\mathbb{C}^k$ with coordinates z,t. If $|J|=k$, then for any $z\in\mathbb{C}^n$ we have on $\{z\}\times\mathbb{C}^k$ the identity

$$\eta_J=W_J(t,z)\mathrm{d}t_1\wedge\dots\wedge\mathrm{d}t_k.$$

Example 2.1 For $k = n = 2$ we have

$$\eta_{(2,0)} = \frac{\mathrm{d}(t_1 - z_1)}{t_1 - z_1} \wedge \frac{\mathrm{d}(t_2 - z_1)}{t_2 - z_1},$$
$$\eta_{(1,1)} = \frac{\mathrm{d}(t_1 - z_1)}{t_1 - z_1} \wedge \frac{\mathrm{d}(t_2 - z_2)}{t_2 - z_2} - \frac{\mathrm{d}(t_2 - z_1)}{t_2 - z_1} \wedge \frac{\mathrm{d}(t_1 - z_2)}{t_1 - z_2}.$$

The hypergeometric integrals (2.10) can be defined in terms of the differential forms η_J:

$$I^{(\gamma)}(z_1, \dots, z_n) = \sum_{J \in \mathcal{I}_k} \left(\int_{\gamma(z)} \Phi_{k,n,m} \eta_J \right) f_J v_m. \tag{2.13}$$

Introduce the logarithmic differential 1-forms

$$\alpha = \sum_{1 \leqslant i < j \leqslant n} \frac{m_i m_j}{2\kappa} \frac{\mathrm{d}(z_i - z_j)}{z_i - z_j} + \sum_{1 \leqslant i < j \leqslant k} \frac{2}{\kappa} \frac{\mathrm{d}(t_i - t_j)}{t_i - t_j} + \sum_{s=1}^{n} \sum_{i=1}^{k} \frac{-m_s}{\kappa} \frac{\mathrm{d}(t_i - z_s)}{t_i - z_s},$$
$$\alpha' = \sum_{1 \leqslant i < j \leqslant k} \frac{2}{\kappa} \frac{\mathrm{d}(t_i - t_j)}{t_i - t_j} + \sum_{s=1}^{n} \sum_{i=1}^{k} \frac{-m_s}{\kappa} \frac{\mathrm{d}(t_i - z_s)}{t_i - z_s}.$$

We shall use the following algebraic identities for logarithmic differential forms.

Theorem 2.2 [11] *On $\mathbb{C}^n \times \mathbb{C}^k$ we have*

$$\alpha' \wedge \eta_J = \sum_{s=1}^{n} (j_s + 1) \frac{m_s - j_s}{\kappa} \eta_{J + \mathbf{1}_s}, \tag{2.14}$$

for any J with $|J| = k - 1$, and

$$\alpha \wedge \sum_{J \in \mathcal{I}_k} \eta_J f_J v_m = \frac{1}{\kappa} \sum_{i<j} \Omega^{(i,j)} \frac{\mathrm{d}(z_i - z_j)}{z_i - z_j} \wedge \sum_{|J|=k} \eta_J f_J v_m. \tag{2.15}$$

Proof Identity (2.14) is the special case of Theorem 6.16.2 in [11] for the Lie algebra $\mathfrak{sl}_2$. Identity (2.15) is a special case of Theorem 7.5.2$''$ in [11] for the Lie algebra $\mathfrak{sl}_2$. □

Corollary 2.3 *On $\mathbb{C}^n \times \mathbb{C}^k$ we have*

$$\sum_{J \in \mathcal{I}_k} \mathrm{d}(\Phi_{k,n,m} \eta_J) f_J v_m = \frac{1}{\kappa} \sum_{i<j} \Omega^{(i,j)} \frac{\mathrm{d}(z_i - z_j)}{z_i - z_j} \wedge \sum_{J \in \mathcal{I}_k} (\Phi_{k,n,m} \eta_J) f_J v_m, \tag{2.16}$$

where the differential is taken with respect to variables z, t.

Now we deduce from identity (2.14) the following formula (2.20). Since $|J| = k-1$, we can write

$$\eta_J = \sum_{l=1}^{k} c_{J,l}(t,z) \mathrm{d}t_1 \wedge \cdots \wedge \widehat{\mathrm{d}t_l} \wedge \cdots \wedge \mathrm{d}t_k + \cdots, \tag{2.17}$$

where the dots denote the terms having differentials $\mathrm{d}z_i$ and $c_{J,l}(t,z)$ are rational functions of the form

$$P_{J,l}(t,z) \left(\prod_{1 \leqslant i < j \leqslant n} (z_i - z_j) \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j) \prod_{l=1}^{n} \prod_{i=1}^{k} (t_i - z_l) \right)^{-1}, \tag{2.18}$$

where $P_{J,l}(t,z)$ is a polynomial in t, z with integer coefficients. Also for any $s = 1, \ldots, n$ we have

$$\eta_{J+\mathbf{1}_s} = W_{J+\mathbf{1}_s} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_k + \cdots, \tag{2.19}$$

where the dots denote the terms having differentials $\mathrm{d}z_i$. Formula (2.14) implies that for any J with $|J| = k-1$ we have the identity

$$\begin{aligned} &\mathrm{d}_t \left(\Phi_{k,n,m} \sum_{l=1}^{k} c_{J,l}(t,z) \mathrm{d}t_1 \wedge \cdots \wedge \widehat{\mathrm{d}t_l} \wedge \cdots \wedge \mathrm{d}t_k \right) \\ &\quad = \sum_{s=1}^{n} (j_s + 1) \frac{m_s - j_s}{\kappa} \Phi_{k,n,m} W_{J+\mathbf{1}_s} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_k, \end{aligned} \tag{2.20}$$

where d_t denotes the differential with respect to the variables t.

Now we deduce from identity (2.16) the following formula (2.23). Fix $i \in \{1, \ldots, n\}$. For any $J \in \mathcal{I}_k$, write

$$\begin{aligned} \Phi_{k,n,m} \eta_J &= \Phi_{k,n,m} W_J \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_k \\ &\quad + \mathrm{d}z_i \wedge \left(\Phi_{k,n,m} \sum_{l=1}^{k} c_{J,i,l}(t,z) \mathrm{d}t_1 \wedge \cdots \wedge \widehat{\mathrm{d}t_l} \wedge \cdots \wedge \mathrm{d}t_k \right) + \cdots, \end{aligned} \tag{2.21}$$

where the dots denote the terms which contain $\mathrm{d}z_j$ with $j \neq i$, and the coefficients $c_{J,i,l}(t,z)$ are rational functions in t, z of the form

$$P_{J,i,l}(t,z) \left(\prod_{1 \leqslant i < j \leqslant n} (z_i - z_j) \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j) \prod_{l=1}^{n} \prod_{i=1}^{k} (t_i - z_l) \right)^{-1}, \tag{2.22}$$

where $P_{J,i,l}(t,z)$ is a polynomial in t, z with integer coefficients.

Formula (2.16) implies that for any $i \in \{1, \dots, n\}$ we have

$$\sum_{J\in\mathcal{I}_k} \Big(\frac{\partial}{\partial z_i}\big(\Phi_{k,n,m} W_J\big)\, \mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_k \\ +\mathrm{d}_t\Big(\Phi_{k,n,m} \sum_{l=1}^{n} c_{J,i,l}(t,z)\mathrm{d}t_1 \wedge \dots \wedge \widehat{\mathrm{d}t_l} \wedge \dots \wedge \mathrm{d}t_k\Big)\Big) f_J v_m \\ = \frac{1}{\kappa} \sum_{j\neq i} \frac{\Omega^{(i,j)}}{z_i - z_j} \sum_{J\in\mathcal{I}_k} \Phi_{k,n,m} W_J \mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_k f_J v_m, \tag{2.23}$$

where d_t denotes the differential with respect to the variables t.

Integrating both sides of Eqs. (2.20) and (2.23) over $\gamma(z)$ and using Stokes' theorem, we obtain Eqs. (2.12) and (2.5) for the vector $I^{(\gamma)}(z)$ in (2.10). Theorem 2.1 is proved.

2.5 Solutions of KZ equations with values in $\mathrm{Sing}L^{\otimes m}[|m| - 2k]$ over $\mathbb{F}_p$

Given $k, n \in \mathbb{Z}_{>0}$, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$, $\kappa \in \mathbb{Q}^\times$, let $p > 2$ be a prime number such that p does not divide the numerator of κ. In this case Eqs. (2.12) and (2.5) are well defined over the field $\mathbb{F}_p$ and we may discuss their polynomial solutions in $\mathbb{F}_p[z_1, \dots, z_n]$.

Choose positive integers M_s for $s = 1, \dots, n$, $M_{i,j}$ for $1 \leqslant i < j \leqslant n$, and M^0, such that

$$M_s \equiv -\frac{m_s}{\kappa}, \qquad M_{i,j} \equiv \frac{m_i m_j}{2\kappa}, \qquad M^0 \equiv \frac{2}{\kappa} \qquad (\mathrm{mod}\ p).$$

Fix integers $q = (q_1, \dots, q_k)$. Let $t = (t_1, \dots, t_k)$, $z = (z_1, \dots, z_n)$ be variables. Define the *master polynomial*

$$\Phi^{(p)}_{k,n,M}(t,z) := \Phi^{(p)}_{k,n,M}(t_1, \dots, t_k, z_1, \dots, z_n) \\ = \prod_{1\leqslant i<j\leqslant n} (z_i - z_j)^{M_{i,j}} \prod_{1\leqslant i\leqslant j\leqslant k} (t_i - t_j)^{M^0} \prod_{s=1}^{n}\prod_{i=1}^{k} (t_i - z_s)^{M_s}. \tag{2.24}$$

Consider the Taylor expansion of the vector

$$\sum_{J\in\mathcal{I}_k} \Phi^{(p)}_{k,n,M}(t,z) W_J(t,z) f_J v_m = \sum_{i_1,\dots,i_k} \bar{I}^{(i_1,\dots,i_k)}(z,q)(t_1 - q_1)^{i_1} \dots (t_k - q_k)^{i_k}. \tag{2.25}$$

Notice that each coordinate $\Phi^{(p)}_{k,n,M}(t,z) W_J(t,z)$ is a polynomial in t, z with integer coefficients due to the positivity of the integers M_s, $M_{i,j}$, M^0 and the definition of the weight functions $W_J(t,z)$. Hence the Taylor coefficients $\bar{I}^{(i_1,\dots,i_k)}(z,q)$ are vectors of polynomials in z with integer coefficients. Let $I^{(i_1,\dots,i_k)}(z,q) \in (\mathbb{F}_p[z])^{\dim L^{\otimes m}[|m|-2k]}$ be their canonical projection modulo p.

Theorem 2.4 *For any integers $q = (q_1, \dots, q_k)$ and positive integers $l = (l_1, \dots, l_k)$, the vector of polynomials $I(z,q) := I^{(l_1 p-1,\dots,l_k p-1)}(z,q)$ satisfies Eqs. (2.12) and (2.5).*

The parameters $q, l_1 p - 1, \dots, l_k p - 1$ are analogs of cycles γ in Sect. 2.3.

Proof To prove that $I^{(l_1 p-1,\dots,l_k p-1)}(z,q)$ satisfies (2.12) and (2.5), consider the Taylor expansions at $(t_1,\dots,t_k) = (q_1,\dots,q_k)$ of both sides of Eqs. (2.20) and (2.23), and divide them by $\mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_k$. Notice that the Taylor expansions are well defined due to formulas (2.18) and (2.22).

Project the Taylor coefficients of $(t_1 - q_1)^{l_1 p-1} \dots (t_k - q_k)^{l_k p-1}$ to $(\mathbb{F}_p[z])^{\dim L^{\otimes m}[|m|-2k]}$. Then the terms coming from the $\mathrm{d}_t(\,)$-summands equal zero since $\mathrm{d}(t_i^{l_i p})/\mathrm{d}t_i = l_i p t_i^{l_i p-1} \equiv 0 \pmod p$, and we obtain Eqs. (2.12) and (2.5). $\square$

Example 2.2 Let $p = 3, \kappa = 4, n = 5, k = 2, m_1 = \dots = m_5 = 1$. Notice that in this case $\kappa \equiv 1 \pmod 3$ and we may set $\kappa = 1$.

The set $\mathcal{I}_k$ consists of ten elements $J = (j_1, \dots, j_5)$ with $j_i \in \{0,1\}$ and $j_1 + \dots + j_5 = 2$. The space $L^{\otimes m}[|m|-2k] = (L_1)^{\otimes 5}[1]$ has basis $f_J v_m = f^{j_1} v_1 \otimes \dots \otimes f^{j_5} v_1$, $J \in \mathcal{I}_k$. We have

$$\begin{aligned}
\Omega^{(1,2)} v_1 \otimes v_1 \wedge \dots &\equiv -v_1 \otimes v_1 \wedge \dots, \\
\Omega^{(1,2)} f v_1 \otimes f v_1 \wedge \dots &\equiv -f v_1 \otimes f v_1 \wedge \dots, \\
\Omega^{(1,2)} f v_1 \otimes v_1 \wedge \dots &\equiv f v_1 \otimes v_1 \wedge \dots + v_1 \otimes f v_1 \wedge \dots, \\
\Omega^{(1,2)} v_1 \otimes f v_1 \wedge \dots &\equiv f v_1 \otimes v_1 \wedge \dots + v_1 \otimes f v_1 \wedge \dots
\end{aligned}$$

(mod 3). The other $\Omega^{(i,j)}$ act similarly. The system of equations (2.12) on $I(z) = \sum_{J \in \mathcal{I}_k} I_J(z) f_J v_m$ consists of five equations. The first is

$$I_{(1,1,0,0,0)}(z) + I_{(1,0,1,0,0)}(z) + I_{(1,0,0,1,0)}(z) + I_{(1,0,0,0,1)}(z) \equiv 0 \pmod 3,$$

where $z = (z_1, \dots, z_5)$; the others are similar. Let $t = (t_1, t_2)$. We may choose the master polynomial

$$\Phi^{(p=3)}_{2,5,M}(t,z) = (t_1 - t_2)^2 \prod_{1 \leqslant i < j \leqslant 5} (z_i - z_j)^2 \prod_{i=1}^{2} \prod_{s=1}^{5} (t_i - z_s)^2.$$

Fix integers $q = (0,0)$ and $l = (4,3)$. Then the vector

$$I^{(11,8)}(z) = \sum_{J \in \mathcal{I}_k} I^{(11,8)}_J(z) f_J v_m \tag{2.26}$$

with

$$I^{(11,8)}_{(1,1,0,0,0)}(z) = -z_3 - z_4 - z_5, \quad I^{(11,8)}_{(1,0,1,0,0)}(z) = -z_2 - z_4 - z_5, \tag{2.27}$$

and similar other coordinates satisfy Eqs. (2.12) and (2.5).

Example 2.3 Let $\kappa = 4$, $n = 2$, $k = 2$, $m_1 = m_2 = 2$. The space $L_2^{\otimes 2}[0]$ has basis $f^2 v_2 \otimes v_2$, $f v_2 \otimes f v_2$, $v_2 \otimes f^2 v_2$. The system of equations (2.12) takes the form:

$$I_{(2,0)}(z) + I_{(1,1)}(z) = 0, \quad I_{(1,1)}(z) + I_{(0,2)}(z) = 0.$$

Let $p = 4l + 3$ for some l. We may choose

$$\Phi^{(p)}_{2,2,M}(t_1, t_2, z_1, z_2) = (z_1 - z_2)^{\frac{p+1}{2}} (t_1 - t_2)^{\frac{p+1}{2}} \prod_{i=1}^{2} \prod_{s=1}^{2} (t_i - z_s)^{\frac{p-1}{2}}.$$

Notice that $\frac{p+1}{2}$ is even, the polynomial $\Phi^{(p)}_{2,2,M}(t_1, t_2, z_1, z_2)$ is symmetric with respect to permutation of t_1, t_2, and the solution

$$\begin{aligned} &I^{(p-1,p-1)}(z_1, z_2) \\ &\quad = (z_1 - z_2)^{\frac{p+1}{2}} \big(c_{(2,0)}(z_1, z_2) f^2 v_2 \otimes v_2 + c_{(1,1)}(z_1, z_2) f v_2 \otimes f v_2 \\ &\qquad + c_{(2,0)}(z_1, z_2) v_2 \otimes f^2 v_2 \big) \end{aligned} \tag{2.28}$$

is nonzero. Here $c_J(z_1, z_2)$ are the polynomials determined by the construction of Sect. 2.5.

For example, for $p = 3$,

$$I^{(2,2)}(z) = (z_1 - z_2)^2 (f^2 v_2 \otimes v_2 - f v_2 \otimes f v_2 + v_2 \otimes f^2 v_2). \tag{2.29}$$

2.6 Relation of solutions to integrals over $\mathbb{F}_p^k$

For a polynomial $F(t_1, \dots, t_k) \in \mathbb{F}_p[t_1, \dots, t_k]$ and a subset $\gamma \subset \mathbb{F}_p^k$ define the integral

$$\int_\gamma F(t_1, \dots, t_k) := \sum_{(t_1, \dots, t_k) \in \gamma} F(t_1, \dots, t_k).$$

Theorem 2.5 *Fix $x_1, \dots, x_n \in \mathbb{F}_p$. Consider the vector of polynomials*

$$F(t) := \Phi^{(p)}_{k,n,M}(t_1, \dots, t_k, x_1, \dots, x_n) \sum_{J \in \mathcal{I}_k} W_J(t_1, t_2, x_1, \dots, x_n) f_J v_m,$$

of formula (2.25). Assume that $\deg_{t_i} F(t_1, \dots, t_k) < 2p-2$ for $i = 1, \dots, k$. Consider the solution $I^{(p-1,\dots,p-1)}(z, q)$ of Eqs. (2.12) and (2.5), described in Theorem 2.4.

Then

$$I^{(p-1,\dots,p-1)}(x_1,\dots,x_n,q) = (-1)^k \int_{\mathbb{F}_p^k} F(t_1,\dots,t_k). \tag{2.30}$$

This integral is a p-analog of the hypergeometric integral (2.11).

Proof Theorem 2.5 is a straightforward corollary of formula (1.17), cf. the proof of Theorem 1.3. □

Example 2.4 The polynomial $F(t_1,t_2)$ of Example 2.3 satisfies the inequalities $\deg_{t_i} F(t_1,t_2) < 2p-2$ for $i=1,2$.

2.7 Example of a p-analog of skew-symmetry

For $J \in \mathcal{I}_k$, the differential forms $W_J(t,z)\mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_k$ are skew-symmetric with respect to permutations of $t_1,\dots,t_k$. Here is an example of a p-analog of that skew-symmetry. For another demonstration of the skew-symmetry, see Example 2.5.

Consider the KZ differential equations with parameters $n,k,\kappa,m_1,\dots,m_n \in \mathbb{Z}_{>0}$, where $\kappa, m_1,\dots,m_n$ are even, $\kappa = 2\kappa'$, $m_1 = 2m_1',\dots,m_n = 2m_n'$. Assume that κ' is even and the prime p is such that $\kappa' \big| (p-1)$ and $(p-1)/\kappa'$ is odd, cf. Example 2.5. Choose

$$\begin{aligned}\Phi^{(p)}_{k,n,M}(t,z) &= \prod_{1\leqslant i<j\leqslant n}(z_i-z_j)^{M_{i,j}} \prod_{1\leqslant i<j\leqslant k}(t_i-t_j)^{p-\frac{p-1}{\kappa'}} \prod_{i=1}^{k}\prod_{s=1}^{n}(t_i-z_s)^{m_s'\frac{p-1}{\kappa'}}\\ &= \prod_{1\leqslant i<j\leqslant n}(z_i-z_j)^{M_{i,j}} \Bigg(\prod_{1\leqslant i<j\leqslant k}(t_i-t_j)^{\kappa'-1}\prod_{i=1}^{k}\prod_{s=1}^{n}(t_i-z_s)^{m_s'}\Bigg)^{\frac{p-1}{\kappa'}} \prod_{1\leqslant i<j\leqslant k}(t_i-t_j).\end{aligned} \tag{2.31}$$

Notice that

$$\varphi(t,z) := \prod_{1\leqslant i<j\leqslant k}(t_i-t_j)^{\kappa'-1}\prod_{i=1}^{k}\prod_{s=1}^{n}(t_i-z_s)^{m_s'} \tag{2.32}$$

as well as the product $\prod_{1\leqslant i<j\leqslant k}(t_i-t_j)$ are skew-symmetric with respect to permutations of $t_1,\dots,t_k$.

Let a be a generator of the cyclic group $\mathbb{F}_p^\times$. Let $x=(x_1,\dots,x_n)\in\mathbb{F}_p^n$. For $\ell = 1,\dots,\kappa'$, denote

$$\gamma_\ell(x) = \left\{ t\in\mathbb{F}_p^k \mid \varphi(t,x)^{\frac{p-1}{\kappa'}} = a^{\ell\frac{p-1}{\kappa'}} \right\}, \qquad \gamma_0(x) = \left\{ t\in\mathbb{F}_p^k \mid \varphi(t,x)=0 \right\}. \tag{2.33}$$

The partition of $\mathbb{F}_p^k$ by subsets $(\gamma_\ell(x))_{\ell=0}^{\kappa'}$ is invariant with respect to the action of the symmetric group S_k of permutations of $t_1,\dots,t_k$. For every ℓ, the subset $\gamma_\ell(x)$ is invariant with respect to the action of the alternating subgroup $A_k \subset S_k$. For $J \in \mathcal{I}_k$ the restriction of the function $W_J(t,x)\prod_{1\leqslant i<j\leqslant k}(t_i-t_j)$ to the set $\gamma_\ell(x)$ is A_k-invariant. We have

$$\int_{\mathbb{F}_p^k} \Phi^{(p)}_{k,n,M}(t,z)W_J(t,x)$$
$$= \prod_{1\leqslant i<j\leqslant n}(z_i-z_j)^{M_{i,j}} \sum_{\ell=1}^{\kappa'/2} 2a^{\ell\frac{p-1}{\kappa'}} \int_{\gamma_\ell(x)} W_J(t,x) \prod_{1\leqslant i<j\leqslant k}(t_i-t_j).$$

2.8 Relation of solutions to surfaces over $\mathbb{F}_p$

Example 2.5 For distinct $x_1, x_2 \in \mathbb{F}_p$ let $\Gamma(x_1,x_2)$ be the closure in $P^1(\mathbb{F}_p)\times P^1(\mathbb{F}_p)$ of the affine surface

$$y^2 = (t_1-t_2)(t_1-x_1)(t_2-x_1)(t_1-x_2)(t_2-x_2), \tag{2.34}$$

where $P^1(\mathbb{F}_p)$ is the projective line over $\mathbb{F}_p$. For a rational function $h:\Gamma(x_1,x_2)\to\mathbb{F}_p$ define the integral

$$\int_{\Gamma(x_1,x_2)} h = \sum_{P\in\Gamma}{}' h(P), \tag{2.35}$$

as the sum over all points $P\in\Gamma(x_1,x_2)$, where $h(P)$ is defined.

Recall

$$W_{(2,0)}(t_1,t_2,x_1,x_2) = \frac{1}{t_1-x_1}\frac{1}{t_2-x_1}, \qquad W_{(0,2)}(t_1,t_2,x_1,x_2) = \frac{1}{t_1-x_2}\frac{1}{t_2-x_2},$$
$$W_{(1,1)}(t_1,t_2,x_1,x_2) = \frac{1}{t_1-x_1}\frac{1}{t_2-x_2} + \frac{1}{t_2-x_1}\frac{1}{t_1-x_2}.$$

Theorem 2.6 *Let $p=4l+3$ for some l. Let*

$$c_{(2,0)}(z_1,z_2)f^2v_2\otimes v_2 + c_{(1,1)}(z_1,z_2)fv_2\otimes fv_2 + c_{(2,0)}(z_1,z_2)v_2\otimes f^2v_2$$

be the vector of polynomials appearing in the solution (2.28) of the KZ equations of Example 2.3. Then

$$c_{(2,0)}(x_1,x_2) = \int_{\Gamma(x_1,x_2)} \frac{t_1-t_2}{(t_1-x_1)(t_2-x_1)},$$
$$c_{(1,1)}(x_1,x_2) = \int_{\Gamma(x_1,x_2)} \frac{t_1-t_2}{(t_1-x_1)(t_2-x_2)} + \int_{\Gamma(x_1,x_2)} \frac{t_1-t_2}{(t_2-x_1)(t_1-x_2)},$$
$$c_{(0,2)}(x_1,x_2) = \int_{\Gamma(x_1,x_2)} \frac{t_1-t_2}{(t_1-x_2)(t_2-x_2)}. \tag{2.36}$$

Proof The values of $W_J(t_1, t_2, x_1, x_2)$ at infinite points of $\Gamma(x_1, x_2)$ are equal to zero, so the integrals are sums over points of the affine surface. We prove the first equality in (2.36). We have

$$\begin{aligned}
&\int_{\Gamma(x_1,x_2)} \frac{t_1 - t_2}{(t_1 - x_1)(t_2 - x_1)} \\
&= \sum_{t_1,t_2 \neq x_1} \frac{t_1 - t_2}{(t_1 - x_1)(t_2 - x_1)} \\
&\quad + \sum_{t_1,t_2} \frac{t_1 - t_2}{(t_1 - x_1)(t_2 - x_1)} \Big((t_1 - t_2) \prod_{i=1}^{2} \prod_{s=1}^{2} (t_i - x_s) \Big)^{\frac{p-1}{2}} \\
&= \sum_{t_1,t_2 \in \mathbb{F}_p} [(t_1 - x_2)^{p-2} - (t_1 - x_1)^{p-2}] \\
&\quad + \sum_{t_1,t_2 \in \mathbb{F}_p} \sum_{i_1,i_2} c^{i_1,i_2}(x_1, x_2) t_1^{i_1} t_2^{i_2} = c_{(2,0)}(x_1, x_2).
\end{aligned}$$

□

Remark Consider the projection $\Gamma(x_1, x_2) \to \mathbb{F}_p^2$, $(t_1, t_2, y) \mapsto (t_1, t_2)$. For any distinct $t_1, t_2 \in \mathbb{F}_p$ exactly one of the two points (t_1, t_2), (t_2, t_1) lies in the image of the projection, since $(t_1 - t_2)(t_1 - x_1)(t_2 - x_1)(t_1 - x_2)(t_2 - x_2)$ is skew-symmetric in t_1, t_2 and -1 is not a square if $p = 4l + 3$.

3 Resonances in $\mathfrak{sl}_2$ KZ equations

3.1 Resonances in conformal field theory over $\mathbb{C}$

Let $m_1, \dots, m_n, k \in \mathbb{Z}_{>0}$, $L^{\otimes m} = L_{m_1} \otimes \dots \otimes L_{m_n}$. Assume that $\kappa > 2$ is an integer. Assume that

$$0 \leqslant m_1, \dots, m_n, m_1 + \dots + m_n - 2k \leqslant \kappa - 2.$$

Consider the positive integer

$$\ell = \kappa - 1 - |m| + 2k. \tag{3.1}$$

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ with distinct coordinates define

$$B_{k,n,m}(z) = \Big\{ w \in L^{\otimes m} \mid h.w = (|m| - 2k)w,\ e.w = 0,\ (ze)^\ell w = 0 \Big\},$$

where $ze : L^{\otimes m} \to L^{\otimes m}$ is the linear operator defined by the formula

$$w_1 \otimes \dots \otimes w_n \ \mapsto \ \sum_{s=1}^{n} z_s w_1 \otimes \dots \otimes e w_s \otimes \dots \otimes w_n,$$

for any $w_1 \otimes \cdots \otimes w_n \in L^{\otimes m}$. This vector space is called the *space of conformal blocks*.

Example 3.1 Let $k = 1$, $|m| = \kappa$, $\ell = 1$, Then

$$B_{k,n,m}(z) = \Big\{ \sum_{s=1}^{n} I_s v_{m_1} \otimes \cdots \otimes f v_{m_s} \otimes \cdots \otimes v_{m_n} \,\Big|\, \sum_{s=1}^{n} m_s I_s = 0,\ \sum_{s=1}^{n} z_s m_s I_s = 0 \Big\}.$$

Theorem 3.1 [4,5] *The family of subspaces*

$$B_{k,n,m}(z) \subset \mathrm{Sing} L^{\otimes m}[|m| - 2k],$$

depending on z, is invariant with respect to the KZ equations. □

Theorem 3.2 [4,5] *All the hypergeometric solutions of the KZ equations with values in* $\mathrm{Sing} L^{\otimes m}[|m| - 2k]$*, constructed in Sect.* 2.3*, take values in the subspaces of conformal blocks.*

Proof Theorem 3.2 is proved in [4]. Another proof for arbitrary simple Lie algebras is given in [5]. Let $I^{(\gamma)}(z) = \sum_{J\in\mathcal{I}_k} I_J^{(\gamma)}(z) F_J v_m$ be a hypergeometric solution. We need to check that $(ze)^{\ell} I^{(\gamma)}(z) = 0$. This equation is a system of algebraic equations on the coefficients $(I_J^{(\gamma)}(z))_{J\in\mathcal{I}_k}$. The equations of the system are labeled by basis vectors of $L^{\otimes m}[|m| - 2(k-\ell)]$. Namely, for any $Q \in \mathcal{I}_{k-\ell}$ one calculates the coefficient of $F_Q v_m$ in $(ze)^{\ell} I^{(\gamma)}(z)$ and equates that coefficient to zero, cf. the second equation in Example 3.1. Such an equation follows from a cohomological relation. Namely, the corresponding differential k-form, whose integral over $\gamma(z)$ has to be zero, equals the differential with respect to the t-variables of some differential $k-1$-form $\eta_{n,k,\ell,Q}(t, z)$. Then the desired equation holds by Stokes' theorem, see this reasoning on pp. 182–184 in [4]. This proves Theorem 3.2. □

Remark That $k - 1$-form $\eta_{n,k,\ell,Q}(t, z)$ is determined by the numbers n, k, ℓ and the index Q and has the form

$$\eta_{n,k,\ell,Q}(t,z) = \frac{\Phi_{k,n,m}(t,z)}{\prod_{1\leqslant i<j\leqslant n}(z_i - z_j)\prod_{1\leqslant i<j\leqslant k}(t_i - t_j)\prod_{i=1}^{k}\prod_{s=1}^{n}(t_i - z_s)}\,\mu_{n,k,\ell,Q}(t,z), \tag{3.2}$$

where $\mu_{n,k,\ell,Q}(t, z)$ is a polynomial differential $k - 1$-form in t, z with integer coefficients determined by n, k, ℓ, Q only, see pp. 182–184 in [4].

3.2 Resonances over $\mathbb{F}_p$

Given $k, n \in \mathbb{Z}_{>0}$, $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>0}$, $\kappa \in \mathbb{Z}_{>0}$, let $p > 2$ be a prime number such that p does not divide κ. Choose positive integers M_s for $s = 1, \ldots, n$, $M_{i,j}$ for

$1 \leqslant i < j \leqslant n$, M^0 and K such that

$$M_s \equiv -\frac{m_s}{\kappa}, \quad M_{i,j} \equiv \frac{m_i m_j}{2\kappa}, \quad M^0 \equiv \frac{2}{\kappa}, \quad K \equiv \frac{1}{\kappa} \qquad (\text{mod } p).$$

Fix integers $q = (q_1, \dots, q_k)$. As in Sect. 2.5 for any nonnegative integers $l_1, \dots, l_k$ define the vector $I^{(i_1,\dots,i_k)}(z, q) \in (\mathbb{F}_p[z])^{\dim L^{\otimes m}[|m|-2k]}$.

Theorem 3.3 *Let $\ell \in \mathbb{Z}_{>0}$ be such that*

$$(\ell - 1)K - \sum_{s=1}^{n} M_s - (k-1)M^0 \equiv 1 \pmod{p}. \tag{3.3}$$

Then for any integers $q = (q_1, \dots, q_k)$ and positive integers $l = (l_1, \dots, l_k)$, the vector of polynomials $I^{(l_1 p-1,\dots,l_k p-1)}(z, q)$ satisfies the equation

$$(ze)^\ell I^{(l_1 p-1,\dots,l_k p-1)}(z, q) = 0. \tag{3.4}$$

Remark The resonance equation (3.1) has the form

$$\frac{\ell - 1}{\kappa} = 1 - \frac{|m|}{\kappa} + \frac{2}{\kappa}(k-1).$$

Equation (3.3) is the reduction modulo p of that equation.

Proof The proof of Theorem 3.3 is similar to the proof of Theorem 2.1 and uses the universal differential $k-1$-forms $\eta_{n,k,\ell,Q}(t, z)$ of Sect. 3.1 instead of the differential $k-1$-forms $\eta_J(t, z)$ in (2.17). □

Example 3.2 Let $p = 3$, $\kappa = 4$, $n = 5$, $k = 2$, $m_1 = \dots = m_5 = 1$. Consider the vector $I^{(11,8)}(z) = \sum_{J\in\mathcal{I}_k} I_J^{(11,8)}(z) f_J v_m$ of Example 2.2, which is a solution of (2.5) and (2.12). The resonance equation (3.3) in this case takes the form $\ell + 1 \equiv 0$ (mod 3) and is satisfied for $\ell = 2$. The condition $(ze)^2 I^{(11,8)}(z) = 0$ means

$$\sum_{J=(j_1,\dots,j_5)\in\mathcal{I}_k} I_J^{(11,8)}(z) \prod_{i=1;\, j_i=1}^{5} z_i \equiv 0 \pmod{3}. \tag{3.5}$$

Equation (3.5) takes the form

$$-z_1 z_2(z_3 + z_4 + z_5) - \dots - z_4 z_5(z_1 + z_2 + z_3) = -3 \sum_{1\leqslant i<j<k\leqslant 5} z_i z_j z_k \equiv 0$$

(mod 3).

4 KZ equations over $\mathbb{F}_p$ for other Lie algebras

The KZ equations are defined for any simple Lie algebra $\mathfrak{g}$ or more generally for any Kac–Moody algebra, see for example [11]. Similarly to what was done in Sects. 2 and 3, one can construct polynomial solutions of those KZ equations over $\mathbb{F}_p$ as well as of the singular vector equations and resonance equations over $\mathbb{F}_p$.

The construction of the polynomial solutions over $\mathbb{F}_p$ in the $\mathfrak{sl}_2$ case was based on the algebraic identities for logarithmic differential forms (2.14), (2.15) and the associated cohomological relations (2.20), (2.23) as well as on the cohomological relations associated with the differential forms $\eta_{n,k,\ell,K}(t,z)$ in (3.2). For an arbitrary Kac–Moody algebra the analogs of the algebraic identities in (2.14) and (2.15) are the identities of Theorems 6.16.2 and 7.5.2$''$ in [11], respectively. For an arbitrary simple Lie algebra, the construction of analogs of the cohomological identities for the differential forms $\eta_{n,k,\ell,K}(t,z)$ is the main result of [5].

Remark For the $\mathbb{F}_p$-analogs of multidimensional hypergeometric integrals associated with arrangements of hyperplanes, see [14]. For Remarks on the Gaudin model and Bethe ansatz over $\mathbb{F}_p$, see [15].

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